Math 251A Fall 2024: Homework 2. Due 10/30.

1. For a bounded, open domain U with C^1 boundary and with exterior ball condition, let $f \in C(U)$ and $g \in C(\partial U)$. Show that if any two smooth (in $C^2(U) \cap C^1(\bar{U})$) solutions exist for the Neumann problem

$$(P) \qquad \begin{cases} -\Delta w = f \text{ in } U; \\ \partial_\nu w = g \text{ on } \partial U, \end{cases}$$

where ν denotes the outward normal vector on ∂U with respect to U, differs only by a constant.

2. Let $u \in C^2(U)$, $U = \mathbb{R}^d_+ := \{x : x_d > 0\}$ solve (P). We aim to find a representation formula for u using the corresponding Green's function, assuming that u decays sufficiently fast as $|x| \to \infty$. This content follows the argument given in section 2.2.4. Note that this formula for the Dirichlet problem is given by (30) in section 2.2.4 of Evans.

(a) What should be the corresponding equation for the corrector $\phi^x(y)$ in U and on ∂U, for which the Green's function will be given as $G(x, y) := \Phi(y - x) - \phi^x(y)$? Explain.

(b) Can we explicitly find ϕ^x?

3. Show that if there exists $w \in A := C^2(\bar{U}) \cap C(U)$ minimizing

$$E(w) = \int_U (\frac{1}{2}|Dw|^2 - fw)dx$$

over all functions in $w \in \mathcal{A}$, then u solves (P) with $g = 0$.

4. Let $U \subset \mathbb{R}^d$ be open. Show that if $u \in L^1(U \times [0, T])$ solves the heat equation in distribution sense, namely

$$\int_{U \times (0,T)} (\phi_t + \Delta\phi) u dx dt = 0 \ \text{ for any } \phi \text{ in } C_c^\infty(U \times (0, T)),$$

then $u \in C^\infty(U \times [0, T])$.

5. Consider $u \in C^2_1(\mathbb{R}^n \times (0, \infty)) \cap C(\mathbb{R}^d \times [0, \infty))$ solution of

$$u_t - \Delta u = u(1 - u) \quad \text{ in } \mathbb{R}^n \times (0, \infty).$$

with the property that $u \to 0$ as $|x| \to \infty$. Show that if $u(x, 0) \in [0, 1]$ in $\mathbb{R}^d$, then $u \in [0, 1]$ in $\mathbb{R}^d \times [0, \infty)$.

6. [Long-time behavior] Let $U := \{|x| < 1\}$ and let $g \in C(\partial U)$, $f \in C(\bar{U})$. Suppose that u is a sufficiently smooth solution of the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad U \times (0, \infty); \\ u(x,t) = g(x) & \text{on} \quad \partial U \times [0, \infty); \\ u(x,0) = f(x) & \text{in} \quad U. \end{cases}$$

(a) Show that $v = (u_t)^2$ solves $v_t - \Delta v \leq 0$ in $U \times (0, \infty)$.

(b) Show that v decays to zero exponentially fast as $t \to \infty$.

(c) Does the limit $u^*(x) := \lim_{t\to\infty} u(x,t)$ exist in $\bar{U}$? Can we have a representation formula for u^*?

7. Evans (2nd edition) p88, Problem 15.