Exercises from week 1:

1. Show that for any $\varepsilon > 0$ and $0 < \alpha < 1$, there is $C_\varepsilon = C(\varepsilon, \alpha, d) > 0$ such that for any $u \in C^{2,\alpha}(B_1)$ we have

$$\|D^2u\|_{L^\infty(B_1)} \leq \varepsilon[D^2u]_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}.$$

and

$$\|u\|_{C^2(B_1)} \leq \varepsilon[D^2u]_{C^\alpha(B_1)} + C_\varepsilon\|u\|_{L^\infty(B_1)}.$$

To prove above inequalities, you may first want to prove that

$$\|u\|_{C^2(B_1)} \leq C_1[D^2u]_{C^\alpha(B_1)} + C_2\|u\|_{L^\infty(B_1)}.$$

2. (To be discussed next week) Let $B_r^+ := \{x \in B_r : x_n \geq 0\}$. If $u \in C^{2,\alpha}(B_1^+)$ solves $-\Delta u = f$ in $B_1 \cap \{x_n > 0\}$ and $u = 0$ on $\{x_n = 0\} \cap B_1$, then show that

$$\|u\|_{C^{2,\alpha}(B_{1/2}^+)} \leq C(\|u\|_{L^\infty(B_1^+)} + \|f\|_{C^\alpha(B_1^+)}).$$