Math 251A Fall 2022: Homework 3, due 11/9.

1. Let $u \in C^1(B_1)$ be a H^1-solution to $-\nabla \cdot (A(x)\nabla u) = f$ with $f \in L^d(B_1)$ and continuous a_{ij} with ellipticity constants λ, Λ. Show that for any $0 < \alpha < 1$ we have

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}),$$

where C only depends on $\alpha, d, \lambda, \Lambda$ and the mode of continuity for a_{ij}. Explain why the proof does not apply to the case to $\alpha = 1$.

2. [Boundary regularity] For a given $\alpha \in (0,1)$, let $u \in C^1(B_1) \cap C^\alpha(\overline{B_1})$ be a H^1 solution to $-\nabla \cdot (A(x)\nabla u) = 0$ in B_1, with continuous a_{ij} in $\bar{B}_1$ with ellipticity constants λ and Λ, and boundary data $g \in C^\alpha(\partial B_1)$. Show that

$$\|u\|_{C^\alpha(\overline{B_1})} \leq C\|g\|_{C^\alpha(\partial B_1)},$$

where C depends on the same terms as in problem 1.

$\star$ For problem 1-2, one can use the following lemma (we will prove it in class on 11/7):

Lemma. *Let u and f be as given in Problem 1. Then*

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)}),$$

where $C = C(d, \lambda, \Lambda)$.

3. [Existence by fixed point theorem] We will show that there exists at least one solution $u \in C^2(B_1) \cap C(\overline{B_1})$ of the following problem

$$\Delta u = \sin u \text{ in } B_1, \quad u = g \text{ in } \partial B_1,$$

with $g \in C^1(\partial B_1)$. Consider the map $T : X \to X$ with $X = C^{1/2}(\overline{B_1})$, where $T(w) = u$ with u being the unique solution of

$$\Delta u = \sin w \text{ in } B_1, \quad u = g \text{ in } \partial B_1.$$

(a) For $\alpha = 1/2$, show that $T(X)$ is contained in a compact subset of X.

(b) Apply the Schauder fixed point theorem to show that T has a fixed point.