Math 251A Fall 2024: Homework 4. Due 12/6

1. Let F be smooth and $g \in L^\infty(I\!R^d)$.

(a) Let w be uniformly Lipschitz in $I\!R \times [0,\infty)$ satisfying $w_t + F(w_x) = 0$ a.e. with $w(x,0) = \int_0^x g(y)dy$. Show that $u := w_x$ (defined a.e., as an element of $L^\infty(I\!R \times (0,\infty))$), is a weak (integral) solution of $u_t + (F(u))_x = 0$ in $I\!R \times [0,\infty)$ with initial data g. Please argue carefully how Lipschitz continuity of w plays the role in the proof.

(b) Show that if w satisfies

$$w(x+z,t) + w(x-z,t) - 2w(x,t) \leq C(1+1/t)|z|^2 \quad \text{for any } x, t \text{ and } z > 0,$$

then $u \in L^\infty(I\!R \times [0,\infty))$ satisfies

$$u(x+z,t) - u(x,t) \leq C(1+1/t)z \text{ for a.e. } x, z, t \text{ with } z > 0.$$

2. Let $\vec{b} \in C^2(I\!R^d)$. Show that if $u, v \in L^\infty(I\!R^d \times [0,\infty))$ satisfy

$$\int_0^\infty \int_{I\!R^d} [w(\varphi_t + \vec{b}\cdot D\varphi)](x,t)dx = \int u_0(x)\varphi(x,0)dx \text{ for any } t > 0 \text{ and for any } \varphi \in C_c^1(I\!R^d \times [0,\infty)),$$

then we have $u = v$ a.e. in $I\!R^d \times [0,\infty)$. Please refer to your solution of homework 1 to re-use any previous arguments.

3. Suppose that $u(x+z) - u(x) \leq Mz$ for almost every $x \in I\!R$ and $z > 0$. Let $u^\epsilon := u * \eta_\epsilon$, where η_ϵ is the usual mollifier. Show that $u^\epsilon_x \leq M$.

4. Consider the conservation law

$$\begin{cases} u_t + (u^4)_x = 0 & \text{in} \quad I\!R \times (0,\infty); \\ u = g & \text{on} \quad I\!R \times \{t = 0\}, \end{cases}$$

where $g = 1$ if $x < 0$.

(a) Let us assume in addition that $g = 2$ if $x \geq 0$. Find the entropy solution of the above problem. You must show that it is an entropy solution.

(b) Now suppose that $g = \max(1-x, 0)$ if $x \geq 0$. Find the first time where the entropy solution develops a discontinuity.

5. Show that only one of the following functions is a viscosity solution of $u_t + |Du|^2 = 0$ in $I\!R \times (0,\infty)$:

(a) $u_1(x,t) = |x| - t$;

(b) $u_2(x,t) = -|x| - t$.

6. [Existence of smooth solutions for viscous Hamilton-Jacobi equation] Let v satisfy (in the distribution sense)

$$v_t - \Delta v = f \text{ in } I\!R^d \times [0,T], \quad v(x,0) = g(x),$$

with a fixed $T > 0$ and a uniformly Lipschitz and bounded function g. If $f \in L^\infty(I\!R^d \times [0,T])$, then v is bounded, and uniformly Lipschitz in $I\!R^d \times [0,T]$ with

$$\|v\|_{L^\infty}, Lip_v \leq C(T\|f\|_\infty + \|g\|_\infty + Lip_g).$$

Moreover, for any $\tau > 0$, the following Schauder estimates hold for v:

(a) For any $0 < \alpha < 1$, v is locally uniformly $C^{1,\alpha}$ in space variable and C^1 in time variable in $I\!R^d \times (\tau, T]$.

(b) For $k = 0,1,2,...$ if $f \in C^{k,\alpha}(I\!R^d \times (0,T])$, then v is $C^{k+2,\alpha}$ in space variable and $C^{k+1,\alpha}$ in time variable in $I\!R^d \times (\tau, T]$. (See for example, Evans Chapter 5, p254-255, for definition of $C^{k,\alpha}$ spaces.)

Let $H(p,x) : I\!R^d \times I\!R^d \to I\!R$ be C^∞. Using the above information and Schauder's fixed point theorem (stated below), show that, for any $\epsilon > 0$, there is a solution u which is smooth in $I\!R^d \times (0,\infty)$, and also is uniformly Lipschitz and bounded in $I\!R^d \times [0,\infty)$, of

$$u_t^\epsilon - \epsilon \Delta u + H(Du^\epsilon, x) = 0 \text{ in } I\!R^d \times [0,\infty); \quad u(x,0) = g(x). \tag{1}$$

[Schauder's fixed point theorem] Let X be a Banach space, and let K be a convex, bounded, and closed subset of X. Then a continuous map $\Phi : K \to K$ has a fixed point if $\Phi(K)$ is compact in X.

7. Let u^ϵ be obtained from problem 6. If $|H(p,x) - H(p,y)| \leq C|x-y|$ for all $p \in I\!R^d$, show that $u^\epsilon(\cdot,t)$ is uniformly Lipschitz for any fixed $t > 0$, independent of $\epsilon > 0$. (Hint: first show that for two solutions u_i^ϵ, $i = 1,2$ of (1) corresponding to g_i, we have

$$\max_{x \in I\!R^d} |u_1^\epsilon(x,t) - u_2^\epsilon(x,t)| \leq \max_{x \in I\!R^d} |g_1(x) - g_2(x)|.$$