**Math 251A Fall 2024: Homework 3 Due 11/15.**

1. Let $u : I\!R \times [0,\infty) \to I\!R$ be a smooth solution to

$$\left\{ \begin{array}{lll} u_{tt} - u_{xx} = f & & \text{in } I\!R \times (0,\infty), \\ u(x,0) = g(x), & u_t(x,0) = h(x) & \text{in } I\!R^d. \end{array} \right.$$

(a) When $f = 0$, prove that

$$\sup |\partial_x u| \leq \max |g'| + \max |h|.$$

(b) Prove the corresponding statement when $f \neq 0$.

2. Let $u$ be a smooth solution to

$$\left\{ \begin{array}{lll} u_{tt} - \Delta u + u^3 = 0 & & \text{in} \quad I\!R^3 \times (0,\infty), \\ u(x,0) = g(x), & u_t(x,0) = h(x). & \end{array} \right.$$

Suppose that $u(x,0) = u_t(x,0) = 0$ in $|x| < 1$. Describe the maximal region where $u(x,t)$ must be zero.

3. Let $u$ solve

$$\left\{ \begin{array}{lll} u_{tt} - \Delta u = 0 & & \text{in} \quad I\!R^d \times (0,\infty), \\ u(x,0) = g(x), & u_t(x,0) = h(x). & \end{array} \right.$$

Suppose that $g$ and $h$ are compactly supported. Represent $\int u(x,t)dx$ in terms of $g$ and $h$. Be careful to justify the integrability of $u$ and any other functions you integrate over the whole $I\!R^d$.

4. Let $u(x,y) : I\!R^2 \to I\!R$ be harmonic in the upper half plane $y \geq 0$, with $u(x,0) = g(x)$. We suppose that $u, g \in L^1(I\!R^d)$. Find an operator $N$ that sends $\hat{g}$ to $\hat{h}$, where $h(x) := \partial_y u(x,0)$, and $\hat{}$ is the Fourier transform in $x$-variable. In particular conclude that if $g' \in L^2(I\!R)$ then $h \in L^2(I\!R)$.

5. For a smooth $L(v,x) : I\!R^d \times I\!R^d \to I\!R$ that is convex and uniformly superlinear (namely $\liminf_{|v|\to\infty} \inf_{x \in I\!R^d} \frac{L(v,x)}{|v|} = \infty$), and for a Lipschitz continuous function $g$, let us define

$$u(x,t) := \inf_{w \in \mathcal{A}} \{ \int_0^t L(\dot{w}(s), w(s))ds + g(w(0)) \}, \tag{1}$$

where $\mathcal{A} := \{w \in C^1([0,t]), w(t) = x\}$. In class we have shown that $u(\cdot,t)$ is uniformly Lipschitz in $x$-variable for any fixed $t > 0$.

Show that $u$ is uniformly Lipschitz in both $x$ and $t$-variable in $\mathbb{R}^d \times [0,t]$, with $u(x,0) = g(x)$ in $\mathbb{R}^d$. You may use the dynamic programming principle that we covered in class. Here we do not assume that the optimal path exists.

6. Let $u$ be given by above formula, and now suppose that the optimal path exists in $\mathcal{A}$ for all $(x,t)$. Assuming that $L(v,x)$ and the corresponding $H(p,x)$ is smooth, and that $u$ is differentiable at $(x,t)$, explain why the optimal path $w(s) : [0,t] \to \mathbb{R}$ is given by the Hamiltonian ODE system

$$\begin{cases} \dot{w}(s) = D_p H(p,w) \text{ for } [0,t]; \\ \dot{p}(s) = -D_x H(p,w) \text{ for } [0,t]; \\ w(t) = x, \quad \dot{w}(t) = D_p H(Du(x,t),x). \end{cases}$$

as long as the solution of the ODE exists. You may refer to any material we covered in the class.

7. Let $u$ be given by the Hopf-Lax formula associated with $u_t + \frac{|u_x|^2}{2} = 0$ in $\mathbb{R} \times (0,\infty)$. If $g$ is smooth, can you always expect $u$ to be smooth? Explain.