Math 251A Fall 2024: Homework 1, due 10/16.

0. [Optional] This exercise is a careful review of how we derived continuity equation in class from volume-preserving transport property, since it is not in the textbook. Let $\vec{b} : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$ be uniformly C^2 in both space and time variables, and let $X(x,\cdot)$ solve the ODE in time variable:

$$\begin{cases} \dot{X}(x,t) = \vec{b}(X(x,t),t) \text{ for } t > 0, \\ X(x,0) = x. \end{cases}$$

We can use the classical fact that for each fixed x there is a unique solution $X(x,t)$ of the ODE that is continuous in space variable, for any uniformly Lipschitz vector field $\vec{b}$.

(a) Show that $X(\cdot,t) : \mathbb{R}^d \to \mathbb{R}^d$ is invertible (let us denote this inverse as $X^{-1}(x,t)$) at each t.

(b) Show that $X(\cdot,t)$ is C^2 if $\vec{b}$ is C^2 in space variable.

(c) Set $u(x,t) := u_0(X^{-1}(x,t))(detDX(x,t))^{-1}$ such that it satisfies, for any given $t > 0$ and for any $\varphi \in C_c^\infty(\mathbb{R}^d \times (0,\infty))$,

$$\int_{\mathbb{R}^d} \varphi(x,t)u(x,t)dx = \int_{\mathbb{R}^d} \varphi(X(x,t),t)u_0(x)dx.$$

Show that then, if u_0 is C^1,

$$\int_{\mathbb{R}^d} \varphi(u_t + \nabla \cdot (\vec{b}u))\,(x,t)dx = 0 \text{ for any } t > 0 \text{ and for any } \varphi \in C_c^\infty(\mathbb{R}^d \times (0,\infty)),$$

and thus conclude that u is a C^1-solution of the continuity equation $u_t + \nabla \cdot (\vec{b}u) = 0$ with $u(x,0) = u_0(x)$.

1. [Backward-in-time Uniqueness] Let $\vec{b} \in C^2(\mathbb{R}^d)$ and X be as given above.

(a) Show that, for any $f \in C^1(\mathbb{R}^d \times [0,\infty))$ and $h_0 \in C^1(\mathbb{R}^d)$,

$$h(x,t) := h_0(y(x,t)) + \int_0^t f(X(y(x,t),s),s)ds \text{ with } y(x,t) := (X^{-1}(\cdot,t))(x)$$

solves $h_t + \vec{b} \cdot Dh = f$ in $\mathbb{R}^d \times (0,\infty)$ with $h(\cdot,0) = h_0$. (Hint: evaluate the formula of h at $x = X(x,t)$ and vary t.)

(b) Using (a), show that if both $u, v \in C(I\!R^d \times (0,\infty))$ satisfies, with $w = u$ or $w = v$,

$$\int_0^\infty \int_{I\!R^d} [w(\varphi_t + \vec{b}\cdot D\varphi)](x,t)dx = \int u_0(x)\varphi(x,0)dx \text{ for any } t > 0 \text{ and for any } \varphi \in C_c^1(I\!R^d \times [0,\infty)),$$

then for any $T > 0$ we have $u = v$ in $[0,T]$ if $u(\cdot,T) = v(\cdot,T)$.

2. Show that the fundamental solution $\Phi(x) = \alpha_d |x|^{2-d}$ that we constructed in class for the Laplace's equation satisfies

$$\int_{I\!R^d} \Phi(x)\Delta\varphi(x)dx = -\varphi(0) \text{ for any } \varphi \in C_c^\infty(I\!R^d) \text{ for } d \geq 3.$$

Namely $-\Delta\Phi = \delta_{x=0}$ in the distribution sense.

3. $u \in L^1_{loc}(\Omega)$ with $\Omega \subset I\!R^d$ is called *weakly harmonic* in Ω if

$$\int_\Omega u\Delta\phi dx = 0$$

for all $\phi \in C_c^\infty(\Omega)$ (Or we say that $-\Delta u = 0$ in the distribution sense.) Show that a weakly harmonic function in Ω is (except on a set of measure zero) harmonic in Ω.

4. [Generalized Liouville theorem] Show that if $u : I\!R^d \to I\!R$ is a harmonic function and if there is an integer k such that $\limsup_{R\to\infty} \frac{\sup_{|x|\leq R} |u|(x)}{R^k}$ is bounded, then show that u is a polynomial of degree at most k.

5. [Schwarz reflection principle] Let $B_1 := B_1(0) = \{|x| < 1\}$ in $I\!R^n$, and let $B_1^+ := B_1 \cap \{x_n > 0\}$. Consider a harmonic function $u \in C^2(B_1^+) \cap C(\overline{B_1^+})$ such that $u = 0$ on $\partial B_1^+ \cap \{x_n = 0\}$. Show that

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, ..., x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

is a harmonic function in B_1.

6. [Uniqueness for unbounded domains]

(a) Show that there is a unique bounded harmonic function in $I\!R^+_n$ with continuous boundary data on $x_n = 0$.

(b) Does the uniqueness result hold for bounded harmonic functions in $C^2(\Omega) \cap C(\overline{\Omega})$ with given continuous boundary data on $\partial\Omega$, if Ω is unbounded and connected? Please give an example if the answer is no.

7. Let $u \in C^2(U)$ with a bounded domain $U \subset I\!R^d$. solve the Neumann boundary problem

$$\begin{cases} Lu = f \text{ in } U, \\ \\ u = g \text{ on } \partial U, \end{cases}$$

where L is an elliptic operator $Lu = -\Delta u + \vec{b} \cdot Du$ with $\vec{b} \in C(U) \cap L^\infty(U)$. Show that

$$\sup_U |u| \leq C(\sup_U |f| + \sup_{\partial U} |g|),$$

where C only depends on $\|\vec{b}\|_{L^\infty(U)}$ and the dimension d.