THE STEFAN PROBLEM AND FREE TARGETS OF OPTIMAL BROWNIAN MARTINGALE TRANSPORT

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Abstract. We formulate and solve a free target optimal Brownian stopping problem from a given distribution while the target distribution is free and is conditioned to satisfy a given density height constraint. The solutions to this optimization problem then generates solutions to the Stefan problem for both supercooled fluid freezing (St_1) and ice melting (St_2), depending on the type of cost for optimality. The freezing (St_1) case has not been well understood in the literature beyond one dimension, while our result gives a well-posedness of weak solution in general dimensions, with a naturally chosen initial data. The cost is a Lagrangian type integral along path, where the Lagrangian function satisfies strict time monotonicity, increasing (Type (I) or decreasing (Type (II)). Type (I) case corresponds to the freezing fluid while type (II) to the melting ice. The optimal stopping time is characterized by the hitting time to a certain monotone barrier set in the space time, while the optimal target distribution saturates the density constraint. The barrier sets are determined by the type of the cost and the initial distribution, and give the space-time free boundaries of the flow for the Stefan problems (St_1), (St_2). The free target optimization problem exhibits monotonicity, from which a remarkable universality follows in the sense that the optimal target distribution is independent of the cost or its type. This gives a new connection between the freezing and melting Stefan problems.

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1. Introduction

Given a configuration μ of agents subject to a random motion, what would happen if they try to stay under a given allowed population density height f, in particular, while trying to optimize their collective dynamics? In the present paper we consider the Brownian motion W_t in $\mathbb{R}^d$, and the collective dynamics is controlled only by a stopping time τ, that is, when each random particle stops. Precisely, we consider the following:

Problem 1.1. *Consider a nonnegative function $f \in L^\infty(\mathbb{R}^n)$. Given an absolutely continuous, compactly supported measure μ on $\mathbb{R}^n$, find a stopping time τ^* and its distribution $W_{\tau^*} \sim \nu^*$, that solve*

$$\mathcal{P}_f(\mu) := \inf_{(\tau,\nu)} \{\mathcal{C}(\tau) \mid W_0 \sim \mu,\ W_\tau \sim \nu,\ \nu \le f,\ \ \nu \text{ is compactly supported}\}. \tag{1.1}$$

Notice that here we do not assume that μ, ν are probability measures. So the expression '$W_0 \sim \mu$ and $W_\tau \sim \nu$' should be understood as for each measurable set E,

$$\nu[E] = \int Prob[W_\tau \in E \mid W_0 = x] d\mu(x).$$

The problem (1.1) obviously requires some condition for f because if f is too small there will be no admissible ν. We consider those cases where there is at least one admissible ν. Our model case is $f \equiv 1$ on $\mathbb{R}^d$, however a much more general f can be considered. We are particularly interested in the cost $\mathcal{C}(\tau)$ of the 'Lagrangian form':

$$\mathcal{C}(\tau) = \mathbb{E}\left[\int_0^\tau L(B_t, t) dt\right] \text{ for a continuous } L : \mathbb{R}^d \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}. \tag{1.2}$$

This problem is in the spirit of what is studied in "constrained transport", in particular, for analyzing congested crowd motion (see e.g. [46, 31, 1]) and tumor growth (see e.g. [42, 34]), which is subject to contact inhibition; the problem (1.1) can be viewed as Brownian martingale version of the optimal transport free target problem with upper bound of [31]. In those cases the dynamics have less restricted to allow free drift, making the optimal dynamics deterministic. Additional noise to their dynamics would generate stochastic aspects. In our problem the dynamics is restricted only to the Brownian motion but with freedom for stopping time; such dynamics is called Brownian martingale.

Throughout this paper we make assumptions on the cost $\mathcal{C}$ (1.2):

$$0 \le L \le D \text{ for some constant } D \text{ and,} \tag{1.3}$$

$$t \mapsto L(\cdot, t) \text{ is either strictly increasing (Type (I)), or strictly decreasing (Type (II)).} \tag{1.4}$$

We obtain existence, uniqueness, and characterization of the optimal solution of Problem 1.1; see Theorem 7.4. Remarkably the problem exhibits universality (see Theorem 7.6), namely, the resulting optimal target ν^* does not depend on a particular choice of the function L, as long as L satisfies either of the monotonicity assumptions. These are due to monotonicity properties of the problem regarding the optimal stopping τ^* and target ν^* (Theorem 7.1), which also give L^1 contraction (Theorem 7.8) and BV estimates for ν^* (Theorem 7.10). Such monotonicity is a novel feature, resulting from not fixing the target distribution ν but treating it as the solution of the problem. Another important feature of Problem 1.1 is that the optimal solution ν^*, also the mass flow μ_t of W_t toward it, saturates the constraint f (Theorem 8.3) wherever possible.

The free target feature of Problem 1.1 distinguishes it from the more well studied so-called optimal Skorokhod problem (see e.g. [10, 28, 27, 30, 29] as well as the 1 dimensional results considered by [36, 16, 25, 18]) where optimal stopping times are studied for given *fixed* initial and target measures μ, ν and *without* the density constraint f, that is,

$$\mathcal{P}(\mu,\nu) := \inf_{\tau}\{\mathcal{C}(\tau) \mid W_0 \sim \mu,\ W_\tau \sim \nu\}. \tag{1.5}$$

Solutions to Problem (1.5) solve the Skorokhod problem [48, 43, 6, 44, 4, 23, 41, 40] for which a connection to optimal transport was hinted in the work of [33], and established in [10] who used optimal transport theory (see e.g. the books [50, 51, 45]) and randomized stopping times to unify all the previous know solutions of the Skorokhod problem as the solutions of (1.5). We also mention that optimal Skorokhod problem is a special case of the optimal martingale problem (see e.g. [49, 8, 22, 9, 32, 26, 5, 27] and references therein) which has been recently popularized in mathematical finance.

As a main application we combine the above mentioned results for Problem 1.1 with the PDE methods developed in [28] for (1.5) to make progress in the supercooled Stefan problem, a free boundary problem of the heat equation, which has been poorly understood in the literature.

The supercooled Stefan problem describes freezing of supercooled water into ice, and can be written in divergence form as

$$(\eta - \chi_{\{\eta>0\}})_t - \Delta\eta = 0. \tag{St_1}$$

Here $-\eta$ denotes the temperature of the supercooled water, and thus the set $\{\eta > 0\}$ is the region for the supercooled water. The solution of (St_1) is famously known to exhibit irregular, fractal-like interfaces that could jump discontinuously over time. This is in sharp contrast to the Stefan problem for melting ice into water,

$$(\eta + \chi_{\{\eta>0\}})_t - \Delta\eta = 0, \tag{St_2}$$

which can be viewed as a singular nonlinear parabolic equation of the form $[\beta(\eta)]_t - \Delta\eta = 0$ with an increasing function β; see [2] where one proves comparison and contraction properties of weak solutions based on this property.

While (St_2) is known to generate stable and regularizing solutions ([2, 3, 15]), solutions of (St_1) are shown to develop discontinuity and non-uniqueness in finite time, even in one space dimension and with smooth initial data ([47, 14]). For higher dimensions, the global existence of weak solutions for (St_1) has stayed largely open.

It is natural to consider these problems as generated by a particle system, which diffuses but stops its motion when it hits the interface between water and ice. In one space dimension, there has been several works which construct solutions of (St_1) as a continuum limit of such particle systems, including [12, 13, 35, 19, 20]. In higher dimensions the global-time existence of solutions for (St_1) has remained open, due to the difficulty of obtaining a stable deterministic limit [39]. Such an existence theorem is what we establish in this paper.

Similar to aforementioned works, we also consider solutions based on Brownian particles with hitting times, but with optimization. Indeed a novelty of our approach lies in the optimization structure given by Problem (1.1), for the choice of stopping time τ^* and the final target distribution ν^*. The optimality gives a certain 'regularizing effect' to the solution of Problem (1.1), and it generates a stable and physically meaningful solutions of both (St_1) and (St_2). The method and results are valid for general dimensions. In particular, for (St_1) we obtain the following results:

(a) (Theorem 9.3 and Corollary 9.11) For μ uniformly larger than 1 in its support, there exists a unique weak solution of (St_1) that vanishes in finite time.

(b) (Theorem 9.6 and Corollary 9.12) For compactly supported μ, there exists a unique weak solution of (St_1) that vanishes in finite time except in the support of μ.

Similar results hold for (St_2) (Theorem 9.2). The weak solutions in the above results will correspond to the distribution of active Brownian particles associated with the optimal stopping times obtained from (1.1), with $f = 1$ for (a) and $f = \chi_{\mathbb{R}^n \setminus \mathrm{supp}\mu}$ for (b). These results rely on a rigorous connection that we establish between the weak solution and the optimal Brownian stopping (see Theorems 5.4 and 5.6). Our results also explain the well-known non-uniqueness for the freezing case; see the discussion in Section 6 especially, Proposition 6.3.

In connection to the Stefan problem, we can understand our upper constraint problem (1.1) in the context of freezing and melting. For costs of type (I) the initial distribution μ can be viewed as the freezing energy of supercooled liquid, or in other words the negative temperature distribution of the supercooled heat particles. For costs of type (II) we view μ as some distribution of energy that activates the 'heat' particles. The upper constraint f should stand for the "latent supercooling or heat energy" corresponding to the available transition energy that the flow from μ needed to yield for the freezing or melting to occur. In other words, f prescribes the maximal amount of stopped Brownian particles a given location can accommodate.

The monotonicity types of the cost, (I), (II), correspond to (St_1), (St_2), respectively. While (I) and (II) generate different optimal stopping times, we prove that their target measure ν^* is independent of the cost, or even of its monotonicity type (Theorem 7.6). Using this rather remarkable *universality*, we are able to characterize the initial trace of the solutions given in (a) and (b) above: we refer to Section 9 for further discussions.

1.1. **The connection between the Stefan problem and Problem 1.1.** The present paper relies on the previous works on the optimal Skorokhod problem (1.5). Notice that after getting the optimal target ν^* in Problem 1.1, finding the optimal stopping time τ^* is reduced to solving (1.5) with μ and ν^*. It is shown [10] [28] that the solution to (1.5) is the first hitting time of (W_t, t) to a space-time barrier R, which enjoys time-monotonicity. Such a barrier set can be defined by a barrier function s, such that $R := \{(x,t) \mid s(t) \leq t\}$ for type (I) and $R := \{(x,t) \mid s(t) \geq t\}$ for type (II). Our results above imply that the free boundary of the Stefan problem, at time t, corresponds to the boundary of the time t-slice of R, that is $\{x \mid (x,t) \in R\}$. In fact, to derive our well-posedness of the Stefan problem $(St_1), (St_2)$, it is crucial for us to verify that R is a closed set, so its complement R^c is open. We are able to handle this by showing Theorem 4.14, which connects the barrier set of τ^* to a closed set generated by the potential function U_{μ_t} of the mass flow of W_t under τ^*. Interestingly its proof essentially uses the Eulerian method in [28].

The PDE methods of [28] is based on the correspondence between stopping time τ and its Eulerian flow (η, ρ) given on the space-time, that describes the distribution η of (W_t, t) before stopping by τ, and the stopped distribution of mass ρ of (W_τ, τ). This (η, ρ) solves weakly

$$\eta_t - \frac{1}{2}\Delta\eta = -\rho.$$

Moreover, under the assumptions (1.3)-(1.4) the (η, ρ) resulting from a solution of $\mathcal{P}(\mu, \nu)$ (1.5) yields that ρ is formally supported on the boundary of the active region $\{\eta > 0\}$ [28]. Assuming sufficient regularity, ρ should be concentrated on the boundary of the barrier set, which is also the boundary of the active region $\{\eta > 0\}$. Thus $\rho = \nu(\chi_{\{\eta>0\}})_t$ or $\rho = \nu(\chi_{\{\eta>0\}})_t$ depending on the monotonicity of the set $\{\eta > 0\}$ over time, and we end up with the weighted Stefan problem,

$$(\eta \pm \nu\chi_{\{\eta>0\}})_t - \frac{1}{2}\Delta\eta = 0, \tag{1.6}$$

where ν is the target measure for the optimal stopping time. This equation has been formally considered for a given fixed target measure ν in [28], and our Theorem 5.4 and 5.6 provide a rigorous justification.

The saturation result (Theorem 8.3) is crucial for the connection between Problem 1.1 and the Stefan problem $(St_1), (St_2)$. It gives $\nu = 1$ (when $f \equiv 1$) in the active region, so it reduces the above equation (1.6) to $(St_1), (St_2)$ for type (I), (II), respectively.

1.2. **Further remarks.** A novel element of our approach lies in combination of Lagrangian and Eulerian point of view, as we use both probabilistic arguments using Brownian paths, and PDE arguments using their Eulerian coordinates, to obtain basic topological properties. It seems that the closedness of the barrier set R for type (I) is new, while understanding further regularity of the free boundary for the corresponding supercooled Stefan problem (St_1), is a wide open problem. In this direction, we prove a strict monotonicity result for the barrier function s, a version of comparison principle, which may shed a light on the challenging question; see Theorems 10.1 and 10.2. This is a novel feature of Problem 1.1 as it is a consequence of the optimization of the free target distribution.

As a byproduct of our approach we define the notion of *subharmonically generated sets* in Section 6, which connects the Brownian stopping time and existence of the solution to the supercooled Stefan problem (St_1). It is a pair (Σ, E) of sets determined by a certain type of Brownian stopping time from an initial measure μ; Definition 6.1. The solvability of (St_1) with the initial data (μ, E) is determined by whether there is Σ such that (Σ, E) is subharmonically generated from μ; Theorem 6.2. This notion is used to describe unique solvability of (St_1) and also to characterize vanishing in finite time solution of it in Section 9; Theorem 9.9 and Corollaries 9.11 and 9.12.

The main results in this paper appear rather surprising, given the lack of understanding in the literature on multi-dimensional solutions of (St_1). In a subsequent work, we will further develop our method to extend well-posedness theory for (St_1) to more general initial boundaries. Also, the connection (Remark 9.5) between (St_1) and (St_2) that we find in this paper seems to be unexpected, and understanding this connection at a more physical level, is an interesting open problem. It would also be interesting to understand the physics behind why type (I) corresponds to freezing and type (II) to melting.

In this paper our method for finding the optimal target ν is solely based on optimizing with probability measures. We do not consider duality which is a powerful tool for understanding optimal Brownian stopping, though we heavily rely on the results from [28] that are obtained via solving the dual problem of the fixed target case, to find the barrier set of the optimal stoping time. Our free target problem gives a new aspect to the duality that we will discuss in a subsequent work.

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2. Preliminaries

Throughout this section we assume for simplicity, that the measures μ and ν are compactly supported in $\mathbb{R}^d$. Much of the discussion (and notation) in this section is borrowed from [28] which gives foundational results for the development given in this paper. Note that the purpose of this section is to collect the precise definitions and results necessary for the discussion in the subsequent sections. We try to include the least amount of information.

2.1. Notation.

- W_t denotes the Brownian motion, while W_t^y denotes the Brownian motion with $W_0 = y$.
- $B_r(x)$ denotes the ball of radius r centered at x.
- $a \wedge b := \min[a, b]$.
- for each Borel measurable set $S \subset \mathbb{R}^d$, $|S|$ denotes its d-dimensional Lebesgue measure.
- LSC = the set of lower semi-continuous functions on $\mathbb{R}^d$.
- H_0^1 = the Sobolev space on $\mathbb{R}^d$.

2.2. **Randomized stopping times and optimal Skorokhod problem in $\mathbb{R}^n$.** Related to Problem 1.1, it is important to understand the notion of stopping times and randomized stopping times.

2.2.1. *Stopping times and randomized stopping times.* Consider the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P}^\mu)$ where $\Omega = C(\mathbb{R}_{\geq 0}; \mathbb{R}^d)$, $\mathbb{P}^\mu$ the Wiener measure with initial distribution μ, and $\{\mathcal{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$ the natural filtration of the Brownian motion. Let $\mathcal{M}(\mathbb{R}_{\geq 0})$ denote the space of Radon measures on $\mathbb{R}_{\geq 0}$. A *randomized stopping time* α [7, 38] is a measure-valued random variable $\alpha : \Omega \ni \gamma \mapsto \alpha_\gamma \in \mathcal{M}(\mathbb{R}_{\geq 0})$ such that as a measure on $\mathbb{R}_{\geq 0}$, $\alpha_\gamma \geq 0$, $\alpha_\gamma(\mathbb{R}_{\geq 0}) = 1$, and the map $\gamma \mapsto \alpha_\gamma([0, t])$ is $\mathcal{F}_t$-measurable $\forall\, t$. Along each Brownian path, mass is dropped according to the distribution on $\mathbb{R}_{\geq 0}$ determined by the randomized stopping time. A *stopping time* $\tau : \Omega \to \mathbb{R}_\gamma$ is a random variable such that for each t, $\tau \wedge t$ is $\mathcal{F}_t$-measurable; it can be understood as a special case of randomized stopping time, where the measure α_γ on each path γ is a Dirac mass at the value $\tau(\gamma)$ in $\mathbb{R}_{\geq 0}$. To distinguish the different between randomized stopping time and stopping time, we often call the latter 'nonrandomized'.

