Math 251A Fall 2022: Homework 4. Due Dec. 2nd

1. For a smooth vector field $\vec{b} : \mathbb{R}^d \to \mathbb{R}^d$, let $\Phi : \mathbb{R}^d \to \times[0,\infty)\mathbb{R}^d$ solve the ODE

$$\dot{\Phi}(x,t) = \vec{b}(\Phi(x,t),t), \qquad \Phi(x,0) = x, \tag{1}$$

where $\dot{}$ denotes the time derivative.

(a) Show that $J\Phi := \det D\Phi$ satisfies the ODE

$$\dot{J\Phi}(x,t) = \nabla \cdot \vec{b}(\Phi(x,t),t) J\Phi(x,t).$$

(b) Show that

$$\exp^{-\int_0^t \sup_x [\nabla\cdot\vec{b}(x,s)]_- ds} \le J\Phi(x,t) \le \exp^{\int_0^t \sup_x [\nabla\cdot\vec{b}(\cdot,s)]_+ ds}.$$

2. Consider (1) wth $d = 1$ and $\vec{b}(x,t) = b_0(x) = |x|^{1/2}$.

(a) Find a regularized Lagrangian flow Φ for the ODE. Is it unique?

(b) Construct a family of approximating smooth vector fields b_k that approximates b locally uniformly and the corresponding ODE flow Φ_k converges to Φ.

(c) Construct a family of approximating smooth vector fields b_k that approximates b locally uniformly and the corresponding ODE flow Φ_k converge to a solution Ψ of (1) with $b = b_0$ that is different from Φ.

3. This problem finishes the proof of stability property for regularized Lagrangian flows. Suppose $\Phi_k : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ are locally uniformly bounded in $L^\infty(\mathbb{R}^d \times [0,\infty))$ and satisfies

$$\lim_{k\to\infty} \int \phi(\Phi_k(x,t),t)\zeta_0(x)dxdt = \int \phi(\Phi(x,t),t)\zeta_0(x)dxdt$$

for any $\phi \in C_c^\infty(\mathbb{R}^d \times [0,\infty))$ and $\zeta_0 \in L^\infty(\mathbb{R}^d)$. Show that Φ_k converges strongly to Φ in $L^1_{loc}(\mathbb{R}^d \times [0,\infty))$. You may want to first establish weak convergence in appropriate L^p spaces, and then upgrade it to strong convegence in L^1_{loc}.

4. Let $M_\lambda f := \sup_{0<r<\lambda} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy$. Show that, for $f \in W^{1,p}(I\!R^d)$ with $p \geq 1$, we have

$$\frac{1}{R} \int_{B_R(x)} \frac{|\nabla f(y)|}{|x-y|^{d-1}} dy \leq c_d M_R Df(x).$$

5. Consider the initial configuration $f_0 : [0,1] \to \{0,1\}$ such that, for some $0 < \kappa < \frac{1}{2}$,

$$\kappa\epsilon < \int_y^{y+\epsilon} f(x)dx \leq (1-\kappa)\epsilon \quad \text{for all } y \in [0, 1-\epsilon].$$

We define $V(s)$ as the minimum cost of any sequence of left-to-right transpositions $f_0, f_1, ..., f_n$ such that f_n equals 1 on at least one single interval of length $s \in [0, \kappa]$. Show that

$$V(s) \geq \min_{0<\sigma<s} [V(s-\sigma) + V(\sigma)] + k^2 s \quad \text{for } \epsilon < s \leq \frac{\kappa}{2}.$$

For recalling definitions, wee https://arxiv.org/pdf/math/0302228.pdf