Math 251A Fall 2022: Homework 1. Due Wed Oct 12th

1. [Weak solutions for Laplace's equation] An integrable function in a domain Ω is called *weakly harmonic* in Ω if

$$\int_\Omega u \Delta \phi dx = 0$$

for all smooth functions with compact support in Ω. Show that a weakly harmonic function in Ω is (except on a set of measure zero) harmonic in U.

2. [Schwarz reflection principle] Let $B_1 := B_1(0) = \{|x| < 1\}$ in $\mathbb{R}^n$, and let $B_1^+ := B_1 \cap \{x_n > 0\}$. Consider a harmonic function $u \in C^2(B_1^+) \cap C(\overline{B_1^+})$ such that $u = 0$ on $\partial B_1^+ \cap \{x_n = 0\}$. Show that

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, ..., x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

is a harmonic function in B_1.

3. [Uniqueness for unbounded domains]

(a) Show that there is a unique bounded harmonic function in $\mathbb{R}^+_n$ with continuous boundary data on $x_n = 0$.

(b) Does the uniqueness result hold for bounded harmonic functions in $C^2(\Omega) \cap C(\overline{\Omega})$ with given continuous boundary data on $\partial\Omega$, if Ω is unbounded and connected? Please give an example if the answer is no.

4. [Harnack inequality] Let B_r denote $B_r(0)$ and define $\text{osc}_{B_r} u := \sup_{B_r} u - \inf_{B_r} u$.

(a) Show that $\sup_{B_{1/2}} u \leq C_n \inf_{B_{1/2}} u$ if u is a positive harmonic function in B_1.

(b) Using (a), show the oscillation decay of harmonic functions:

$$\text{osc}_{B_{1/2}} u \leq \frac{C_n - 1}{C_n + 1} \text{osc}_{B_1} u, \tag{1}$$

where u is harmonic but not necessarily positive.

(c) explain how (b) yields that u is Hölder continuous in $\bar{B}_{1/2}$.

5. Let $U_T := U \times (0,T]$ and let $\partial_p U_T$ denote the parabolic boundary of U_T.Show that if $u \in C^{2,1}_{x,t}(U_T) \to I\!R$ satisfies

$$u_t - \Delta u + \sin u = 0 \text{ in } U_T, \quad u = 0 \text{ on } \partial_p U_T,$$

then u is identically zero.

6. [Long-time behavior] Let $U := \{|x| < 1\}$. Suppose that u is a sufficiently smooth solution of the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad U \times (0,\infty); \\ u = g(x) & \text{on} \quad \partial U \times [0,\infty); \\ u(x,0) = f(x) & \text{in} \quad U. \end{cases}$$

(a) Show that $v = (u_t)^2$ solves $v_t - \Delta v \le 0$ in $U \times (0,\infty)$.

(b) Show that v decays to zero exponentially fast as $t \to \infty$.

(c) Does the limit $u^*(x) := \lim_{t\to\infty} u(x,t)$ exist in $\bar{U}$? Can we have an explicit formula for u^*?