We will say that a subset $Q \subset \mathbb{R}^+ \times \Omega$ is almost sure for a randomized stopping time if

$$\mathbb{E}\Big[\int_{\mathbb{R}^+} \mathbf{1}\{t \in Q\} d\alpha\Big] = 0$$

where $\mathbf{1}$ is the indicator function of the set. Often this will appear instead as an abbreviated form, i.e. $\mathbf{1}\{t \leq \eta\} := \mathbf{1}\{t \in Q_\eta\}$ where η is a stopping time and $Q_\eta = \{(t, \omega);\ t \leq \eta(\omega)\}$. We abbreviate a randomized stopping time by $\tau \sim \alpha$ in the sense that

$$\mathbb{E}\big[f(\tau)\big] = \mathbb{E}\Big[\int_{\mathbb{R}_{\geq 0}} f(t)\alpha(dt)\Big].$$

The we say ν is the distribution of the (randomized) stopping time τ, that is, $W_\tau \sim \nu$, if

$$\mathbb{E}\big[g(W_\tau)\big] = \int g(z)\nu(dz),\ \forall \text{ continuous } g.$$

From now on we abuse notation and use τ denote either stopping time or randomized stopping time.

2.2.2. *Subharmonic order and optimal Skorokhod problem.* Subharmonic order relates two measures from the point of the Brownian motion. First, f is a subharmonic function on an open set O in $\mathbb{R}^d$ if it is upper-semicontinuous with values on $\mathbb{R} \cup \{-\infty\}$, and for every $x \in O$ and every closed ball B in O with center at x, it satisfies

$$f(x) \leq \frac{1}{|B|}\int_B f(y)dy.$$

When f is C^2, the latter condition is equivalent to $\Delta f \geq 0$ on O, where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian. The reason why we specify the domain O is because subharmonic functions in O may not necessarily be extended to the whole set $\mathbb{R}^d$.

Two measures μ, ν on $\mathbb{R}^d$ with $|\mu| = |\nu|$ is said to be in *subharmonic order* $\mu \leq_{SH} \nu$, if for each open set O containing $supp(\mu + \nu)$,

$$\int \varphi(x)d\mu(x) \leq \int \varphi(x)d\nu(x) \quad \text{ for every smooth subharmonic function on } O.$$

See e.g. [27, Definition 1.2]. It is known (see e.g. [27, Theorem 1.5]) that for compactly supported μ and ν as we assume throughout the paper, we have $\mu \leq_{SH} \nu$ if and only if there exists a randomized stopping time τ for the Brownian motion with $W_0 \sim \mu$ and $W_\tau \sim \nu$, with $\mathbb{E}[\tau] < \infty$. For such μ and ν, for any lower semicontinuous cost functional $\mathcal{C}$ for the stopping times, one can find an optimal randomized time between μ and ν using the compactness of the space of randomized stopping times; this has been observed by Beiglböck, Cox, and Huesmann [10]. In fact, it is a nontrivial problem to see that such optimal randomized stopping time is indeed a stopping time (non-randomized) to solve the problem $\mathcal{P}(\mu,\nu)$ (1.5). This has been proved under the assumptions on the cost (1.3) and (1.4). (For other types of cost, for example the one based on the distance, $\mathcal{C}(\tau) = \mathbb{E}[|W_0 - W_\tau|]$, similar results have been proved in [30]; see also [29].)

Theorem 2.1 (Existence/Uniqueness of Optimal Skorokhod Problem [10, 28])**.** *Given compactly supported measures μ, ν on $\mathbb{R}^d$ with $\mu \leq_{SH} \nu$ and $\mu, \nu \ll$ Leb, the optimal Skorokhod problem $\mathcal{P}(\mu,\nu)$* (1.5) *with the cost $\mathcal{C}$ in* (1.2) *under the assumption* (1.3) *and* (1.4)*, has a unique optimal stopping time τ^*, which is not randomized in the type (I) case, and in the type (II) case, randomized only at $t = 0$; in the latter case $W_0 = x$ and $\tau^* = 0$ occurs with probability $\nu(x)/\mu(x)$. Moreover, there exists a space-time barrier set $R^* \in \mathbb{R}^d \times \mathbb{R}_{\geq 0}$ such that τ^* is given by the hitting time to R^*, that is,*

$$\tau^* = \inf\{t \mid (W_t, t) \in R^*\} \quad \textit{(in type (II) case, it holds for those paths with } \tau^* > 0\textit{).}$$

For our purpose it is important to understand the barrier set R^*. It can be characterized by using the dual solutions of $\mathcal{P}(\mu,\nu)$. In [28]) they used dynamic programming principle to establish the duality: $\mathcal{P}(\mu,\nu) = \mathcal{D}(\mu,\nu)$ where

$$\mathcal{D}(\mu,\nu) := \sup_{\psi \in LSC} \Big\{ \int_{\mathbb{R}^d} \psi(z)\nu(dz) - \int_{\mathbb{R}^d} J_\psi(x,0)\mu(dx) \Big\}.$$

Here, the function $(x,t) \mapsto J_\psi(x,t)$ is called the 'value function' which is the result of dynamic programming:

$$J_\psi(y,t) := \sup_\sigma \Big\{ \mathbb{E}\Big[\psi(W_\sigma^y) - \int_0^\sigma L(t+s, W_s^y) ds \Big] \Big\}$$

where σ are randomized stopping times. Then, it is shown [28] that under the assumptions in Theorem 2.1 (in fact for more general μ and ν), there exists an optimal dual function $\psi^* \in LSC \cap H_0^1$. The barrier set R^* then is given by

$$R^* = \{(x,t) \mid J_{\psi^*}(t,x) = \psi^*(x)\} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}. \tag{2.1}$$

The strict monotonicity assumption on L (1.4) for type (I) and (II) implies (forward or backward) time-monotonicity of the set R^*, namely,

$$R^* = \{(x,t) \mid t \geq s(x)\} \text{ for type (I)}, \quad R^* = \{(x,t) \mid t \leq s(x)\} \text{ for type (II)},$$

where the function s, called the barrier function, and the stopping time τ^* is given by

$$s(x) = \inf\{t \in \mathbb{R}^+;\ J_{\psi^*}(t,x) = \psi^*(x)\} \quad \& \quad \tau^* = \inf\{t \mid t \geq s(W_t)\} \qquad \text{for type (I)},$$
$$s(x) = \sup\{t \in \mathbb{R}^+;\ J_{\psi^*}(t,x) = \psi^*(x)\} \quad \& \quad \tau^* = \inf\{t \mid t \leq s(W_t)\} \qquad \text{for type (II)}.$$

2.3. **Eulerian formulation.** It is important for us to relate the Brownian motion with stopping time τ, with its corresponding mass flow, which we call Eulerian flow; considering such Eulerian formulation and effectively using it to analyze optimal stopping times is one of the main innovations of [28].

An Eulerian flow for us is a pair of measures (η, ρ) on $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$ such that in the weak sense,

$$\rho(t,x) + \partial_t \eta(t,x) = \frac{1}{2}\Delta\eta(t,x), \tag{2.2}$$

$$\int_{\mathbb{R}^+} d\rho = \nu, \qquad \eta(x,0) = \mu(x).$$

For a more precise description, let $O \subset \mathbb{R}^d$ be a bounded open convex set that contains the supports of μ and ν. Pick $\gamma < \lambda$ the Poincaré constant of O. We consider

$$C_{-\gamma}(\mathbb{R}_{\geq 0} \times \overline{O}) := \{w : C(\mathbb{R}^+ \times \overline{O}) \mid \mathrm{e}^{-\gamma t} w(t,x) \to 0 \text{ as } t \to \infty, \text{ uniformly in } x\}.$$

Its dual space $\mathcal{M}_\gamma(\mathbb{R}^+ \times \overline{O})$ is the finite Radon measures with γ-exponential decay. We let $C^{1,2}_{-\gamma}(\mathbb{R}^+ \times \overline{O})$ denote the functions whose first derivative in time and second derivatives in space lie in $C_{-\gamma}(\mathbb{R}^+ \times \overline{O})$. Then, following [28] we shall say that (η, ρ) is *an admissible pair*, provided $\eta, \rho \in \mathcal{M}_\gamma(\mathbb{R}^+ \times \overline{O})$ and $\eta, \rho \geq 0$, and they satisfy the following two equations:

$$\int_{\overline{O}} u(z)\,\nu(dz) = \int_{\overline{O}}\int_{\mathbb{R}^+} u(x)\rho(dt,dx) \quad \text{for all } u \in C(\overline{O}). \tag{2.3}$$

$$-\int_{\overline{O}} w(0,y)\,\mu(dy) = \int_{\overline{O}}\int_{\mathbb{R}^+} \Big[\frac{\partial}{\partial t}w(t,x) + \frac{1}{2}\Delta w(t,x)\Big]\eta(dt,dx) - \int_{\overline{O}}\int_{\mathbb{R}^+} w(t,x)\rho(dt,dx) \quad \text{for all } w \in C^{1,2}_{-\gamma}(\mathbb{R}^+ \times \overline{O}). \tag{2.4}$$

Let us translate the Brownian stopping into Eulerian flow:

Proposition 2.2 (see [28] Proposition 2.2)**.** *Given μ and ν compactly supported with $\mu, \nu \ll$ Leb. Suppose τ is a stopping time with $W_0 \sim \mu$, $W_\tau \sim \nu$. Then, there is an admissible pair (η, ρ) of measures on $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$, such that for every $g \in C_c(\mathbb{R}^d \times \mathbb{R}_{\geq 0})$*

$$\mathbb{E}\big[g(\tau, W_\tau)\big] = \int_{\overline{O}}\int_{\mathbb{R}^+} g(t,x)\rho(dt,dx),$$

and

$$\mathbb{E}\Big[\int_0^\tau g(t,W_t)dt\Big] = \int_{\overline{O}}\int_{\mathbb{R}^+} g(t,x)\eta(dt,dx).$$

This gives equivalence between the optimization problem for the stopping time and the one for the Eulerian flow [28]:

$$\mathcal{P}(\mu,\nu) = \mathcal{P}_1(\mu,\nu) := \inf_{(\eta,\rho)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^+} L\big(t,x)\eta(dt,dx)$$

where (η, ρ)'s are admissible pairs.

We close this section with an essential lemma for us to consider the Eulerian flow in the context of the barrier of the stopping time. It will be later used in Section 4.

Lemma 2.3 ([28] Lemma 4.5)**.** *Suppose $\mu \ll Leb$ is a compactly supported measure and $R \subset \mathbb{R}^d \times \mathbb{R}_{\geq 0}$. Let (η, ρ) be admissible with the condition $\eta(R) = 0$ and $\rho(R) = 1$.*

We suppose R is a measurable forward-barrier, namely, $(r, x) \in R$ whenever $(t, x) \in R$ with $t \leq r$, and $(t, x) \in R$ if there is $(t_i, x) \in R$ with $t_i \to t$. Then (η, ρ) is unique.

If instead R is a measurable backward-barrier, namely, $(r, x) \in R$ whenever $(t, x) \in R$ with $t \geq r$, and $(t, x) \in R$ if there is $(t_i, x) \in R$ with $t_i \to t$, then (η, ρ) is uniquely determined given the value of ρ on the set $(0, x)$ where $s(x) = 0$.

Remark 2.4. *Let us comment on why the additional condition $\eta(R) = 0, \rho(R) = 1$ to the corresponding PDE (2.4) is sufficient to determine the unique solution. Because $\rho(R) = 1$ for the probability measure ρ, we have $\rho(R^c) = 0$. Therefore, the (2.4) behaves like the heat equation for η in R^c, with the Dirichlet condition which is due to $\eta(R) = 0$; therefore the uniqueness is not surprising, especially if R is closed (so R^c is open). In fact we will show that we can take R as a closed set; see Section 4, Theorem 4.14.*

3. Properties of stopping times

In this section we prove several basic but important results for the properties of hitting times as well as optimal stopping times. We handle the general case where the barriers are merely measurable sets, say, not open or closed. This may be of its own interest. For the present paper, because of Section 4 (see Thereom 4.14), it suffices to consider only the case where the barrier set is closed, in which case some proofs of this section can possibly be simplified.

3.1. **Properties of stopping times with respect to a barrier set.** We first recall the following result from potential theory:

Lemma 3.1. *Let μ_z is the distribution of the stopped Brownian motion starting at z by the first hitting time to the round sphere ∂B_r of radius r centered at 0. Let σ be the surface measure of ∂B_r. Suppose $|z| \geq 2r$ or $|z| \leq \frac{1}{2}r$. Then, there exists a universal constant $C > 0$ depending only on the dimension, such that the Radom-Nikodym derivative satisfies*

$$\frac{d\mu_z}{d\sigma} \leq C.$$

In particular, for the Brownian motion W_t^z starting from z, and for the first hitting time τ_σ to the sphere ∂B_r, we have

$$Prob[W_{\tau_\sigma}^z \in E] \leq C \frac{\sigma[E]}{\sigma[\partial B_r]} \text{ for any } E \subset \partial B_r.$$

Proof. As the Brownian motion is scaling invariant, it is sufficient to consider the case $r = 1$. Then, the desired upper bound should be a standard result of potential theory. This is a consequence of Dahlberg's Theorem [17]. □

Using this lemma we now prove a key lemma, which takes care of a subtle possibility that Brownian paths blocked by a barrier set may possibly drop mass in the set; the lemma says it should not happen when the final distribution is absolutely continuous. Another equivalent interpretation is that if it takes some time for Brownian particles from μ to reach G, then the particles should spread over G before stopping or ν should be singular.

We consider probability measures μ and ν satisfying

$$\mu \ll Leb \text{ and } \nu \leq f \text{ for a bounded measurable function } f \text{ in } \mathbb{R}^d. \tag{3.1}$$

Lemma 3.2. *For probability measures μ and ν satisfying* (3.1)*, let τ be a randomized stopping time with $W_0 \sim \mu$ and $W_\tau \sim \nu$. Let G be a measurable set, and let τ_G be the first hitting time to G. That is, $\tau_G = \inf\{t \mid W_t \in G\}$. Suppose $\tau \leq \tau_G$ and $\mu \wedge (\nu|_G) = 0$. Then $\nu[G] = 0$.*

Remark 3.3. *Notice that the condition $\nu \ll f$, especially $\nu \ll Leb$, is essential in Lemma 3.2. For example, consider the distribution $\nu \sim W_\tau$ of the Brownian motion of the hitting time τ to a sphere G, does not satisfy the result of this lemma.*

Proof. Suppose $\nu[G] > 0$ for contradiction. Then there exists $\delta > 0$ such that the set

$$G_\delta := \{z \in G \mid \frac{d\nu}{dx} \geq \delta\}$$

has $|G_\delta| > 0$. Consider a Lebesgue point x of G_δ, where the Lebesgue density is 1. Then, for each small $\epsilon > 0$, there exists $\bar{r} > 0$ (depending on ϵ) such that the following holds:

$$\nu[B_r(x)] \geq \delta/2|B_r(x)|, \qquad |G_\delta \cap B_r(x)| \geq (1-\epsilon)|B_r(x)| \qquad \text{for all } 0 < r \leq \bar{r}.$$

From the Fubini's theorem, there are $0 < r_1 < r_2 < \bar{r}$ with $8r_1 < r_2 < 10r_1$ such that

$$\mathcal{H}^{d-1}[G_\delta \cap \partial B_{r_i}(x)] \geq (1-5\epsilon)\mathcal{H}^{d-1}[\partial B_{r_i}(x)], \quad i = 1, 2.$$

From Lemma 3.1, for a dimensional constant C we have

$$Prob[W_\tau \notin B_{r_2}(x) \ |W_t \in B_{4r_1}(x) \ \text{ for some } t \leq \tau] \leq C\epsilon,$$

since for the Brownian path from a point inside $B_{4r_1}(x)$ to go outside $B_{r_2}(x)$, it has to go through $\partial B_{r_2}(x)$ but avoiding hitting G, thus G_δ. Therefore,

$$Prob[W_\tau \in B_{r_2}(x) \ |W_t \in B_{4r_1}(x) \ \text{ for some } t \leq \tau] \geq 1 - C\epsilon.$$

