

Math 251A Fall 2026: Homework 1

1. Let $\vec{b}(x, t) : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ be C^1 in space and continuous in time. Let $X(x, t)$ solve the ODE

$$\dot{X}(x, t) = \vec{b}(X(t), t), \quad X(x, 0) = x.$$

Show that, for any given $t > 0$, $X(\cdot, t)$ is C^1 .

Hint: Pick any unit vector $e \in \mathbb{S}^d$, and consider the ODE for

$$Y_h(t) := \frac{X(x + he, t) - X(x, t)}{h} \text{ for } h > 0.$$

Using Grönwall, show that $Y_h(t) \rightarrow A(t)$ as $h \rightarrow 0^+$, where $A(t)$ satisfies the ODE

$$\dot{A}(t) = D\vec{b}(X(t), t)A(t), \quad A(0, t) = e.$$

2. In the setting of Problem 1, show that $J(\cdot, t) := \det D_x X(\cdot, t)$ satisfies

$$\dot{J}(t) = \nabla \cdot \vec{b}(X(x, t), t)J(t).$$

Thus when $\vec{b}$ is divergence-free, the map $X(\cdot, t)$ is volume-preserving. In this case the transport equation is the same as the continuity equation, and $\vec{b}$ is called *incompressible*.

3. Show that the fundamental solution $\Phi(x) = \alpha_d |x|^{2-d}$ that we constructed in class for the Laplace's equation satisfies

$$\int_{\mathbb{R}^d} \Phi(x) \Delta \varphi(x) dx = -\varphi(0) \text{ for any } \varphi \in C_c^\infty(\mathbb{R}^d) \text{ for } d \geq 3.$$

Namely $-\Delta \Phi = \delta_{x=0}$ in the distribution sense.

4. $u \in L^1_{loc}(\Omega)$ with $\Omega \subset \mathbb{R}^d$ is called *weakly harmonic* in Ω if

$$\int_{\Omega} u \Delta \phi dx = 0$$

for all $\phi \in C_c^\infty(\Omega)$ (Or we say that $-\Delta u = 0$ in the distribution sense.) Show that a weakly harmonic function in Ω is (except on a set of measure zero) harmonic in Ω .

5. [Generalized Liouville theorem] Show that if $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a harmonic function and if there is an integer k such that $\limsup_{R \rightarrow \infty} \frac{\sup_{|x| \leq R} |u|(x)}{R^k}$ is bounded, then show that u is a polynomial of degree at most k .

6. [Schwarz reflection principle] Let $B_1 := B_1(0) = \{|x| < 1\}$ in $\mathbb{R}^n$, and let $B_1^+ := B_1 \cap \{x_n > 0\}$. Consider a harmonic function $u \in C^2(B_1^+) \cap C(\overline{B_1^+})$ such that $u = 0$ on $\partial B_1^+ \cap \{x_n = 0\}$. Show that

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

is a harmonic function in B_1 .

7. Let $u \in C^2(U) \cap C(\bar{U})$ with a bounded domain $U \subset \mathbb{R}^d$ solve

$$\begin{cases} Lu = f \text{ in } U, \\ u = g \text{ on } \partial U, \end{cases}$$

where L is an elliptic operator $Lu = -\Delta u + \vec{b} \cdot Du$ with $\vec{b} \in C(U) \cap L^\infty(U)$. Show that

$$\sup_U |u| \leq C(\sup_U |f| + \sup_{\partial U} |g|),$$

where C only depends on $\|\vec{b}\|_{L^\infty(U)}$, U , and the dimension d .

8. We call u to be *superharmonic* if $-\Delta u \geq 0$.

(a) Show that, if u is superharmonic and obtains a minimum at the interior of a bounded domain Ω , then u is a constant in Ω .

(b) Show that positive superharmonic functions in $\mathbb{R}^d$ are constant when $d = 2$.

(c) Show that the statement is false when $d > 2$.