Math 251A Fall 2022: Homework 2. Due Oct.26th

1. Let us consider the inhomogeneous wave equation

$$(P) \qquad \begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \mathbb{R}^3 \times (0,\infty); \\ u = u_t = 0 & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

Using Duhamel's formula, please answer the following:

(a) How regular should f be to guarantee the existence of solution u in $C^2(\mathbb{R}^3 \times (0,\infty))$?

(b) If $f = f(x,t)$ is supported in $B_r(0) \times [-r,r]$ for some $r > 0$, can we describe the region where u is supported?

2. For u solving

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & \text{in} \quad U_T; \\ u = g(x) & \text{on} \quad \partial_p U_T; \\ u_t = h(x) & \text{on} \quad U \times \{t = 0\}, \end{cases}$$

Show that the energy

$$E(t) := \frac{1}{2}\int_U (u_t)^2 + (u_x)^2 dx$$

satisfies

$$E(t)^{1/2} \leq E(0)^{1/2} + \frac{1}{\sqrt{2}}\int_0^t \|f(\cdot,\tau)\|_{L^2(U)} d\tau.$$

Hint: Use Gronwall's inequality.

3. [Variational method for existence] Let U be a bounded domain in $\mathbb{R}^d$. For each $x \in U$, let $S(x)$ be a positive symmetric $d \times d$ semi-definite matrix satisfying $\lambda(Id)_{d\times d} \leq S \leq \Lambda(Id)_{d\times d}$ for some $\lambda, \Lambda > 0$. Consider the energy

$$E(w) = \int_U |S\nabla w|^2 dx$$

in $\mathcal{A} := \{w \in H^1(U) \text{ with } Tw = g\}$, where $g \in C^1(\bar{U})$.

(a) Show that there is a minimizer u in $\mathcal{A}$.

(b) Show that any minimizer u is the unique H^1 solution of $-\nabla \cdot (A\nabla u) = 0$ in U with boundary data (Trace) g, where $A = S^2$.

In answering these questions, please carefully point out which properties of the space $H^1(U)$ that you are using.

4. [Interpolation exercise] Show that for any sufficiently small $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that the following holds for any $u \in C^{1,\alpha}(\bar{B}_1)$:

$$\sup_{x \in B_1} |\nabla u| \leq C_\epsilon \sup_{x \in B_1} |u| + \epsilon [\nabla u]_{C^\alpha(\bar{B}_1)}.$$