By a similar argument of using the same lemma, we get

$$Prob[W_\tau \in B_{r_1}(x) \mid \ W_t \in B_{4r_1}(x) \setminus B_{2r_1}(x) \ \text{ for some } \ t \leq \tau] \leq C\epsilon.$$

Now let us consider

$$\begin{aligned} \nu[B_{r_1}(x)] &= Prob[W_\tau \in B_{r_1}(x)] \\ &\leq Prob[W_0 \in B_{2r_1}(x)] + Prob[W_\tau \in B_{r_1}(x) \ \& \ W_0 \notin B_{2r_1}(x)] \\ &= I + II. \end{aligned}$$

Notice that

$$I = \mu[B_{2r_1}(x)] \leq \epsilon|B_{2r_1}(x)|$$

from the assumption $\mu \wedge (\nu|_G) = 0$ and the choice of x. On the other hand, from continuity of Brownian paths, for the Brownian path from a point outside $B_{2r_1}(x)$ to arrive in $B_{r_1}(x)$, it first needs to arrive in $B_{4r_1}(x) \setminus B_{2r_1}(x)$. Therefore,

$$\begin{aligned} II &\leq Prob[W_\tau \in B_{r_1}(x) \;\&\; W_t \in B_{4r_1}(x) \setminus B_{2r_1}(x) \;\text{ for some }\; t \leq \tau] \\ &= Prob[W_\tau \in B_{r_1}(x) \mid \; W_t \in B_{4r_1}(x) \setminus B_{2r_1}(x) \;\text{ for some }\; t \leq \tau] \\ &\quad \times Prob[\, W_t \in B_{4r_1}(x) \setminus B_{2r_1}(x) \;\text{ for some }\; t \leq \tau]. \end{aligned}$$

Also, notice that

$$\begin{aligned} \nu[B_{r_2}(x)] = Prob[W_\tau \in B_{r_2}(x)] &\geq Prob[W_\tau \in B_{r_2}(x) \mid W_t \in B_{4r_1}(x) \text{ for some } t \leq \tau] \\ &\quad \times Prob[W_t \in B_{4r_1}(x) \text{ for some } t \leq \tau]. \end{aligned}$$

Combining all these with the previous probability estimates with ϵ, we see that

$$II \leq \frac{C\epsilon}{1 - C\epsilon} \nu[B_{r_2}(x)]$$

which implies

$$\nu[B_{r_1}(x)] \;\leq\; I + II \;\leq\; \epsilon |B_{2r_1}(x)| + \frac{C\epsilon}{1 - C\epsilon} \nu[B_{r_2}(x)].$$

Recall that $\nu[B_{r_1}(x)] \geq \delta |B_{r_1}(x)|$, ν is bounded, and that the volumes $|B_{r_1}(x)|$, $|B_{2r_1}(x)|$, and $|B_{r_2}(x)|$ are all comparable up to a constant factor depending only on the dimension. Then, we get a contradiction by letting $\epsilon \to 0$. This completes the proof. □

There are several important consequences of Lemma 3.2. An immediate consequence is this.

Corollary 3.4. *Let μ, ν, τ be as given in Lemma 3.2. In addition suppose that τ is given as the first hitting time to the barrier function s, that is, a measurable function $s : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, such that*

$$\begin{aligned} &\tau = \inf\{t \mid t \geq s(W_t)\} \textit{ in (I) case, or} \\ &\tau = \inf\{t \mid t \leq s(W_t)\} \textit{ in (II), respectively.} \end{aligned}$$

Let $G = \{x \mid s(x) = 0\}$ in (I) case, $G = \{x \mid s(x) = \infty\}$ in (II) case, respectively. Assume $f \wedge \mu = 0$. Then, $\nu[G] = 0$.

Proof. Notice that $\tau \leq \tau_G = \inf\{t \mid W_t \in G\}$ from the definition of τ and τ_G. Therefore, the result follows from Lemma 3.2. □

Later in this section we will need another technical fact regarding stopping times.

Lemma 3.5. *Let τ be a stopping time. Suppose that G is a measurable set with positive measure, and every point of G has Lebesgue density larger than $a > 0$. Let $D \subset (0, \infty)$ be a countable dense subset such that*

$$\textit{for each fixed constant } t \in D,\ Prob[W_t \in G \;\&\; t < \tau] = 0. \tag{3.2}$$

Let $\tau_G := \inf\{r > 0 \mid W_r \in G\}$. Then, $\tau \leq \tau_G$ almost surely.

Proof. First note that

$$Prob[\tau > \tau_G] = Prob[\exists t', t' < \tau \ \& \ W_{t'} \in G]$$

and

$$Prob[\exists t'' \in D, t'' < \tau \ \& \ W_{t''} \in G]$$

$$\geq Prob[\exists t'' \in D, t'' < \tau \ \& \ W_{t''} \in G \ \mid \exists t', t' < \tau \ \& \ W_{t'} \in G] Prob[\exists t', t' < \tau \ \& \ W_{t'} \in G].$$

We now show the following:

Claim: There exists a dimensional constant $C > 0$ such that

$$Prob[\exists t'' \in D, t'' < \tau \ \& \ W_{t''} \in G \ \ |\exists t', t' < \tau \ \& \ W_{t'} \in G] = C/2 > 0.$$

Proof of Claim. For each fixed $t'' > t'$, and for given $x = W_{t'}$, the probability density σ for $W_{t''}$ is nothing but the heat kernel at around x with time $t'' - t'$, that is,

$$\sigma(z) = \frac{C_d}{(t'' - t')^{d/2}} e^{-|z-x|^2/(t''-t')}.$$

Also note that for fixed $t' < t''$ and for given $W_{t'} = x$, , the event '$t' < \tau$' is independent of the event '$W_{t''} \in G$', from the strong Markov property of the Brownian motion. Hence, for each $x \in G$ and for sufficiently small $r > 0$ that depends on x we have

$$\begin{aligned}
&Prob[W_{t''} \in G \mid t' < \tau \ \& \ W_{t'} = x] \\
&= Prob[W_{t''} \in G \mid W_{t'} = x] \\
&\geq Prob[W_{t''} \in G \cap B_r(x) \mid W_{t'} = x] \\
&\geq \frac{C_d}{(t'' - t')^{d/2}} e^{-r^2/(t''-t')} |G \cap B_r(x)|. \\
&\geq \frac{a}{2} \frac{C_d}{(t'' - t')^{d/2}} e^{-r^2/(t''-t')} |B_r(x)|,
\end{aligned}$$

where the last line follows since G has Lebesgue density larger than $a > 0$.

Now let $r = (t'' - t')^{1/2}$. Then above computation yields

$$\lim_{t'' \to t'+} Prob[W_{t''} \in G \mid t' < \tau \ \& \ W_{t'} = x] \geq C > 0.$$

Since D is dense, this implies that for each $\delta > 0$,

$$Prob \left[\exists t'' \in D \ \& \ \ t' < t'' < t' + \delta \ \ \& \ W_{t''} \in G \ \Big| \ t' < \tau \ \& \ W_{t'} \in G\right] \geq C.$$

Notice that for fixed t',

$$Prob[\tau > t' + \delta \mid t' < \tau \ \& \ W_{t'} \in G] \to 1 \text{ as } \delta \to 0 \text{ for } \delta > 0.$$

and thus

$$Prob[\exists t'' \in D, t'' < \tau \ \& \ W_{t''} \in G \ \ | \ t' < \tau \ \& \ W_{t'} \in G] \geq C/2 > 0.$$

Since C is uniform over t', we conclude that

$$Prob[\exists t'' \in D, t'' < \tau \ \&\ W_{t''} \in G \ \ |\exists t', t' < \tau \ \&\ W_{t'} \in G] \geq C/2$$

as well, verifying the claim. □

The Claim yields that

$$Prob[\exists t'' \in D, t'' < \tau \ \&\ W_{t''} \in G] \geq \frac{C}{2} Prob[\tau > \tau_G].$$

On the other hand, since D is countable, we have

$$Prob[\exists t'' \in D, t'' < \tau \ \&\ W_{t''} \in G] \leq \sum_{\bar{t} \in D} Prob[W_{\bar{t}} \in G \ \&\ \bar{t} < \tau].$$

Therefore, if $Prob[\tau > \tau_G] > 0$, then there should exist $\bar{t} \in D$ such that

$$Prob[W_{\bar{t}} \in G \ \&\ \bar{t} < \tau] > 0.$$

This contradicts (3.2), thus completing the proof. □

Remark 3.6. *In Lemma 3.5 the assumption that G consists of its Lebesgue point is essential. For example, consider in $\mathbb{R}^1$, and the interval $G_1 = [3, 4]$ and let $G = \{0\} \cup G_1$. Let μ be the uniform distribution on the interval $[-1/2, 1/2]$. One can find a stopping time τ and its distribution ν, $W_0 \sim \mu$, $W_\tau \sim \nu$ such that ν is supported on $[-3/2, 3/2]$. Then, because of the support of ν, we have that $\tau \leq \tau_{G_1}$. Therefore, for each $t > 0$, the probability $Prob[W_t \in G \ \&\ t < \tau] = Prob[W_t = 0 \ \&\ t < \tau] = 0$. However, on the other hand, $\tau \not\leq \tau_G$, as Brownian paths will pass through 0 before τ, with positive probability.*

Lemma 3.5 together with Lemma 3.2 give the following useful result.

Corollary 3.7. *Let μ, ν and τ be as given in Lemma 3.2, and let G be a measurable subset of $\mathbb{R}^d$. Let $D \subset (0, \infty)$ be a countable dense subset such that (3.2) holds. Then either the support of μ overlaps with that of $\nu|_G$, or $\nu[G] = 0$.*

Remark 3.8. *Notice that the condition $\nu \ll f$, especially $\nu \ll Leb$, is essential in Corollary 3.7. In particular, if $|G| = 0$, then $Prob[W_t \in G \ \&\ t < \tau] = 0$ for each constant $t > 0$ when $W_0 \sim \mu \ll Leb$. So, if a singular measure ν supported on such zero Lebesgue measure set, for example the distribution $\nu \sim W_\tau$ of the Brownian motion of the hitting time τ to a sphere, does not satisfy the result of this Corollary.*

Proof. The case $|G| = 0$ is obvious since $\nu \ll Leb$. Suppose $|G| > 0$ and $\mu \wedge \nu = 0$ in $\tilde{G}$, we will show that $\nu[G] = 0$. Let $\tilde{G}$ be the set of Lebesgue points of G. Since (3.2) remains valid for $\tilde{G}$, Lemma 3.5 yields $\tau \leq \tau_{\tilde{G}}$ (almost surely), which then implies $\nu[\tilde{G}] = 0$ by Lemma 3.2 as $\mu \wedge \nu|_{\tilde{G}} = 0$. Since $\nu[G] = \nu[\tilde{G}]$ for $\nu \ll Leb$, this completes the proof. □

We now focus on stopping times given as the hitting time to a barrier function $s : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$. Lemma 3.2 and its consequences give a useful characterization for such a stopping time, which will play a key role later in the paper (see for instance Theorem 5.4 and Section 10). Roughly speaking the following proposition endows a certain regularity to the set $\{(x, t) \mid t = s(x)\}$

from the condition (3.1), by ensuring that the Brownian particles stop only at the set, nor above or below (in t), for instance.

Proposition 3.9. *Let $s : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ be a measurable function, and let τ be the hitting time to the barrier s, that is, $\tau := \inf\{t \mid t \geq s(W_t)\}$ in the type (I) case, $\tau := \inf\{t \mid t \leq s(W_t)\}$ in the type (II) case, respectively. Assume $W_0 \sim \mu$, $W_\tau \sim \nu$ where μ, ν satisfies* (3.1)*. In the (II) case further assume that $\mu \wedge \nu = 0$. Then,*

$$\tau = s(W_\tau) \text{ almost surely.} \tag{3.3}$$

Proof. First recall that, by [28, Corollary 3.6], the stopping time occurs only in the barrier set. Thus, if (3.3) is not true, then

$$Prob[\tau > s(W_\tau)] > 0 \text{ in the (I) case,}$$
$$Prob[\tau < s(W_\tau)] > 0 \text{ in the (II) case, respectively,}$$

then, there exists $\mathbb{Q} \ni \bar{t} > 0$ such that

$$Prob[\tau > \bar{t} > s(W_\tau)] > 0 \text{ in the (I) case,}$$
$$Prob[\tau < \bar{t} < s(W_\tau)] > 0 \text{ in the (II) case, respectively.}$$

Let us separate the (I) and (II) cases.

1. (I) case. Consider the set $S := \{x \mid s(x) < \bar{t}\}$ and the distribution $\nu_{\bar{t}}$ of $W_{\bar{t}\wedge\tau}$. Note that $W_{\bar{t}} \in S$ implies that $\tau \leq \bar{t}$. Therefore, $W_{\bar{t}\wedge\tau} \in S$ is equivalent to "$W_\tau \in S$ and $\tau \leq \bar{t}$", which is then equivalent to "$s(W_\tau) < \bar{t}$ and $\tau \leq \bar{t}$ ", whose probability is strictly < 1 due to $Prob[\tau > \bar{t} > s(W_\tau)] > 0$. Therefore, $\nu_{\bar{t}}[S] < 1$, thus the measure $\mu_{\bar{t}} := \nu_{\bar{t}} - \nu_{\bar{t}}|_S$ has a positive total mass $|\mu_{\bar{t}}| > 0$.
We then start the Brownian motion from the distribution $\mu_{\bar{t}}$ and initial time $\bar{t}$; we immediately stop if $\tau \leq \bar{t}$, otherwise we continue the Brownian motion until τ; we call this stopping time $\bar{\tau}$. Let $\bar{\nu}$ denote that distribution of $W_{\bar{\tau}}$, then, from the condition $\nu \ll Leb$, we also have $\bar{\nu} \ll Leb$. Moreover, from the definition of S and the fact $Prob[\tau > \bar{t} > s(W_\tau)] > 0$, we have

$$\bar{\nu}[S] > 0.$$

On the other hand, let τ_S be the first hitting time to the set S for Brownian motion starting from the time $\bar{t}$ with the initial distribution $\mu_{\bar{t}}$. Notice that if $W_t \in S$ (with $t \geq \bar{t}$) then $t \geq \bar{t} > s(W_t)$ by the definition of S, therefore $\tau \leq t$, thus $\bar{\tau} \leq t$ from the construction of the stopping time $\bar{\tau}$. This implies that $\bar{\tau} \leq \tau_S$. From the construction of $\mu_{\bar{t}}$ we see that $\mu_{\bar{t}} \ll Leb$ and $\mu_{\bar{t}} \wedge \bar{\nu}|_S = 0$. Therefore from Lemma 3.2 we see that $\bar{\nu}[S] = 0$, a contradiction. This completes the proof in the (I) case.

2. (II) case. Let $S = \{x \mid s(x) > \bar{t}\}$. Let $\bar{\tau} = \bar{t} \wedge \tau$ and $\tau_S := \inf\{t \mid W_t \in S\}$. Notice that obviously if $t \geq \bar{t}$, then $t \geq \bar{\tau}$. On the other hand, if $t < \bar{t}$ and $W_t \in S$ then $s(W_t) > t$ so $t > \tau$. These imply that $\bar{\tau} \leq \tau_S$. Moreover, let $\bar{\nu}$ be the distribution of $\bar{\tau}$, that is, $W_{\bar{\tau}} \sim \bar{\nu}$. Then from $Prob[\tau < \bar{t} < s(W_\tau)] > 0$ and the assumption $\mu \wedge \nu = 0$ in this (II) case, there exists $S_1 \subset S$ such that $\mu[S_1] = 0$ and $\bar{\nu}[S_1] > 0$. Since $\mu \wedge (\bar{\nu}|_{S_1}) = 0$, and $\bar{\tau} \leq \tau_S \leq \tau_{S_1}$, we have from Lemma 3.2, that $\bar{\nu}[S_1] = 0$, a contradiction. This completes the proof. ☐

4. Stopping times and potential flows.

For each stopping time τ we can consider the time flow of distribution μ_t of $W_{\tau\wedge t}$, $0 \leq t \leq \infty$. These distributions then define potential functions by solving the corresponding Poisson equation. The potential functions can be used back to study the stopping times, which is especially beneficial because they have regularity properties coming from the elliptic regularity and Ito's formula. Our main theorem (Theorem 4.14) of this section utilizes this to show that when τ is given by the hitting time to a barrier of type (I) or (II), the barrier is a closed set. This is a key fact that will be used in Section 5 where we establish the consistency of our probabilistic formulation of the Stefan problem with the PDE formulation.

Using potential functions to study stopping times has been considered in the literature starting from [6] for existence of Brownian stopping times (Skorokhod problem), and more recently for analyzing optimal stopping problems (see [16, 25, 18], for one space dimension, and [28] for higher dimensions).

