

Math 251A Fall 2026: Homework 2. due 10/28

1. [Long-time behavior] Let $U := \{|x| < 1\}$. Suppose that u is a sufficiently smooth solution of the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U \times (0, \infty); \\ u = g(x) & \text{on } \partial U \times [0, \infty); \\ u(x, 0) = f(x) & \text{in } U. \end{cases}$$

- (a) Show that $v = (u_t)^2$ solves $v_t - \Delta v \leq 0$ in $U \times (0, \infty)$.
- (b) Show that v decays to zero exponentially fast as $t \rightarrow \infty$.
- (c) Does the limit $u^*(x) := \lim_{t \rightarrow \infty} u(x, t)$ exist in $\bar{U}$? Can we have an explicit formula for u^* ?

2. Let $f \in L^\infty(\mathbb{R}^d \times (0, \infty))$ and define u by Duhamel's formula,

$$u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(x - y, t - s) f(y, s) dy ds.$$

- (a) Show that $u \in C(\mathbb{R}^d \times [0, \infty))$ with $|u(x, t)| \leq t \|f\|_\infty$.
- (b) Show that u is a *weak solution* of $u_t - \Delta u = f$, namely that for every $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} u(-\varphi_t - \Delta \varphi) dx dt = \int_0^\infty \int_{\mathbb{R}^d} f \varphi dx dt.$$

- (c) Show that u is C^1 in x but not necessarily C^2 in x .

3. Here we will prove existence of a solution to

$$\begin{cases} u_t - \Delta u = u^2 & \text{in } \mathbb{R}^d \times [0, T]; \\ u(\cdot, 0) = g \in C(\mathbb{R}). \end{cases}$$

- (a) For $g \equiv C > 0$, find the solution, and show that it blows up at $t = 1/C$.

(b) Let $X := L^\infty(\mathbb{R}^d \times [0, T])$ and define $F : X \rightarrow X$ by

$$F(u)(\cdot, t) := \Phi(\cdot, t) * g + \int_0^t \Phi(\cdot, t-s) * u^2(\cdot, s) ds.$$

Show that F is a contraction map on a ball in $L^\infty(\mathbb{R}^d \times [0, T])$ that includes g , when $T \leq C_g$.

(c) Conclude that there is a weak solution to (P) for $T \leq C_g$.

(d) (Optional) Show that u is in fact a classical solution to (P) for $T \leq C_g$ by a “bootstrapping” argument.

4. Let $g, h \in L^1_{loc}(\mathbb{R}^d)$. We say that $u \in L^1_{loc}(\mathbb{R} \times [0, \infty))$ is a *weak solution* of

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad u = g, \quad u_t = h \quad \text{on } \mathbb{R}^d \times \{t = 0\},$$

if for every $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, \infty))$

$$\int_0^\infty \int_{\mathbb{R}^d} u (\varphi_{tt} - \Delta \varphi) dx dt = \int_{\mathbb{R}^d} h(x) \varphi(x, 0) dx - \int_{\mathbb{R}^d} g(x) \varphi_t(x, 0) dx. \quad (1)$$

We verify that d’Alembert formula defines a unique weak solution in $d = 1$.

(a) Let $F, G \in \mathcal{D}'(\mathbb{R})$ and $u(x, t) = F(x+t) + G(x-t)$. Show that for every $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty))$,

$$\int_0^\infty \int_{\mathbb{R}} u (\varphi_{tt} - \varphi_{xx}) dx dt = - \int_{\mathbb{R}} (F+G)(x) \varphi_t(x, 0) dx + \int_{\mathbb{R}} (G-F)(x) \varphi_x(x, 0) dx. \quad (2)$$

It follows that d’Alembert’s formula defines a weak solution.

(b) Show that the weak solution is unique: if u_1, u_2 are weak solutions with the same g, h , then $u_1 = u_2$ almost everywhere.

5. Let u be a radial weak solution of $u_{tt} - \Delta u = 0$ in $\mathbb{R}^3$ with $u(\cdot, 0) = g$ and $u_t(\cdot, 0) = 0$ in $L^1_{loc}(\mathbb{R}^3 \times [0, \infty))$.

(a) Obtain a formula for $u(x, t)$ in terms of G .

(b) Show that if $g(x) = (1 - k|x|)_+$, $u(0, t)$ experiences a discontinuity at $t = 1$. What happens to $\|u\|_{L^1}$ over time?

(c) What happens to $\|u(0, t)\|_{L^\infty}$ over time if $g(x) = (1 - k(|x| - 1))_+$?

6. Let u be as given in problem 5. Show that if $\|u(\cdot, t)\|_{L^p} \leq C\|g\|_{L^p}$ where C is independent of g , then p must be 2.

7. Evan p 90, problem 24.