Math 245A Fall 2007 Midterm: Solutions

1. (a) Show that, for any set E with $m^*(E) < \infty$, there exists a measurable set A exists such that $E \subset A$ and $m^*(E) = m(A)$.

Proof We know that

$$m^*(E) = \inf\{m(\mathcal{O}) : \mathcal{O} \text{ is open and contains } E\}. \tag{1}$$

Hence for one can choose a sequence of open sets $\mathcal{O}_n$ containing E such that $m^*(E) \leq m(\mathcal{O}_n) \leq m^*(E) + \frac{1}{n}$.

(The statement (1) is Observation 3 in p.13. Alternatively, one can use a sequence of countable union of closed cubes whose measure is close to the measure of E by $1/n$. Countable intersections of these sets will then be a Borel set.)

Now $\mathcal{O} = \cap_{n=1}^{\infty} \mathcal{O}_n$ is a Borel set (or G_δ-set) and thus is measurable. Moreover $m(\mathcal{O}) \leq m(\mathcal{O}_n) \leq m^*(E) + \frac{1}{n}$ for any n and therefore $m(\mathcal{O}) \leq m^*(E)$. On the other hand $m(\mathcal{O}) \geq m^*(E)$ since $\mathcal{O}$ contains E. Hence $m(\mathcal{O}) = m^*(E)$.

(b) Show that if A is measurable then there exists an increasing sequence of compact sets $K_1 \subset K_2 \subset ... \subset A$ such that

$$m(A - \cup_{k=1}^{\infty} K_j) = 0.$$

Proof There exists a sequence of closed sets $F_n \subset A$ such that $m(A - F_n) \leq \frac{1}{n}$. Let $K_j^n = F_n \cap \{|x| \leq j\}$, so that K_j^n is compact. Now we know that

$$m(A - \cup_{j,n=1}^{\infty} K_j^n) \leq m(A - \cup_j K_j^n) \leq \frac{1}{n} \text{ for } n = 1, 2, ...$$

Hence $m(A - \cup_{j,n} K_j^n) = 0$.

Now take

$$K_l = \cup_{j,n \leq l} K_{j,n}.$$

then $K_1 \subset K_2 \subset ... \subset A$. Furthermore $\cup_{l=1}^{\infty} K_l = \cup_{j,n=1}^{\infty} K_j^n$, hence $m(A - \cup_l K_l) = 0$.

Remark Since $m(A)$ may be infinite, one cannot use limiting arguments such as Corollary 3.3. (ii), Chapter 1.

2. Suppose $f_1, f_2, ...$ are measurable functions defined in $I\!R^n$ and f_k converges to f almost everywhere in $I\!R^n$. Also assume g_1,g_2,... are nonnegative functions in $L^1(I\!R^n)$ that $|f_k| \leq g_k$ almost everywhere. Furthermore suppose $g_k \to g$ almost everywhere, $g \in L^1(I\!R^n)$ and

$$\int g dx = \lim_{k\to\infty} \int g_k dx.$$

Prove that

$$\int f dx = \lim_{k\to\infty} \int f_k dx.$$

Proof Consider the sequence $g_k - f_k$. By Fatou's lemma

$$\int (g-f)dx \leq \liminf_k \int (g_k - f_k)dx = \int g dx - \limsup_k \int f_k dx.$$

Hence $\int f dx \geq \limsup_k \int f_k dx$. On the other hand, the same argument with $g_k + f_k$ yields

$$\int (g+f)dx \leq \int g dx + \liminf_k \int f dx.$$

Therefore $\int f dx \leq \liminf_k \int f dx$. Therefore

$$\int f dx = \liminf_k \int f dx = \limsup_k \int f dx = \lim_k \int f.$$

Remark Indeed one can argue with $g + g_k \pm |f_k - f|$ to deduce that $\lim_k \int |f_k - f| dx = 0$, which would be a stronger statement than what the problem ask you to do.

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is measurable. Show that the graph of f, defined by

$$E = \{(x, y) \in \mathbb{R}^{n+1} : y = f(x)\}$$

is measurable and has Lebesgue measure zero in $\mathbb{R}^{n+1}$.

Proof: First of all E is measurable, since $E = \{\tilde{f}(x, y) = 0\}$, where $\tilde{f}(x, y) = y - f(x) : \mathbb{R}^{n+1} \to \mathbb{R}$ is a measurable function. Therefore

$$\{(x, y) : \tilde{f} = 0\} = \cap_{n=1}^{\infty}\{(x, y) : -\frac{1}{n} < \tilde{f} < \frac{1}{n}\} : \text{ measurable.}$$

Hence χ_E is measurable function defined in $\mathbb{R}^{n+1}$, and is nonnegative. Thus Tonelli's Theorem(Theorem 3.2, Chapter 2) applies to yield

$$m(E) = \int \chi_E dxdy = \int_{\mathbb{R}^n} (\int_{\mathbb{R}} \chi_E(x, y)dy)dx = \int_{\mathbb{R}^n} m(E_x)dx.$$

But $m(E_x) = m(\{f(x)\}) = 0$. Hence $m(E) = 0$.