4.1. **Potential flows, definition and regularity.** In this subsection we define the potential flow U_{μ_t} and establish its space-time continuity (Corollary 4.9).

Let $N(y)$ be the Newtonian potential function on $\mathbb{R}^d$, namely

$$N(y) := \begin{cases} \frac{1}{2}|y| & d = 1, \\ 2\pi \log |y| & d = 2, \\ \frac{1}{d(2-d)\omega_d}|y|^{2-d} & \text{otherwise.} \end{cases}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$ such that

$$\Delta N(y) = \delta_0.$$

Given a measure μ, define the potential U_μ defined as

$$U_\mu(x) := \int N(x-y)d\mu(y).$$

Since $\Delta U_\mu = \mu$, standard elliptic regularity theory yields the following:

Lemma 4.1 (Spatial continuity)**.** *Assume that there are constants $M, R > 0$ such that $\mu \leq M$ and $\mu = 0$ outside B_R. Then $\|U_\mu\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq C$ for any $0 < \alpha < 1$, with $C = C(\alpha, M, R)$.*

Remark 4.2. *Our main focus is on the stopping times τ generating a compactly supported ν, $W_\tau \sim \nu$, with the upper density constraint $\nu \leq f$. For such τ, the measure μ_t, the distribution of $W_{\tau\wedge t} \sim \mu_t$, is uniformly bounded with respect to t, as μ_t is bounded by the greater of the solution to the heat equation (with initial value μ) and f. Also, the support of μ_t is contained in the convex hull of the support of ν since the subharmonic order $\mu_t \leq_{SH} \nu$ implies convex order $\mu_t \leq_C \nu$. In what follows we thus focus on the case where the result of Lemma 4.1 holds.*

We now recall a simple consequence of Ito's formula. For each $0 \leq g \in C_c^1(\mathbb{R}^d)$, consider a subharmonic function $u \in C^2$ such that $\Delta u = g$. Note that one can find such $u \in C^{2,\alpha}$ from elliptic regularity. Using Ito's formula, for each stopping time $\sigma \geq 0$ we have

$$\mathbb{E}[u(W_\sigma) \mid W_0 = y] - u(y) = \mathbb{E}\left[\int_0^\sigma \frac{1}{2}\Delta u(W_t)dt \;\middle|\; W_0 = y\right] = \mathbb{E}\left[\int_0^\sigma \frac{1}{2}g(W_t)dt \;\middle|\; W_0 = y\right].$$

In particular, we can show

Lemma 4.3. *Suppose that μ is a probability measure on $\mathbb{R}^d$ and $W_0 \sim \mu$ and $W_{\tau_i} \sim \nu_i$, $i = 1, 2$ and that the potentials U_{ν_1}, U_{ν_2} are measurable functions. Assume that $\tau_1 \leq \tau_2$. Then, for each closed or open set $E \subset \mathbb{R}^d$, we have for the characteristic function χ_E that*

$$\int_E \left(U_{\nu_2} - U_{\nu_1}\right)(y)dy \;=\; \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}\chi_E(W_t)dt\right]. \tag{4.1}$$

In general, for a measurable E, there exists a monotonically increasing sequence of compact sets $K_n \subseteq E$ with $\lim_{n\to\infty} |E \setminus K_n| = 0$ and monotonically decreasing sequence of open sets with $E \subseteq O_n$ and $\lim_{n\to\infty} |O_n \setminus E| = 0$ such that

$$\int_E \left(U_{\nu_2} - U_{\nu_1}\right)(y)dy = \lim_{n\to\infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}\chi_{K_n}(W_t)dt\right] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \lim_{n\to\infty} \chi_{K_n}(W_t)dt\right],$$

$$\int_E \left(U_{\nu_2} - U_{\nu_1}\right)(y)dy = \lim_{n\to\infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}\chi_{O_n}(W_t)dt\right] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \lim_{n\to\infty} \chi_{O_n}(W_t)dt\right].$$

Proof. 1. First let us consider the case where E is compact or open. For a compact set $E = K$, there exists a monotonically decreasing sequence $0 \leq g_k \in C_c^1(\mathbb{R}^d)$, point-wisely converging to χ_K, that is, $\chi_K = \inf_k g_k$. Indeed, take a continuous function h_k that equals $d(x, E)$ if $d(x, E) \leq 1/k$, 1 if $d(x, E) \geq 2/k$, and the linear interpolation, $(1 - 1/k)(d(x, E) - 1/k) + 1/k$, for $1/k \leq d(x, E) \leq 2/k$. After mollifying, this generates a monotone increasing sequence of C^1 functions that converges to $1 - \chi_E$.

When $E = O$ is open and bounded, we can modify the above construction for $K = (\mathbb{R}^d \setminus O) \cap B_R$ for $R \gg 1$ to find corresponding $\{g_k\}_k$ converging to χ_O.

Now, consider the subharmonic functions $u_k \in C^\infty$ such that $\Delta u_k = g_k$. We have

$$\begin{aligned}
\int_E \left(U_{\nu_2} - U_{\nu_1}\right)(y)dy &= \lim_{k\to\infty} \int \left(U_{\nu_2} - U_{\nu_1}\right)(y)g_k(y)dy \text{ (by the monotone convergence theorem)}\\
&= \lim_{k\to\infty} \int \left(\nu_2 - \nu_1\right)(y)u_k(y)\,dy \text{ (integration by parts)}\\
&= \lim_{k\to\infty} \mathbb{E}[u_k(W_{\tau_2}) - u_k(W_{\tau_1})]\\
&= \lim_{k\to\infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}g_k(W_t)dt\right] \text{ (by Ito's formula)}
\end{aligned}$$

where by the monotone convergence theorem (applied to the Wiener measure on the path space as the monotone convergence $g_k \to \chi_E$ can be extended to the path space), the last line is the same as

$$\mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}\chi_E(W_t)dt\right].$$

2. Next for open or cloesd E, consider $E_R = E \cap B_R$. Notice $\chi_{E_R} \to \chi_E$ monotonically as $R \to \infty$. We can apply arguments in 1. for E_R and apply the monotone convergence theorem as $R \to \infty$.

3. Now for a measurable E, one can find a sequence of increasing compact sets $K_1 \subseteq K_2 \subseteq K_3 \subseteq ....$ contained in E such that $\lim_{n\to\infty} |E \setminus K_n| = 0$ and monotonically decreasing sequence of open sets with $E \subseteq O_n$, $O_1 \supseteq O_2 \supseteq O_3 \supseteq \cdots$, and $\lim_{n\to\infty} |O_n \setminus E| = 0$. Let $\{E_n\}$ denote either the sequence $\{K_n\}$ or $\{O_n\}$. Then, from the monotone convergence theorem applied to χ_{E_n}, we have

$$\int_E (U_{\nu_2} - U_{\nu_1})(y)dy = \lim_{n\to\infty} \int_{E_n} (U_{\nu_2} - U_{\nu_1})(y)dy.$$

Since for the sets $\int_{E_n} (U_{\nu_2} - U_{\nu_1})(y)dy = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}\chi_{E_n}(W_t)dt\right]$ from the previous cases, we get

$$\int_E (U_{\nu_2} - U_{\nu_1})(y)dy = \lim_{n\to\infty} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}\chi_{E_n}(W_t)dt\right] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2} \lim_{n\to\infty} \chi_{E_n}(W_t)dt\right]$$

where for the last equality we applied the monotone convergence theorem to the Wiener measure on the path space as the monotone sequence χ_{E_n} can be extended to the path space. This completes the proof. □

The following is a standard fact, but, we provide its proof for the sake of exposition.

Corollary 4.4. *Suppose μ is a bounded, compactly supported measure on $\mathbb{R}^d$ and that τ_1, τ_2 are stopping times with the given initial distribution μ and let $W_{\tau_i} \sim \nu_i$, $i = 1, 2$. Then*

$$\tau_1 \le \tau_2 \text{ implies } U_{\nu_1} \le U_{\nu_2}.$$

Moreover the equality holds if and only if $\tau_1 = \tau_2$ almost surely.

Proof. The order immediately follows from Lemma 4.3. For equality, suppose $U_{\nu_1} = U_{\nu_2}$. We Lemma 4.3 to $E = \mathbb{R}^d$ to obtain

$$0 = \int_{\mathbb{R}^d} (U_{\nu_2} - U_{\nu_1})(y)dy = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \frac{1}{2}dt\right] = \frac{1}{2}\mathbb{E}[\tau_2 - \tau_1]$$

Hence $\mathbb{E}[\tau_2 - \tau_1] = 0$. Since $\tau_2 \ge \tau_1$, we conclude that $\tau_2 = \tau_1$ almost surely. □

Remark 4.5. *Note that there are many stopping times with the same final distribution. Therefore, $U_{\nu_1} = U_{\nu_2}$ without an assumption like $\tau_1 \le \tau_2$ does not give much information.*

Now we define our potential flow.

Definition 4.6. *For compactly supported measures $\mu, \nu \ll Leb$, let τ be a stopping time between μ and ν, that is $W_0 \sim \mu$ and $W_\tau \sim \nu$. We define μ_t to be the distribution of the stopping time $\tau \wedge t$, that is, $W_{\tau\wedge t} \sim \mu_t$. We then call $\{U_{\mu_t}\}_{t>0}$ as the* potential flow *associated with τ.*

Note that $\mu = \mu_0$ and $\nu = \mu_\infty$. From Corollary 4.4 we immediately have

Corollary 4.7. *(a) U_{μ_t} is monotonically nondecreasing in t;*
(b) $U_\mu \le U_{\mu_t} \le U_\nu$ for all t;

(c) U_{μ_t} pointwise converges to U_μ and U_ν respectively as $t \to 0^+$ and $t \to \infty$.

We now prove continuity of the potential flow in time.

Lemma 4.8 (Time continuity)**.** *Suppose $C := \|\mu_t\|_{L^\infty(\mathbb{R}^d \times [0,\infty)} < \infty$, and that the support of μ_t is bounded uniformly in t. Then U_{μ_t} is Lipschitz in time, more precisely, for each x,*

$$0 \leq U_{\mu_{t'}}(x) - U_{\mu_t}(x) \leq \frac{C}{2}[t' - t] \quad \text{for all } 0 \leq t < t'.$$

Proof. Recall from Lemma 4.3 that for each open set E,

$$\int_E \left(U_{\mu_{t'}} - U_{\mu_t}\right)(y)dy = \mathbb{E}\left[\int_{\tau \wedge t}^{\tau \wedge t'} \frac{1}{2}\chi_E(W_t)dt\right].$$

The first inequality then follows from the spatial continuity of U_{μ_t}. For the second inequality, apply Fubini in above equality to obtain

$$\int_E \left(U_{\mu_{t'}} - U_{\mu_t}\right)(y)dy = \frac{1}{2}\int_t^{t'} Prob[W_r \in E \;\&\; r < \tau]dr. \tag{4.2}$$

Notice that for each $r > 0$, by definition of μ_t,

$$Prob[W_r \in E \;\&\; r < \tau] \leq \mu_r[E].$$

Since $\mu_r \leq C$, this and (4.2) implies

$$\frac{1}{|E|}\int_E \left(U_{\mu_{t'}} - U_{\mu_t}\right)(y)dy \leq \frac{C}{2}(t' - t).$$

Now set $E = B_r(x)$ and let $r \to 0^+$, which completes the proof. □

From Lemma 4.1, Corollary 4.4 and Lemma 4.8, the following holds:

Corollary 4.9 (Space-time Lipschitzness of the potential)**.** *Assume that there exists a constant $C > 0$ such that $\mu_t \leq C$ for all $t \geq 0$, and that the support of μ_t is bounded uniformly in t. Then U_{μ_t} is uniformly $C^{1,\alpha}$ in space and Lipschitz in time in $\mathbb{R}^d \times [0,\infty)$. In particular, the convergence in Corollary 4.7 (c) is uniform.*

4.2. **Hitting times to monotone barriers and potential functions.** In this section we investigate the relation between the stopping time τ and the hitting time to a barrier set generated by the potential U_{μ_t}.

Definition 4.10. *For the potential flow $\{U_{\mu_t}\}_{t\geq 0}$ given in Definition 4.6, we define the time forward/backward stopping times*

$$\tau^{U,f} := \inf\{t \mid U_{\mu_t}(W_t) = U_\nu(W_t)\}, \quad \tau^{U,b} := \inf\{t > 0 \mid U_{\mu_t}(W_t) = U_\mu(W_t)\}.$$

We also define the corresponding barrier functions and the barrier sets as follows:

$$s^{U,f}(x) := \inf\{t \mid U_{\mu_t}(x) = U_\nu(x)\},\ R^{U,f} := \{(x,t) \mid t \geq s^{U,f}(x)\}$$

and

$$s^{U,b}(x) := \sup\{t \mid U_{\mu_t}(x) = U_\mu(x)\},\ R^{U,b} := \{(x,t) \mid t \leq s^{U,b}(x)\}.$$

Note that because of the time-monotonicity of U_{μ_t} as in Corollary 4.7, we have

$$\tau^{U,f} = \inf\{t \mid s^{U,f}(W_t) \leq t\}, \quad \tau^{U,b} = \inf\{t \mid s^{U,b}(W_t) \geq t > 0\}.$$

Notice that the condition $t > 0$ for $\tau^{U,b}$ is necessary: otherwise $\tau^{U,b} \equiv 0$.

The potential flow U_{μ_t} can be defined for any stopping time, even for randomized stopping time, so the hitting times $\tau^{U,f}, \tau^{U,b}$ can be associated to any (randomized) stopping time τ. On the other hand it is easy to see that τ and either of $\tau^{U,f}, \tau^{U,b}$ do not coincide in general. Even with a first hitting time τ, it will not be equal to $\tau^{U,f}, \tau^{U,b}$ unless its barrier is monotone in time. In case τ is indeed a hitting time to a time-monotone barrier, we verify that it is in fact equal to $\tau^{U,f}, \tau^{U,b}$, respectively, depending on its monotonicity type; this is proved in Theorem 4.14 below.

To see why such equivalence is useful, observe the following consequence of the continuity of potential flow.

Lemma 4.11. *Assume μ_t and its support is uniformly bounded for all $t \geq 0$. Then $s^{U,f}$ and $-s^{U,b}$ are lower semicontinuous. Thus, the sets $R^{U,f}$ and $R^{U,b}$ are closed in $\mathbb{R}^d \times [0, \infty)$.*

Proof. From Corollary 4.9. □

The closeness of the barrier sets is essential in Section 5 after we verify below that the optimal stopping time τ coincides with $\tau^{U,f}$ or $\tau^{U,b}$, depending on type (I) or (II) of τ. It will imply closedness of the barrier set for τ, which then allow us to apply maximum principle in the PDE formulation in Section 5.

Note that the equivalence between τ and $\tau^{U,f}$ or $\tau^{U,b}$, is hinted in the formula (4.1) which is a consequence of Ito's formula. One sees there that more time spent for the Brownian motion increases the potential function, which is precisely controlled but, only in integral/expected value sense, thus, it is not obvious how to use such a control on average quantities to get information for each Brownian path. We bypass this difficulty in two important steps. First, we show that the barrier set R of τ (for example, the set R^* for the optimal stopping time τ^* as in Section 2) contains (in a measure theoretic sense) the barrier set $R^{U,f}$ for costs of type (I) (respectively, $R^{U,b}$ for costs of type (II)). This is basically a result of the Ito's formula via Lemma 4.3.

Proposition 4.12. *Let μ, ν and τ be as given in Definition 4.6. Suppose τ is characterized as $\tau = \inf\{t \mid t \geq s(W_t)\}$ for costs of type (I), $\tau = \sup\{t \mid 0 < t \leq s(W_t)\}$ for costs of type (II), for a measurable function $s : \mathbb{R}^d \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$. Then we have*

$$\nu[\{x \mid s(x) < s^{U,f}(x)\}] = 0 \textit{ for } \text{(I)} \quad \textit{and } \nu[\{x \mid s(x) > s^{U,b}(x)\}] = 0 \textit{ for } \text{(II)}.$$

Proof. We will prove for costs of type (I), since the argument is parallel for the other case. Suppose not, that is, $\nu(\{s < s^{U,f}\}) > 0$. We can then find a constant $\bar{t} \in [0, \infty)$ such that

$$\nu[E] > 0 \text{ where } E := \{x \mid s(x) \leq \bar{t} < s^{U,f}(x)\}.$$

Since $\nu \ll Leb$, this also means $|E| > 0$. Let us define $w(x,t) := U_\nu(x) - U_{\mu_t}(x)$. From Lemma 4.4 and definition of E we have $w(\cdot, \bar{t}) > 0$ on E, yielding

$$\int_E w(y,\bar{t})dy > 0.$$

At the moment we only know that E is a measurable set; for example, we do not know continuity of s or $s^{U,f}, s^{U,b}$. But, from Lemma 4.3, there exists monotonically increasing sequence of compact sets $K_n \subseteq E$ such that

$$\begin{aligned} \int_E w(y,\bar{t})dy &= \lim_{n\to\infty} \mathbb{E}\left[\int_{\bar{t}\wedge\tau}^{\tau} \tfrac{1}{2}\chi_{K_n}(W_t)dt\right] \\ &= \lim_{n\to\infty} \tfrac{1}{2}\int_{\bar{t}}^{\infty} Prob[W_r \in K_n \;\&\; r < \tau]dr, \end{aligned}$$

where the second equality is from Fubini's theorem.

On the other hand, recall that $\tau = \inf\{t : t \geq s(x)\}$. Hence if $W_r \in E$ and $r \geq \bar{t}$, then $s(W_r) \leq \bar{t} \leq r$ and thus $\tau \leq r$. Therefore

$$Prob[W_r \in K_n \;\&\; r < \tau] \leq Prob[W_r \in E \;\&\; r < \tau] = 0 \text{ for each } n.$$

Back to the previous integrals and the limits, this then implies that

$$\int_E w(y,\bar{t})dy = 0,$$

a contradiction, completing the proof. □

Definition 4.13. *We let τ^U commonly denote $\tau^{U,f}$ and $\tau^{U,b}$ depending on whether μ_t is generated from type (I) or (II) stopping time τ. Likewise we use R^U as the corresponding barrier sets $R^{U,f}$ and $R^{U,b}$. and s^U corresponding to $s^{U,f}$ and $s^{U,b}$.*

Proposition 4.12 and (3.3) together imply that $\tau \geq \tau^U$. It then implies that we may choose the barrier R of τ in such a way that $R \subset R^U$. The opposite inclusion will show the equivalence between τ and τ^U. Unfortunately we fall short of proving this, due to unknown nature of regularity for the distribution of τ^U, in spite of the fact that most likely it is ν. In particular we are unable to use Lemma 2.3 due to the unknown regularity for the corresponding Eulerian variables. Still, it is possible to utilize the potential barrier to derive that the set $\{\eta > 0\}$ is open, which is important for the consistency results in Section 5. We employ the Eulerian formulation that connects the Brownian motion with stopping time with a parabolic flow.

Theorem 4.14. *Let μ, ν and τ be as given in Proposition 4.12. In type (II), assume that $\tau > 0$ almost surely. Then the eulerian variable (η, ρ) that corresponds to τ satisfies*

$$\{\eta > 0\} = (R^U)^C. \tag{4.3}$$

In addition, when $0 < a < \nu$ on its support for a constant a, we have

$$s = s^U \quad \nu\text{-a.e.} \tag{4.4}$$

Remark 4.15. *In type (II) the assumption $\tau > 0$ almost surely, is necessary. For example, if $\mu \wedge \nu \neq 0$ then the optimal stopping time τ randomizes at the initial time to stop the common mass $\mu \wedge \nu$ then proceed with the remaining mass $\mu - \mu \wedge \nu$ for positive time; in this case τ^U can still be > 0, making $\tau \neq \tau^U$.*

Proof of Theorem 4.14. From Proposition 2.2, the stopping time τ induces its Eulerian flow (η, ρ) which satisfies (2.2) with $\eta[R] = 0$ and $\rho[R] = 1$. Furthermore from [28] we have $\eta \in L^2(\mathbb{R}^+; H_0^1(\mathbb{R}^n))$.

Let us denote $w := U_\nu - U_{\mu_t}$ for (I), and $w := U_{\mu_t} - U_\mu$ for (II). For the rest of the proof we focus on type (I), as (II) follows a parallel proof.

Since $\Delta w = \nu - \mu_t$ at each $t \geq 0$, w satisfies

$$\Delta w(x) = \nu(x) - \eta(t,x) - \int_0^t \rho(ds, x) \text{ in } \mathbb{R}^d, \text{ for type (I).} \tag{4.5}$$

Now, for $g \in C(\mathbb{R}^+ \times \mathbb{R}^d)$ with $g(t, \cdot) \in C_c(\mathbb{R}^d)$ for each $t > 0$, consider φ solving $-\Delta\varphi(t, \cdot) = g(t, \cdot)$ for each $t > 0$, with decay at infinity. Using (2.2) for (η, ρ) and (4.5), we integrate by parts to obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^+} g(x,t)\frac{\partial}{\partial t}w(x)dtdx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^+} \varphi(-\partial_t\eta - \rho)dtdx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^+} \varphi(-\frac{1}{2}\Delta\eta)dtdx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^+} -\frac{1}{2}g(x,t)\eta(x,t)dtdx.\end{aligned}$$

Therefore we have

$$\partial_t w = -\tfrac{1}{2}\eta \text{ (in type (II), the sign is opposite.)} \tag{4.6}$$

Let us continue arguing with (I). Integrating (2.2) for (η, ρ) in time from zero to infinity, using the fact that η, a subsolution to heat equation with bounded initial data, vanishes as $t \to \infty$, yields that

$$\frac{1}{2}\Delta(\int_0^\infty \eta(x,s))ds = \nu - \mu = w(x,0). \tag{4.7}$$

From this and (4.6), it follows that $w(x,t) = \frac{1}{2}\int_t^\infty \eta(x,s)ds$. Since η is nonnegative, it follows that the set $\{w > 0\}$ includes $\{\eta > 0\}$, namely $\{\eta > 0\} \subset (R^U)^C$. On the other hand, we already had $\tau \geq \tau^U$, which follows from Proposition 4.12 and (3.3). This means that without loss of generality we can assume that $R \subset R^U$, or in other words, the barrier R of τ can be chosen to satisfy $R \subset R^U$. In particular this implies that $\rho((R^U)^C) = 0$, and thus η solves the heat equation in $(R^U)^C$. In particular it follows that η is positive in the connected open component of $(R^U)^C$ that contains $\{\mu > 0\} \times \{t = 0\}$. Since $(R^U)^C$ decreases in time and is open, so is any of its connected component. It follows that the set $\{\eta(\cdot, t) > 0\}$ decreases in time, and thus from (4.7) we conclude that $\{w > 0\} = \{\eta > 0\} = (R^U)^C$ and thus (4.3).

Next we proceed to show (4.4). We will only show the case for type (I), since parallel arguments hold for (II). Note that by (4.3) we have $R^U = \{w = 0\} = \{\eta = 0\}$, and $(R^U)^c = \{w > 0\}$. Hence, combining (4.6) and (4.5) we have

$$w_t - \frac{1}{2}\Delta w = -\frac{1}{2}\nu\chi_{\{w>0\}}, \quad w(\cdot,0) = U_\nu - U_\mu. \tag{4.8}$$

If $0 < a < \nu < b$ for some constants a, b in its support, then by standard arguments for parabolic obstacle problem (see Lemma 5.2 and section 5.3 of [11]), $\partial\{w > 0\} = \partial\{\eta > 0\}$ has zero space-time Lebesgue measure. We will use this fact in the next paragraph.

Since $R \subset R^U$, we have $s^U \leq s$. Due to Proposition 3.9 ρ can only be supported in $\{t = s(x)\}$. On the other hand $\rho = 0$ where $\{\eta > 0\}$ as shown above and from the (η, ρ) equation $\rho = 0$ in any open set where $\eta = 0$. Therefore, it follows that $\rho = 0$ outside of $\partial\{\eta > 0\}$, that is, ρ is concentrated in the set $\partial\{\eta > 0\}$. Since $\int \rho(x,s)ds = \nu(x)$, Proposition 3.9 implies that

$$\{t = s(x)\} \subset \partial\{\eta > 0\} \text{ for } \nu\text{-a.e. } x.$$

Since $\{\eta > 0\} = (R^U)^C = \{t < s^U(x)\}$, it follows that both points $(s^U(x), x), (s(x), x)$ are in $\partial\{\eta > 0\}$. Since the set $\{\eta > 0\}$ decreases in time, it follows that the whole segment $[s^U(x), s(x)] \times \{x\}$ are in $\partial\{\eta > 0\}$. Hence, if $\nu(s^U < s) > 0$ then it contradicts the fact that the space-time measure of $\partial\{\eta > 0\}$ is zero. Now we can conclude (4.4).

In the case of (II), $w_t = -\frac{1}{2}\eta$ with (4.8) changes to $w_t = \frac{1}{2}\eta$ with

$$w_t - \frac{1}{2}\Delta w = -\frac{1}{2}\nu\chi_{\{w>0\}} + \frac{1}{2}\mu, \quad w(\cdot,0) = 0. \tag{4.9}$$

Here we use the fact that $\tau > 0$. The rest of the argument is parallel to that of (I). □

Theorem 4.14 states that if τ is the hitting time to a monotone-in-time barrier set, then the barrier can be given by R^U where the Brownian motion reaches. Note that the barrier sets cannot be unique, as wherever the Brownian motion does not reach, one can modify the barrier set without changing the hitting time τ from μ and the final distribution ν. We however can characterize the set R^U is the largest barrier set for τ, as any barrier set of τ should be contained in R^U by Proposition 4.12. Because of this we regard R^U as the canonical barrier associated to τ, μ and ν. From now on we mean R^U whenever we say a monotone barrier R for such hitting times τ as in Proposition 4.12.

We can in fact show that R^U (so R in our convention) is not dependent on τ but only on μ and ν as long as the stopping time is given by the hitting time to a monotone barrier; this is not straightforward because R^U is determined by the potential flow U_{μ_t} which then is determined by the distribution $\mu_t \sim W_{\tau\wedge t}$. In particular this yields the uniqueness of the hitting time τ (hitting a forward or backward barrier set) for given μ and ν. When R is finely closed, such uniqueness of the barrier is known (see e.g. [10, Remarks 2.3 and 6.19]). We remove this restrictive assumption in the proof below, where we make use of the parabolic obstacle problem that the w-variable solves.

Lemma 4.16. *Let μ, ν and τ be as given in Proposition 4.12 with ν uniformly positive in its support. In type (II), assume that $\tau > 0$ almost surely. Then for given μ and ν, the barrier set R (more precisely, the barrier function s of R) is unique ν-a.e.*

Proof. Let (η, ρ) and w be as given in the proof of Theorem 4.14.

Note that the above parabolic obstacle problem (4.8) and (4.9) each has a unique solution due to comparison principle. Indeed, suppose w_1 and w_2 solve the same parabolic obstacle problem with the same initial data. Then $(w_1 - w_2)_+$ is a subsolution of the heat equation with zero initial data, and thus it is zero. Hence w and $\{w > 0\} = (R^U)^c$ is determined only by μ and ν. Hence we conclude by (4.4). □

Definition 4.17. *For μ, ν as given in Lemma 4.16, $(R, s(x))$ is each* the barrier set *and* the barrier function *associated to (μ, ν) for the cost type (I) or the cost type (II) if $R = R^U$ is the barrier set uniquely given (ν-a.e.) in Lemma 4.16, in the form of $R := \{(x,t) : t \geq s(x)\}$ for cost (I), or $R := \{(x,t) : t \leq s(x)\}$ for cost (II).*

For the given barrier R, the corresponding Eulerian flow is uniquely determined, due to Lemma 2.3. This justifies the following:

Definition 4.18. *We say the pair (η, ρ) is the Eulerian variables associated with (μ, ν) for the cost type (I) (or (II)) if it is the unique pair (η, ρ) solving* (2.2) *in the weak sense with the property $\eta \in L^2([0,\infty); H_0^1(B_R))$ with $\eta(R) = 0$ and $\rho(R) = 1$ for the ν-a.e. unique barrier $R = R^U$ determined by μ, ν.*

4.3. **Remarks on the potential approach and the parabolic obstacle problem.** We have verified in the proof of Lemma 4.16 that the function $w(x,t) := U_\nu - U_{\mu_t}$ ($w := U_{\mu_t} - U_\mu$ for (II)) solves the obstacle problem (4.8)-(4.9). They are in the form of the parabolic obstacle problem that has been actively studied in the literature. In fact, this connection between the parabolic obstacle problem and the Stefan problem has been used in [21] to introduce a notion of weak solution solely based on (4.8) for a regularized version of (St_1). However, even ignoring the regularization, such notion of weak solutions has its limitations due to lack of sufficient regularity of w-variable to track the problem back to (St_1). We refer to [11] and [35] for available results on regularity and singularity for solutions of (4.8). We point out that the low regularity of w is due to the fact that w_t is non-positive in our setting. When w_t is nonnegative, which corresponds to the costs of type (II) and thus (St_2), much stronger regularity results holds for the parabolic obstacle problem: see e.g. [24].

5. Consistency with the Stefan problem

Let (η, ρ) be the Eulerian variables associated with (μ, ν), given in Definition 4.18. In this section we will show that η solves the Stefan problems, with initial distribution μ and weight ν, and vice versa. This connection has been indicated in [28] with formal analysis.

Let us define weak solutions of the (weighted) Stefan problems $(St_1)_\nu$ and $(St_2)_\nu$ with initial density η_0 and initial domain E:

$$(\eta \pm \nu\chi_{\{\eta>0\}})_t - \frac{1}{2}\Delta\eta = 0, \qquad \eta(\cdot,0) = \eta_0 \in L^1(\mathbb{R}^n), \quad E := \limsup_{t\to 0^+}\{\eta(\cdot,t) > 0\}. \tag{5.1}$$

Definition 5.1. *A nonnegative function $\eta \in L^1(\mathbb{R}^n \times [0,\infty))$ is a weak solution of $(St_1)_\nu$* (or $(St_2)_\nu$) *with initial data (η_0, E) if*

(a) the set $\{\eta(\cdot,t)\} > 0$ decreases (or increases) in t;

(b) $E = \limsup_{t\to 0^+}\{\eta(\cdot,t)>0\}$;

(c) for any test function $\varphi \in C_c^\infty(\mathbb{R}^n\times[0,\infty))$,

$$\int_0^\infty \int_{\mathbb{R}^d}[(\eta - (\text{ or }+)\nu\chi_{\{\eta>0\}})\varphi_t + \frac{1}{2}\eta\Delta\varphi]dxdt = \int_{\mathbb{R}^d}[(\eta_0 + (\text{ or }-)\nu\chi_E)\varphi](x,0)dx \tag{5.2}$$

We say that η is a weak solutions of (St_i) if it is for $(St_i)_\nu$ with $\nu=1$, for $i=1,2$.

Remark 5.2. *One can check that η is a weak solution of $(St_1)_{\nu_i}$ if $\nu_1=\nu_2$ in the set $\{x : (x,t)\in\partial\{\eta>0\}$ for some $t>0\}$. To see this, note that*

$$\int\int \nu\chi_{\{\eta>0\}}\varphi_t\, dxdt = \int \nu(x)\int_0^{s(x)}\varphi_t dtdx = \int \nu(x)(\varphi(x,s(x)) - \chi_E\varphi(x,0))dtdx,$$

where $s(x) := \sup\{t : \eta(x,t)>0\}$. So the only value of ν that matters is at $(x,s(x))$, a.e. x.

Remark 5.3. *Note that the weak solution for $(St_1)_\nu$ or $(St_2)_\nu$ requires specifying not only the initial data η_0 but the initial domain E to be solved as an initial-value problem. With this information we can find a unique solution of (St_2) by comparison principle, for instance see* [2]*. On the other hand, even with specified η_0 and E, (St_1) can exhibit a high degree of non-uniqueness, as we will see in Section 6.1.*

Theorem 5.4. *Let $\mu,\nu\in L^1(\mathbb{R}^d)$, and assume that $\nu>a>0$ for some constant a in its support. Let (η,ρ) be the associated Eulerian flow given in Definition 4.18 for either (I) or (II). If the optimal stopping time associated with (μ,ν) is strictly positive, the following holds:*
- *(a) For (I), η is a weak solution $(St_1)_\nu$ with initial data (μ,E), where E is a set containing the support of ν.*
- *(b) For (II), η is a weak solution $(St_2)_\nu$ with initial data (μ,E) where E is the support of μ.*

Proof. Let R be the *barrier set* associated with (μ,ν) for either the cost of type (I) and (II) given in Definition 4.17. Note that the following holds from Proposition 3.9 and Theorem 4.14:
- (i) ρ is supported on $\{t=s(x)\}$;
- (ii) $\{\eta>0\}=\{t<s(x)\}$ (ν,dt)-a.e.

For the next set of computations we focus on (I). Let $\varphi\in C^\infty(\mathbb{R}^n\times[0,\infty))$ with compact support in space. Recall that (η,ρ) then satisfies (2.2), namely

$$\int_0^\infty\int \eta(\varphi_t+\frac{1}{2}\Delta\varphi)dxdt = \int_0^\infty\int\rho\varphi dxdt + \int\mu\varphi(x,0)dx.$$

We would like to see that η satisfies the weak equation for $(St_1)_\nu$. To this end observe that

$$\int_0^\infty\int\rho\varphi dxdt = \int(\int_0^\infty\rho(x,t)\varphi(x,t)dt)dx = \int\int\nu(x)\varphi(x,s(x))dx,$$

where the second equality comes from (i) and the fact that $\int_0^\infty\rho(x,t)dt=\nu(x)$ a.e. x.

Next, from (ii) we have $\{\eta > 0\} = \{t < s(x)\}$ (ν, dt)-a.e. Thus by Fubini's theorem

$$\begin{aligned}\int \nu(x)\varphi(x,s(x))dx &= \int \nu(x)\int_0^{s(x)} \varphi_t(x,t)dtdx + \int \nu(x)\varphi(x,0)dx \\ &= \int\int_{\{\eta>0\}} \nu(x)\varphi_t(x,t)dtdx + \int \nu(x)\varphi(x,0)dx.\end{aligned}$$

Hence (2.2) can be written as

$$\int_0^\infty \int \eta(\varphi_t + \frac{1}{2}\Delta\varphi)dxdt = \int \nu(x)\chi_{\{\eta>0\}}\varphi_t(x,t)dx - \int [\nu(x) - \mu(x)]\varphi(x,0)dx.$$

Hence η is a weak solution $\eta \in L^1(\mathbb{R}^d \times [0,\infty))$ of $(St_1)_\nu$, with initial data $\eta_0 = \mu$ and the initial trace of the set $\{\eta > 0\}$ equal to a set E containing the support of ν.

For (II), since $\{\eta > 0\} = \{s(x) < t\}$ (ν, dt)-a.e., again by Fubini's theorem

$$\int \nu(x)\varphi(x,s(x))dx = -\int \nu(x)\int_{s(x)}^\infty \varphi_t(x,t)dtdx = -\int\int_{\{\eta>0\}} \nu(x)\varphi_t(x,t)dtdx, \tag{5.3}$$

Since $\tau > 0$, we have $\nu = 0$ in $\{\eta(\cdot, 0+) > 0\}$. Hence (2.2) for case (II) can then be written as

$$\int_0^\infty \int \eta(\varphi_t + \frac{1}{2}\Delta\varphi)dxdt = -\int_{\{\eta>0\}} \nu(x)\varphi_t(x,t)dxdt - \int \mu(x)\varphi(x,0)dx,$$

which is the weak expression for $(St_2)_\nu$ with initial data μ and the initial support the same as μ.

□

Remark 5.5. *In the case $\tau = 0$, the theorem will hold with the revised initial data $\mu - \mu_0$ and revised target measure $\nu - \mu_0$, where μ_0 is the portion of μ with $\tau = 0$, namely*

$$\mu_0(x) := \text{Prob}[\tau = 0 \,|\, W_0 = x]\, \mu(x).$$

Next we consider the reverse direction.

Theorem 5.6. *Let $\mu, \nu \geq 0, L^\infty(\mathbb{R}^n)$ with compact support. Suppose that there exists a weak solution $\eta \in L^1(\mathbb{R}^n \times (0,\infty))$ of $(St_1)_\nu$ with initial data (μ, E), where E is bounded. Let us define s and ρ by*

$$s(x) := \sup\{t : \eta(x,t) > 0\} \text{ and}$$

$$\int\int \rho(x)\varphi(x,t)\, dxdt = \int \nu(x)\varphi(x,s(x))dx \tag{5.4}$$

for any test function $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$. Then $\mu \leq_{SH} \tilde{\nu} := \nu\chi_{\{s(x)<\infty\}}$. Moreover (η, ρ) is the Eulerian variables between $\eta_0 = \mu$ and $\tilde{\nu}$, generated by the optimal stopping time with costs of type (I).

Proof. By definition of $s(x)$, $\{\eta > 0\} = \{t < s(x)\}$. This allows us to write, for instance in the case of (St_1),

$$\int \nu(x)\varphi(s(x),x)dx - \int \nu(x)\varphi(0,x)dx = \int \nu(x)\int_0^{s(x)} \varphi_t(t,x)dtdx = \int\int_{\{\eta>0\}} \nu(x)\varphi_t(t,x)dtdx.$$

Arguing as in the proof of Theorem 5.4, we can then verify that (η,ρ) satisfies the equation (2.2). Moreover, we have $\eta(R) = 0$ for the barrier set $R := \{(x,t) : t \geq s(x)\}$ by the definition of $s(x)$. Observe also that

$$\rho(R) = \int \rho(x,t)dxdt = \int \eta(x,0)dx = \mu(\mathbb{R}^n) = 1,$$

where the first equality holds since ρ is supported in R, and the second is due to the mass preserving property of the heat equation.

Let us define $\xi := \int_t^\infty \eta(x,t)dt$. Integrating in time the weak equation for (St_1), we see that ξ solves the parabolic obstacle problem

$$\xi_t - \frac{1}{2}\Delta\xi = -\frac{1}{2}\nu\chi_{\{\xi>0\}}.$$

On the other hand, since we have $\eta_t - \Delta\eta = -\rho$ in the distribution sense, and since η vanishes as time tends to infinity, it follows that

$$\mu = \eta(\cdot,0) = \int_0^\infty(-\eta_t) = \int_0^\infty(-\Delta\eta + \rho) = -\Delta(\int_0^\infty \eta) + \tilde{\nu}.$$

Hence $\mu - \tilde{\nu} = -\Delta\xi(\cdot,0)$. Thus, by uniqueness of the parabolic obstacle problem, $\xi(x,t) = w(x,t) = \int_t^\infty \tilde{\eta}(x,s)ds$, where $\tilde{\eta}$ is the Eulerian variable generated by the optimal stopping time between μ and $\tilde{\nu}$. In particular $\{\tilde{\eta} > 0\} = \{\eta > 0\}$.

To conclude, note that $\eta \in L^2([0,\infty), H_0^1(\mathbb{R}^n))$, due to the fact that η is a subsolution of the heat equation with compactly supported initial data. Hence taking their zero set as R (where clearly both ρ and ρ_0 are supported), we can apply Lemma 2.3 to conclude that $\eta = \tilde{\eta}$.

□

We finish this section with the corresponding statement for $(St_2)_\nu$. The proof is parallel to $(St_1)_\nu$.

Theorem 5.7. *Let $\mu, \nu \geq 0, L^\infty(\mathbb{R}^n)$ with compact support. Suppose that there exists a weak solution $\eta \in L^1(\mathbb{R}^n \times (0,\infty))$ of $(St_2)_\nu$ and initial data (μ, E), where E is bounded. Let us define s and ρ by*

$$s(x) := \inf\{t : \eta(x,t) > 0\}$$

and (5.4)*. Then (η,ρ) is the Eulerian variables between μ and ν generated by the optimal stopping time with costs of type (II), where*

$$\mu = \eta_0 \text{ and } \tilde{\nu} = \nu\chi_{E^C}.$$

In the next section, we will discuss a specific class of the target measures, from which solutions to the classical Stefan problems (St_1) and (St_2) are generated.

6. Subharmonically generated sets

This section defines a central notion of the present paper which characterizes a pair of sets by existence of a certain stopping time. This notion then is connected to solvability of the supercooled Stefan problem (St_1) the one without weight ν.

Definition 6.1. *We say the pair* (Σ, E) *is* subharmonically generated *by* μ*, if there is a stopping time* τ *with the corresponding Eulerian flow* (η, ρ) (2.2)*, such that* $\tau > 0$ *almost surely,* $W_0 \sim \mu$ *and* $W_\tau \sim \nu = \chi_\Sigma$ *and* $E = \{x \mid \eta(x,t) > 0$ *for some* $t > 0\}$.

This definition is motivated by the following equivalence result.

Theorem 6.2. *Consider a compactly supporeted measure* $\mu \ll Leb$ *on* $\mathbb{R}^d$*. Then for a given set* E*, the following are equivalent.*

- *(a) There exists a weak solution* η *of* (St_1) *with initial data* (μ, E)
- *(b) There exists a measurable set* Σ *such that* (Σ, E) *is subharmonically generated by* μ*.*

Moreover, $\Sigma = \{z(x) < \infty\}$ *with* $z(x) := \sup\{t : \eta(x,t) > 0\}$*.*

Proof. If (a) holds, then (b) follows from Theorem 5.6, with $\Sigma := \{s(x) < \infty\}$ with $s(x) := \sup\{t : \eta(x,t) > 0\}$. Let us remark that, since the set $\{\eta(\cdot,t) > 0\}$ decreases as t increases, we have

$$\limsup_{t\to 0^+}\{\eta(\cdot,t) > 0\} = \cup_{t>0}\{\eta(\cdot,t) > 0\} = E.$$

Similarly, if (b) holds, then (a) follows from Theorem 5.4. □

We mention that obtaining ν as a characteristic function χ_Σ does not guarantee a corresponding solution for (St_1) unless the stopping time is strictly positive. Even with the stopping time strictly positive, for a given μ and E there can be many Σ such that (Σ, E) is subharmonically generated by μ. Below we will discuss an example that illustrates these points. In the next section we will introduce an optimization problem for the target measure ν, that uniquely generates a subharmonically generated pair (E, E) for each given μ, with E explicitly given in terms of μ. More discussion on the Stefan problem is to follow in Section 9, in conjunction with this optimal target problem. We will see therein that our optimal target scheme provides a mechanism to construct solutions of (St_1) in a stable, and physically reasonable manner.

6.1. Example for nonuniquenss for (St_1).

The weak solution of (St_1) is known to have non-uniqueness with given initial data. Here we give an example that yields infinitely many weak solutions to (St_1) with the same initial data (μ, E): these are solutions that does not vanish in finite time.

Proposition 6.3. *Let* $\mu = \frac{1}{|B_\epsilon|}\chi_{B_\epsilon}$ *and* $E = \{|x| \le 1\}$*. Then, there are infinitely many weak solutions to* (St_1) *with initial data* (μ, E)*.*

Proof. Consider the set $\Sigma := A_1 \cup A_2 \subset \mathbb{R}^d$ consisting of the union of two annuli,

$$A_1 := \{2\epsilon \le |x| \le r\}, \quad A_2 := \{r' \le |x| \le 1\}, \qquad \text{where } 0 < 2\epsilon < r < r' < 1.$$

Assume $|\Sigma| = 1$ by choosing appropriate ϵ, r, and r'. Let $\nu = \chi_\Sigma$. We will prove

Claim: there exists a randomized stopping time τ, with $W_0 \sim \mu$ and $W_\tau \sim \nu$.

After verifying this claim, we can find the optimal stopping time τ^* for the optimal Skorokhhod problem $\mathcal{P}(\mu, \nu)$ (1.5). Then, the corresponding Eulearian flow (η, ρ) as given in Propositoin 2.2 gives a solution to $(St_1)_\nu$ via Theorem 5.4, with the initial set

$$E = \limsup_{t \to 0^+} \{\eta(\cdot, t) > 0\} = \{|x| \leq 1\}.$$

This is because, to reach the outer annulus A_2, the Eulerian variable η should be active in the whole set $\{|x| \leq 1\}$. From Remark 5.2, the solution we constructed here for $(St_1)_\nu$, for $\nu = \chi_\Sigma$, actually solves the original supercooled Stefan (St_1). Hence, by choosing different combinations of r, r', but still making $|\Sigma| = 1$, we can generate infinitely many solutions to (St_1) with the above (μ, E). In this case, $\{s(x) = \infty\} = \{|x| \leq 2\epsilon\} \cup \{r \leq |x| \leq r'\}$, the outside part of the two annuli in $E = \{|x| \leq 1\}$.

It remains to prove the claim. We first show that there exists a randomized stopping time τ' with $W_0 \sim \delta_{\{x=0\}}$, such that $W_{\tau'} \sim \nu$. To construct such stoping time, let τ_r be the first hitting time to $S(r) := \{|x| = r\}$. Then its distribution ν_r is a uniform $d-1$ dimensional measure along the set $S(r)$, that is, $\nu_r = C_r \delta_{S(r)}$ with some constant $C_r > 0$. Thus we can randomize at $t = 0$ to find a stopping time τ_A with $W_0 \sim \delta_{\{x=0\}}$ whose distribution ν_r is given by $\nu_A = \int_{r_1}^{r_2} \nu_r f(r) dr$ for some weight $f(r)$ that can be controlled by the randomization at $t = 0$. In particular, we can find such a randomization at $t = 0$ so that ν_A becomes a uniform measure on the annulus $A = \{r_1 \leq |x| \leq r_2\}$. Now, for such τ_{A_1} and τ_{A_2} we can consider the randomized stopping time τ' given as

$$\tau' = \begin{cases} \tau_{A_1} & \text{with probability } \frac{|A_1|}{|A_1|+|A_2|}, \\ \tau_{A_2} & \text{with probability } \frac{|A_2|}{|A_1|+|A_2|}. \end{cases}$$

Then it follows that $W_{\tau'} \sim \nu$. Moreover, by the Markov property of the Brownian motion, we can change the randomization at $x = 0$ to a randomization at $S(\epsilon)$, that is, when the Browinian motion from the origin first hits $S(\epsilon)$ we make a probabilistic choice how to move from there.

One can find a randomized stopping time τ'' as in the above, with $W_0 \sim \delta_{\{x=0\}}$ and $W_{\tau''} \sim \frac{1}{|B_\epsilon|} \chi_{B_\epsilon}$ for the ball B_ϵ. Note that $\tau' > \tau_\epsilon$ almost surely. Clearly $\tau'' \leq \tau_\epsilon$, so $\tau'' \leq \tau'$. This means that for the measure $\mu = \frac{1}{|B_\epsilon|} \chi_{B_\epsilon}$ there exists a randomized stopping time τ with $W_0 \sim \mu$ and $W_\tau \sim \nu$, verifying the claim. □

7. Optimal target problem

From this section on, we consider the optimal target problem (1.1) proposed in our introduction, namely solving for the optimal target $\bar{\nu}$ minimizing the cost function $\mathcal{C}(\tau)$, under the upper density constraint $\nu \leq f$, for a given bounded and measurable function f. One can view this as a projection problem in the space of probability measures. Given $\mu \in P(\mathbb{R}^n)$, what is the closest measure ν in the constraint set $\{\nu \leq f\}$, under the condition that we have the subharmonic order $\mu \leq_{SH} \nu$. Here, the closeness between the two measures is measured by the cost $C(\tau)$ for an optimal stopping time τ between them. Once we have the optimal target ν then we can apply Theorem 2.1, obtaining the optimal stopping time τ that is given by the hitting time to a barrier given by a barrier function s.

Let us point out that it is not easy to explicitly construct the optimal target even in simple cases. For instance when $f \equiv 1$ and $\mu = \delta_{x=0}$, one may guess that the optimal stopping time τ is given by the constant time, when the heat kernel $K(t,y)$ becomes $K \leq 1$. This is not the case as we see below, for the either cost type (I) or (II).

For the rest of the paper will see that the optimal target comes with many interesting characteristics, based on the two main features: monotonicity and saturation property. These two properties allow us to connect the optimal target problem (1.1) with global solvability of the Stefan problem; see Section 9.

We begin our discussion with the monotonicity property.

7.1. **Monotonicity.** The following theorem demonstrates an order between optimal targets of Problem (1.1) and the corresponding optimal stopping times.

Theorem 7.1 (monotonicity)**.** *Consider* $(\mu_i, \nu_i, s_i, \tau_i)$*,* $i = 1, 2$ *such that* τ_i *are stopping times given by the barrier function* s_i*, with initial and target distribution* μ_i *and* ν_i*. Further assume that* $\nu_i \leq f$ *and* $\mu_1 \leq \mu_2$*. Then the following holds:*

(1) If ν_1 *is a solution to* (1.1) *for* μ_1*, while* s_1, τ_1 *are the corresponding uniquely determined (uniquely* ν_1*-a.e. sense) barrier and the corresponding optimal stopping time, then, as the stopping times from the initial distribution* μ_1*, we have* $\tau_1 \leq \tau_2$ *almost surely. (Notice that from* $\mu_1 \leq \mu_2$*, the stopping time* τ_2 *from* μ_2 *can be restricted to the initial distribution* μ_1*.) Here, we do not necessarily assume that* ν_2 *is a solution to* (1.1)*.*

(2) If both ν_i *is a solution to* (1.1) *for* μ_i*, while* s_i, τ_i *are the corresponding uniquely determined (uniquely* ν_i*-a.e. sense) barrier and the corresponding optimal stopping time, respectively, then* $\nu_1 \leq \nu_2$*. In particular there is at most one solution to* (1.1)*.*

Remark 7.2. *We point out that for (1) we only need* τ_1 *to be optimal, not* τ_2*. This in particular characterizes* τ_1 *as the smallest stopping time that starts from* μ_1 *among eligible target distributions with the constraint* f*.*

Proof. To show (1), we first prove that $\tau_1 \leq \tau_2$, when τ_2 is restricted to the initial distribution μ_1. For this it only needs optimality of ν_1.

(I) case. In this case τ_i, $i = 1, 2$, are the first hitting time to the set

$$R_i := \{(x,t) \mid t \geq s_i(x)\}; \quad \tau_i = \inf\{t \mid (W_t, t) \in R_i\}.$$

Let $\bar\tau := \tau_1 \wedge \tau_2$. Recall that from Proposition 3.9 we have $(W_{\tau_i}, \tau_i) \in R_i$ almost surely, $i = 1, 2$. Therefore $\bar\tau \geq \min[s_1(W_{\bar\tau}), s_2(W_{\bar\tau})]$ almost surely. Now, for a Brownian path, if $W_{\bar\tau} \in \{x \mid s_1(x) \leq s_2(x)\}$ then $\bar\tau \geq \tau_1$ so $\bar\tau = \tau_1$. Similarly, if $W_{\bar\tau} \in \{x \mid s_2(x) \leq s_1(x)\}$ then $\bar\tau = \tau_2$.

Therefore, $\bar\nu$ denoting the distribution of $\bar\tau$, we have

$$\bar\nu|_{\{s_1\le s_2\}} \le \nu_1|_{\{s_1\le s_2\}} \le f \text{ and } \bar\nu|_{\{s_2\le s_1\}} \le \nu_2|_{\{s_2\le s_1\}} \le f;$$

so $\bar\nu \le f$ and $\bar\tau$ is admissible.

Now recall that τ_1 is the minimizer for $\mathcal{C}$, which implies $\bar\tau = \tau_1$ almost surely, as otherwise if $Prob[\bar\tau < \tau_1] > 0$ then from the strict monotonicity of $\mathcal{C}$, we see $\mathcal{C}(\bar\tau) < \mathcal{C}(\tau_1)$, a contradiction. This immediately implies that $\tau_1 \le \tau_2$, where we understand τ_2 as the stopping time restricted to the initial distribution μ_1.

(II) case. In this case τ_i, $i = 1, 2$, are randomized at the initial time to give the distribution $f \wedge \mu_i$, $i = 1, 2$, and the rest Brownian particles stops when they first hit to the set $R_i := \{(x, t) \mid t \le s_i(x)\}$, which gives the corresponding final distribution $\nu_i - f \wedge \mu_i$ (from those Brownian paths with $\tau_i > 0$), that satisfies $\le f - f \wedge \mu_i$, $i = 1, 2$. Notice that $f - f \wedge \mu_2 \le f - f \wedge \mu_1$ due to $\mu_1 \le \mu_2$.

From now on until the end of the proof of this case, we restrict τ_2 to the initial distribution μ_1 and call it still τ_2. Define a randomized stopping time $\bar\tau$ as follows: For the portion $\mu_1 \wedge f$ stop immediately, and for the rest $\mu_1 - \mu_1 \wedge f$, follow the Brownian motion until $\tau_1 \wedge \tau_2$. Notice that $\bar\tau \le \tau_1$ from its construction. The rest of proof is similar to (I) case with additional consideration for stopping at the initial time. Details follow.

We show that $\bar\tau$ is admissible, namely, if we let $\bar\nu$ be its distribution (with the initial distribution μ_1), $W_{\bar\tau} \sim \bar\nu$, then $\bar\nu \le f$. To see this, first write $\bar\nu = \mu_1 \wedge f + \bar\nu^1$ where $\bar\nu^1$ is the distribution of the rest. For Brownian particles with positive time $\bar\tau > 0$, that is, those accounting for $\bar\nu^1$, if $W_{\bar\tau} \in \{x \mid s_1(x) \ge s_2(x)\}$ then $\bar\tau \ge \tau_1$ so $\bar\tau = \tau_1$. Similarly, if $W_{\bar\tau} \in \{x \mid s_2(x) \ge s_1(x)\}$ then $\bar\tau \ge \tau_2$ so $\bar\tau = \tau_2$. Therefore,

$$\bar\nu^1|_{\{s_1\ge s_2\}} \le (\nu_1 - f \wedge \mu_1)|_{\{s_1\ge s_2\}} \le (f - f \wedge \mu_1)|_{\{s_2\ge s_1\}}$$

as well as

$$\bar\nu^1|_{\{s_2\ge s_1\}} \le (\nu_2 - f \wedge \mu_2)|_{\{s_2\ge s_1\}} \le (f - f \wedge \mu_2)|_{\{s_2\ge s_1\}} \le (f - f \wedge \mu_1)|_{\{s_2\ge s_1\}}.$$

From these, we have

$$\bar\nu = \mu_1 \wedge f + \bar\nu^1 \le \mu_1 \wedge f + (f - f \wedge \mu_1) = f, \quad \text{verifying admissibility of } \bar\tau.$$

Now recall that τ_1 is the minimizer for $\mathcal{C}$, which implies $\bar\tau = \tau_1$ almost surely, as otherwise if $Prob[\bar\tau < \tau] > 0$ then from the strict monotonicity of $\mathcal{C}$ and from $\bar\tau \le \tau_1$, we see $C(\bar\tau) < C(\tau_1)$ for admissible $\bar\tau$, a contradiction. This immediately implies that $\tau_1 \le \tau_2$ as desired.

We now show (2): this requires optimality of both ν_1 and ν_2.

Suppose not, i.e. $\int_{\{\nu_1 > \nu_2\}} f(x)dx > 0$. For a small $\epsilon > 0$, define

$$E_\epsilon := \{x \mid \nu_1(x) > \nu_2(x) + 2\epsilon\nu_1(x)\}.$$

and $\mu_2^\epsilon := \mu_2 - \epsilon\mu_1 \geq 0$. Define an auxiliary randomized stopping time $\tilde\tau$, from the initial distribution μ_2 as follows:

$$\tilde\tau := \begin{cases} \tau_2 & \text{from } \mu_2^\epsilon \ , \\ \tau_1 & \text{if } W_{\tau_1} \in E_\epsilon \text{ from } \epsilon\mu_1, \\ \tau_2 & \text{if } W_{\tau_1} \notin E_\epsilon \text{ from } \epsilon\mu_1 \ . \end{cases}$$

Let us explain the meaning of this. Here the wording 'from μ_2^ϵ' or 'from $\epsilon\mu_1$' should be understood as that at the initial time the Brownian particles belong to the distribution μ_2^ϵ or $\epsilon\mu_1$, respectively. Notice that such a decomposition in the initial distribution is allowed for the randomized stopping time as it is like randomizing at the initial time. Notice that we already proved that $\tau_1 \leq \tau_2$ for Brownian motion starting from μ_1, so the second and third lines in the definition of $\tilde\tau$ is well defined.
Now let us show that $\tilde\tau$ is admissible. Define the distribution $\tilde\nu$ of $\tilde\tau$, that is, $W_{\tilde\tau} \sim \tilde\nu$. From the definition of $\tilde\tau$ we see that

$$\tilde\nu|_{E_\epsilon^c} = \text{ the distribution of } W_{\tau_2} \text{ from } \mu_2^\epsilon \text{ and from } \epsilon\mu_1, \text{ restricted to the set } E_\epsilon^c.$$

Therefore $\tilde\nu|_{E_\epsilon^c} \leq \nu_2$.
On the other hand,

$$\begin{aligned} \tilde\nu|_{E_\epsilon} = & \text{ the distribution of } W_{\tau_1} \text{ from } \epsilon\mu_1 \\ & \quad + \text{ the distribution of } W_{\tau_2} \text{ from } \mu_2 = \mu_2^\epsilon + \epsilon\mu_1, \text{ restricted to the set } E_\epsilon \\ & \leq \epsilon\nu_1 + \nu_2 \text{ on } E_\epsilon \\ & \leq \nu_1 \text{ on } E_\epsilon \text{ by the definition of } E_\epsilon \\ & \leq f \end{aligned}$$

We thus have verified that $\tilde\nu \leq f$, therefore, $\tilde\tau$ is admissible as the stopping time from the initial distribution μ_2.
Now notice that by the construction $\tilde\tau \leq \tau_2$, where the latter is optimal . From the strict monotonicity of the cost C, we see that $\tilde\tau = \tau_2$.

To see why this leads to a contradiction, notice that it implies that for those Brownian particles starting from $\epsilon\mu$ with $W_{\tau_1} \in E_\epsilon$ we have $\tau_1 = \tilde\tau = \tau_2$. Since $\epsilon\nu_1$ is the final distribution for τ_1 starting from $\epsilon\mu_1$ we have

$$\begin{aligned} \epsilon\nu_1|_{E_\epsilon} \leq & \text{ the distribution for } \tau_2 \text{ starting from } \epsilon\mu_1, \text{ restricted to } E_\epsilon. \\ \leq & \text{ the distribution for } \tau_2 \text{ starting from } \epsilon\mu_2, \text{ restricted to } E_\epsilon \text{ (because } \mu_1 \leq \mu_2) \\ = & \ \epsilon\nu_2|_{E_\epsilon}. \end{aligned}$$

Note that this is a contradiction to the definition of E_ϵ, thus proves the claim that $\nu_1 \leq \nu_2$. □

Remark 7.3. *Note that $\tau_1 \leq \tau_2$ should be understood to hold for those Brownian paths starting from the initial points x distributed as the common mass $\mu_1 = \mu_1 \wedge \mu_2$ for $\mu_1 \leq \mu_2$. Because of Markov property of the Brownian motion, the stopping time τ_1 can be applied to the initial mass of μ_2 in the region where $\mu_1 > 0$. From the region $\mu_1 = 0$, it should be understood that the stopping time $\tau_1 = 0$ because, from where $\mu_1 = 0$ there is no motion for τ_1. In this sense,* *the inequality* $\tau_1 \le \tau_2$ *can be applied to all Brownian paths from the initial distribution* μ_2 *not only from* μ_1.

The rest of the section discusses important consequences of the monotonicity property.

7.2. **Existence.** Here we show the well-posedness of the optimization problem (1.1).

Theorem 7.4. *Let* μ *be bounded and compactly supported, and suppose the set*

$$\mathcal{A} := \{\nu : \mu \le_{SH} \nu, \quad \nu \text{ is compactly supported and } \nu \le f\}$$

is nonempty for $f \in L^\infty(\mathbb{R}^n)$. *Then, there exists unique optimal target measure* ν *for the problem* (1.1).

Proof. Once existence is established, uniqueness is a direct consequence of Theorem 7.1.

First let us consider the case where f is compactly supported. In this case, basically the relevant domain for the measures $\nu \in \mathcal{A}$ and Brownian motion between μ and ν is compact (in particular, a subset of the convex hull of $\operatorname{supp} f$), therefore existence of a minimizer ν follows easily from the weak-* compactness of the minimizing sequence, since that the condition $\nu \le f$ is closed with the weak-* convergence.

Now, for the general case, let $f_R = f \wedge 1_{B_R}$, then for $R \gg 1$, the corresponding admissibility set $\mathcal{A}_\mathcal{R}$ with f_R is nonempty. Therefore, for the initial measure μ, and for each $1 \ll R < R'$, applying the compactly supported case, we find $\nu_R, \nu_{R'}$ be the optimal solutions with the constraint f_R and $f_{R'}$, respectively. Let $\tau_R, \tau_{R'}$ be the corresponding optimal stopping times. Notice that ν_R also satisfies the density constraint $f_{R'}$ ($\ge f_R$). We can then apply the monotonicity, Theorem 7.1(1), and get $\tau_{R'} \le \tau_R$. It implies that the support of $\nu_{R'}$ is contained in the convex hull of the support of ν_R. Since in the support of ν_R we have $f_R = f_{R'}$, so, $\nu_{R'}$ should be an optimal solution for the constraint f_R, because for the cost we have $\mathcal{C}(\tau_{R'}) \le \mathcal{C}(\tau_R)$. From uniqueness of optmal solution in the compactly supported constraint case, we have $\nu_R = \nu_{R'}$. This implies that the optimal ν_R is independent of R as long as R is sufficiently large. That ν_R, $R \gg 1$, is the optimal target for f. To see this notice that for any compactly supported target measure $\nu' \le f$, there exists $R > 0$ such that $\operatorname{supp} \nu' \subset B_R$, so $\nu' \le f_R$ and the cost for ν_R is less than or equal to that of ν'. □

Remark 7.5. *We note that those* f *with* $\mathcal{A} \neq \emptyset$ *are plenty. For example,* $f \equiv 1$ *on* $\mathbb{R}^d$, *or any* $f \ge 0$ *which has a positive lower bound on a ball* B_R, $R \gg 1$, *will work. Or any* $f \ge 0$ *that has a positive lower bound in the annulus* $\{R_1 \le |x| \le R_2\}$, *with* $1 \ll R_2 < R_2 \le \infty$ *works. Of course, here how large* R, R_1, R_2 *need to be depends on* μ. *These are some particular types of* f *we focus on in this paper, especially in Section 9.*

7.3. **Universality of the optimal target.** A remarkable consequence of this monotonicity is that for a given initial measure μ and upperbound constraint f, the optimal target ν is unique regardless of the cost function $\mathcal{C}$ as long as $\mathcal{C}$ satisfies either (I) or (II). We exploit the fact that Theorem 7.1(1) requires optimality only for one target. This result will have interesting consequences later in the context of Stefan problem (see Section 9).

Theorem 7.6 (Universality). *For a given μ, assume that ν_i, $i = 1, 2$, is an optimal solution to* (1.1) *with the same upperbound constraint f, but, for costs $\mathcal{C}_i$, $i = 1, 2$ where $\mathcal{C}_i$ satisfies (I) or (II). Then $\nu_1 = \nu_2$.*

Proof. Notice tat $\nu_1, \nu_2 \leq f$. Let τ_i, $i = 1, 2$ are the corresponding optimal stopping times, for $\mathcal{C}_i$, $i = 1, 2$, respectively.

For the cost $\mathcal{C}_1$ consider the optimal stopping time τ_2' for the given target ν_2. Then, from Theorem 7.1(1) and optimality of τ_1, we have $\tau_1 \leq \tau_2'$. This implies that $\nu_1 \leq_{SH} \nu_2$.

Similarly, for the cost $\mathcal{C}_2$, consider the optimal stopping time τ_1' for the given target ν_1. Then, from Theorem 7.1(1) and optimality of τ_2, we have $\tau_2 \leq \tau_1'$. This implies that $\nu_2 \leq_{SH} \nu_1$.

The two subharmonic orders imply $\nu_2 = \nu_1$, completing the proof. □

Remark 7.7. *Above universality is rather surprising since, by putting different weights on the location x for L, we may expect different optimal target distribution ν. Our result says this is not the case, namely that the cost function $\mathcal{C}(\tau)$ depends only on the time spent by the Brownian paths.*

7.4. **L^1 contraction and BV estimate.** Next we show the L^1 contraction, which is a consequence of the monotonicity and the fact that the total mass of μ is the same as that of ν.

Theorem 7.8 (L^1-Contraction). *Assume either (I) or (II). Let (μ_1, ν_1), (μ_2, ν_2) be the pairs of the initial distribution and optimal solution of the problem* (1.1)*. Then,*

$$\|(\nu_1 - \nu_2)_+\|_{L^1} \leq \|(\mu_1 - \mu_2)_+\|_{L^1}.$$

Proof. Let $\tilde{\mu} = \mu_1 \wedge \mu_2$; notice that $(\mu_1 - \mu_2)_+ = \mu_1 - \tilde{\mu}$. Let $\tilde{\nu}$ be an optimal solution of the corresponding problem (1.1) with the initial distribution $\tilde{\mu}$. As $\tilde{\mu} \leq \mu_i$, $i = 1, 2$, we have from the monotonicity (Theorem 7.1) that

$$\tilde{\nu} \leq \nu_i, \; i = 1, 2.$$

Now, let $E_+ = \{x \mid \nu_1(x) - \nu_2(x) \geq 0\}$. Then,

$$(\nu_1 - \nu_2)_+ = (\nu_1 - \nu_2)\chi_{E_+} \leq (\nu_1 - \tilde{\nu})\chi_{E_+} \leq \nu_1 - \tilde{\nu}.$$

Therefore,

$$\|(\nu_1 - \nu_2)_+\|_{L^1} \leq \|\nu_1 - \tilde{\nu}\|_{L^1} = \|\mu_1 - \tilde{\mu}\|_{L^1} = \|(\mu_1 - \mu_2)_+\|_{L^1}$$

as desired. Notice that the first equality follows from the fact that the problem preserves the total mass, i.e. $\|\mu_1\|_{L^1} = \|\nu_1\|_{L^1}$, $\|\tilde{\mu}\|_{L^1} = \|\tilde{\nu}\|_{L^1}$. □

Remark 7.9. *The monotonicity and L^1 contraction seem to hold if we add the constraint $\tau \leq t$ in the problem* (1.1)*, namely, for a given t,*

$$\operatorname*{argmin}_{\nu \in P(\mathbb{R}^n) \ \& \ \nu \leq f} \{C(\tau) \mid \tau \leq t, \quad W_0 \sim \mu \ \& \ W_\tau \sim \nu\} \tag{7.1}$$

This will be useful for us when we try to use the Markov property of the problem, and do iteration of the L^1 contraction and consequent BV estimate for η.

We prove the BV estimate as an immediate corollary of the L^1 contraction theorem. For (II), when $f = 1$ and when $\mu = (1+\mu_0)\chi_E$ with $\mu_0 > 0$, this was shown by Meirmanov [37].

Theorem 7.10 (BV estimate)**.** *Let ν be the solution of* (1.1) *with a constant f. Then,*

$$\|\nu\|_{BV} \leq \|\mu\|_{BV}.$$

Proof. Let us first consider the case $\mu \in C^1$. Let us define $\nu^\delta := \nu * \eta_\delta$, where η is a standard nonnegative compactly supported mollifier with total mass one. Since ν is L^1 and ν is compactly supported, ν^δ uniformly converges to ν in L^1. From lower semi-continuity of BV norm under L^1 convergence it is enough to prove that

$$\|\nu^\delta\|_{BV} \leq \|\mu\|_{BV} \text{ for small } \delta > 0.$$

Observe that for each $y \in B_1$, $\epsilon > 0$, we have

$$\int |\nu^\delta(x+\epsilon y) - \nu^\delta(x)|dx \leq \int |\nu(w+\epsilon y) - \nu(w)|dw.$$

Notice that $f \equiv 1$ is translation invariant, therefore from the uniqueness result, $\nu(\cdot + \epsilon y)$ is the optimal solution of (1.1) for the initial measure $\mu(\cdot + \epsilon y)$. We apply then the L^1 contraction (Theorem 7.8) and get for any $\epsilon > 0$,

$$\int \frac{|\nu^\delta(x+\epsilon y) - \nu^\delta(x)|}{\epsilon}dx \leq \int \frac{|\mu(x+\epsilon y) - \mu(x)|}{\epsilon}dx.$$

Now for a fixed δ, ν^δ and μ are C^1 with compact supports, and so by sending $\epsilon \to 0$ in above expression we obtain

$$\int |D\nu^\delta(x)\cdot y|dx \leq \int |D\mu(x)\cdot y|dx.$$

We then integrate both sides in $y \in B_1$,

$$\int_{B_1}\int |D\nu^\delta(x)\cdot y|dxdy \leq \int_{B_1}\int |D\mu(x)\cdot y|dy$$

and apply the Fubini's theorem,

$$\int\int_{B_1} |D\nu^\delta(x)\cdot y|dydx \leq \int\int_{B_1} |D\mu(x)\cdot y|dydx$$

Observe that for any unit vector e_1, we have

$$\int_{B_1} |D\nu^\delta(x)\cdot y|dy = |D\nu^\delta(x)|\int_{B_1} |e_1\cdot y|dy$$

and similarly for μ, which leads to

$$\int |D\nu^\delta(x)|\left(\int_{B_1} |e_1\cdot y|dy\right)dx \leq \int |D\mu(x)|\left(\int_{B_1} |e_1\cdot y|dy\right)dx$$

Therefore, we get the inequality

$$\int |D\nu^\delta(x)|dx \leq \int |D\mu(x)|dx.$$

Hence we showed that $\|\nu^\delta\|_{BV} \leq \|\mu\|_{BV}$ as desired.

Let us now consider the general μ in BV. We consider a sequence of measures $\mu_k \in C^1$ that converges as $k \to \infty$, to μ in BV (also in L^1); in particular $\|\mu_k\|_{BV} \to \|\mu\|_{BV}$. Consider the corresponding ν_k, namely, the solution of (1.1) with initial μ_k. . Apply Step 1, and get

$$\|\nu_k\|_{BV} \leq \|\mu_k\|_{BV}.$$

Now notice that by the L^1 contraction ν_k converges, as $k \to \infty$, to ν in L^1, therefore, from the lowe-semicontinuity of the BV-norm in L^1 convergence, we get

$$\|\nu\|_{BV} \leq \|\mu\|_{BV}.$$

This completes the proof. □

Remark 7.11. *The BV estimate is a rather well-known consequence of L^1 contraction, when the problem is homogeneous with respect to spatial shift.*

8. Saturation property of the optimal target

Here we prove that the optimal target measure, when obtained with positive stopping time, saturates up to the upper density limit f. We first show a converse to Corollary 3.7 for optimal solutions to (1.1), in the following sense.

Lemma 8.1. *Let μ be a probabillity measure, absolutely continuous with Lebesgue measure. Let ν be the corresponding optimal solution of the problem* (1.1) *for (I) or (II) with bounded and measurable f . Let τ be the optimal stopping time with $W_0 \sim \mu$ and $W_\tau \sim \nu$. Let G be a measurable set such that $f|_G > 0$. Suppose that there exists a constant $t_1 > 0$ such that*

$$Prob[W_{t_1} \in G \ \&\ t_1 < \tau] > 0.$$

Then, $\nu[G] > 0$.

Proof. Suppose for contradiction, that $\nu|_G = 0$. We define a randomized stopping time $\bar\tau$ (starting from the distribution μ) as follows. For those Brownian trajectories with $\tau < t_1$, stop at $\bar t$. For those Brownian trajectories with $\tau > t_1$, if $W_{t_1} \notin G$ just proceed until τ, but, if $W_{t_1} \in G$ then drop a portion of mass at the time t_1, such that the resulting mass is positive but has density $\leq f$ over the set G (note that here we are using the fact that $\nu|_G = 0$), then proceed until τ; note that this is possible because $Prob[W_{t_1} \in G \ \&\ t_1 < \tau] > 0$ as well as $f|_G > 0$ for $f \ll Leb$. To be more precise, notice that the density at x of the distribution of W_{t_1} for those Brownian particles with $t_1 < \tau$, is bounded from above by $K_{t_1}(x)$ of the solution to the heat equation with initial data μ, thus at each $x \in G$ when $W_{t_1} = x \in G$, one can stop the mass with the probability

$$\frac{\min[f(x), K_{\bar t}(x)]}{K_{\bar t}(x)} > 0,$$

and we get the positive stopped density $\leq f(x)$. Therefore this randomized stopping time $\bar\tau$ has distribution $W_{\bar\tau} \sim \bar\nu$ with $\bar\nu \leq f$. Moreover, notice that $\bar\tau \leq \tau$ for each random path almost surely, and $Prob[\bar\tau < \tau] > 0$. Then, the cost $\mathcal{C}(\bar\tau) < \mathcal{C}(\tau)$ from the strictness of the cost functional $\mathcal{C}$. This contradicts optimality of τ, thus completing the proof. □

As an immediate corollary of this lemma and Lemma 3.5, we see that for the optimally stopped Brownian motion for our problem (1.1), any set G with zero mass of the final distribution blocks off the Brownian motion.

Corollary 8.2. *Let μ, ν and τ be as given in Lemma 8.1. Let G is a measurable set with positive measure, and suppose that every point of G has positive Lebesgue density. Assume that $f|_G > 0$ and $\nu[G] = 0$. Then, $\tau \le \tau_G$ almost surely.*

Proof. This is a direct consequence of Lemma 8.1 and Lemma 3.5. □

We can combine ideas used in the proof of Lemma 8.1 and the result of Corollary 3.7 to prove that the optimal solution ν saturates the density upper bound.

Theorem 8.3 (saturation)**.** *Assume $\mu \ll Leb$ and ν be the optimal solution of the problem (1.1) with cost of type (I) or (II) with the optimal stopping time τ. Suppose $\tau > 0$ almost surely. Then the optimal solution ν to Problem (1.1) is given in the form*

$$\nu = f\chi_E \quad \text{for some set } E.$$

More generally, without assuming $\tau > 0$, we have the following results:

(a) For costs of type (I), we have

$$\nu = f\chi_E + \mu|_F \quad \text{for some set } E \text{ and } F \text{ with } |E \cap F| = 0.$$

where in E the Brownian motion does not stop immediately, i.e. $s(x) > 0$ for a.e. $x \in E$, and in F the Brownian paths stop immediately, i.e. $s(x) = 0$, for a.e. $x \in F$.

(b) In the (II) case, the optimal target measure ν is given in the following form.

$$\nu = \tilde{\nu} + f \wedge \mu$$

where $\tilde{\nu}$ the optimal solution from the initial measure $\tilde{\mu} = \mu - f \wedge \mu$, while the upper bound constraint is given by $\tilde{f} = f - f \wedge \mu$; here $\tilde{\nu}$ is given in the form

$$\tilde{\nu} = \tilde{f}\chi_E \quad \text{for some set } E$$

and the corresponding optimal stopping time $\tilde{\tau}$ satisfies $\tilde{\tau} > 0$ almost surely.
Moreover $(f \wedge \mu)(x) = Prob[W_0 = x, \tau = 0]\mu(x)$.

Before proceeding to the proof of above theorem, we state the following characterization of the instantly stopped portion for each cost types using the potential flow.

Corollary 8.4. *For cost (I) and E and F as given in Theorem 8.3 we have*

$$E = \{w > 0\} \text{ and } F = \{w = 0\},$$

where $w \ge 0$ is continuous solution of the obstacle problem

$$-\Delta w + (f - \mu)\chi_{\{w>0\}} = 0. \tag{8.1}$$

Proof. Set $w = U_\nu - U_\mu$. Then $E = \{w > 0\}$ and $F = \{w = 0\}$. To see this, for example apply Lemma 4.3 and continuity of U_ν and U_μ. Therefore, we see that

$$\Delta w = \nu - \mu = f\chi_{\{w>0\}} + \mu\chi_{\{w=0\}} - \mu = (f - \mu)\chi_{\{w>0\}}.$$

This completes the proof. □

One can variationally characterize the problem w solves. Noting that $\{w > 0\}$ contains Ω_0, we can write (8.1) as

$$(f - \mu)\chi_{\{w>0\}} = \Delta w.$$

If we let h solve $\Delta h = f - \mu$, then w minimizes

$$\int |D(\varphi - h)|^2 dx$$

among the functions $\varphi \geq 0$. Thus w can be viewed as $\dot{H}^1(\mathbb{R}^d)$ projection of h onto the space of nonnegative functions.

Remark 8.5. *In type (II), notice that the set $\{x \mid \nu(x) < f(x)\}$ belongs to the set where the Brownian motion under the optimal τ stops immediately.*

Proof of Theorem 8.3: We proceed in several steps.

1. *When $\tau > 0$ almost surely.* For arbitrary $1 > \delta > 0$, let

$$H_\delta := \{x \mid \delta \leq \frac{d\nu(x)}{dx} < f(x) - \delta\}.$$

Our goal is to prove $\nu[H_\delta] = 0$. Suppose not for contradiction, that is, $\nu[H_\delta] > 0$. Then, Corollary 3.7 implies either

- (i) $\mu \wedge (\nu|_{H_\delta}) \neq 0$, or
- (ii) there exists a constant $\bar{t} > 0$ such that

$$Prob[W_{\bar{t}} \in H_\delta \ \&\ \bar{t} < \tau] > 0.$$

For (i), we define a randomized stopping time $\tilde{\tau}$ (starting from the distribution μ) as follows. First, $\tilde{\tau}$ is randomized at the initial time for for those Brownian particles from $\mu \wedge (\nu|_{H_\delta})$, in such a way that the distribution of initially stopped particles from $\mu \wedge (\nu|_{H_\delta})$ has positive mass with density δ. For the remaining particles it follows τ. Then the resulting terminal distribution by $\tilde{\tau}$, say, $\tilde{\nu}$ is $\leq f$ by the definition of the set H_δ. Moreover, from the construction of $\tilde{\tau}$ and strict monotonicity of $\mathcal{C}$ we have $\mathcal{C}(\tilde{\tau}) < \mathcal{C}(\tau)$, which contradicts optimality of τ. Therefore, the case (i) should not happen.

For (ii), we define $\tilde{\tau}$ in such a way that for those Brownian trajectories with $\tau < \bar{t}$, stop at τ. For those Brownian trajectories with $\tau > \bar{t}$, if $W_{\bar{t}} \notin H_\delta$ just proceed until τ, but, if $W_{\bar{t}} \in H_\delta$ drop a portion of mass at time $\bar{t}$, such that the resulting mass is positive but has density $\leq \delta$ over the set H_δ, then proceed until τ; note that this is possible because the density at x of the distribution of $W_{\bar{t}}$ is bounded from above by $K_{\bar{t}}(x)$ of the solution to the heat equation with initial data μ, thus at each $x \in H_\delta$ when $W_{\bar{t}} = x \in H_\delta$, one can stop the mass with the probability

$$\frac{\min[\delta, K_{\bar{t}}(x)]}{K_{\bar{t}}(x)} > 0,$$

and we get the positive stopped density with the total mass $\leq \delta$. Therefore this randomized stopping time $\tilde{\tau}$ has distribution $\tilde{\nu}$ with $\tilde{\nu} \leq f$. Moreover, notice that $\tilde{\tau} \leq \tau$ for each random path almost surely, and $Prob[\tilde{\tau} < \tau] > 0$. Then, the cost $\mathcal{C}(\tilde{\tau}) < \mathcal{C}(\tau)$ from the strictness of the cost functional $\mathcal{C}$. This contradicts optimality of τ, thus completing the proof for the case '$\tau > 0$ almost surely'. We verified that $\nu = f\chi_E$ for some set E for (II).

2. (I) case. Recall that in this case $\tau = \inf\{t \mid t \geq s(W_t)\}$. Notice that the set $F := \{x \mid s(x) = 0\}$ gives a barrier set for W_t , $t \leq \tau$. That is, any Brownian path starting from a point in F with the stopping time τ stops immediately, equivalently, $\tau \leq \tau_F$. As a consequence the resulting final distribution with τ starting from $\mu|_F$ is $\mu|_F$. On the other hand, denote $\bar{\mu} := \mu|_{F^c}$ and apply the stopping time τ for those Brownian particles starting from $\bar{\mu}$. We let $\bar{\tau}$ denote such a restriction of τ. Notice that $\bar{\tau} > 0$ almost surely and it is the optimal stopping time for $\bar{\mu}$ and $\bar{f} = f - \mu|_F$. Let $\bar{\nu}$ be the resulting final distribution from the stopping time $\bar{\tau}$. Then from Case 1, we have that

$$\bar{\nu} = \bar{f}\chi_E \quad \text{for some set } E \text{ where } \bar{f}|_E > 0.$$

Moreover, notice that $\bar{\mu} \wedge (\bar{\nu}|_F) = 0$ from the definition $\bar{\mu} = \mu|_{F^c}$. Therefore, from $\tau \leq \tau_F$(since F is part of the barrier) and Lemma 3.2 we get $\bar{\nu}[F] = 0$. This also implies that $|E \cap F| = 0$, therefore, $\bar{f}\chi_E = f\chi_E$. In summary, we have

$$\nu = \bar{\nu} + \mu|_F = f\chi_E + \mu|_F \quad \text{as desired.}$$

This completes the proof for (I).

3. (II) case. In this case the stopping time τ, which is the optimal stopping time between μ and ν, is randomized at $t = 0$ to give $f \wedge \mu$ and proceed with positive time starting from the rest $\tilde{\mu} = \mu - f \wedge \nu$, which gives the additional final distribution $\tilde{\nu} = \nu - f \wedge \nu$. Let $\tilde{\tau}$ denote the stoping time τ restricted to the initial distribution $\bar{\mu}$. Then its final distribution, say, $\tilde{\nu}$ gives an optimal solution to (1.1) with the upper bound constraint $\tilde{f} = f - f \wedge \mu$. Notice that $\tilde{\tau} > 0$ almost surely. Thus from Case 1, we have $\tilde{\nu} = \tilde{f}\chi_E$ for a set E. Notice that $\nu = \tilde{\nu} + f \wedge \mu$. All things combined we verified the result in Case 3.

This completes the proof for (II). □

8.1. **Upper bound for the optimal stopping time in the (I) case.** One of the applications of the saturation result is the following:

Theorem 8.6. *Let μ is a probability measure with compact support and $\mu \ll Leb$, and let f be a bounded measurable function with $f \geq \delta$ for a constant $\delta > 0$. Then the following holds for costs of type (I):*

Let τ be the optimal stopping time for Problem 1.1. Then there exists a constant $\bar{T} = \bar{T}(\delta, \|\mu\|_{L^1})$ such that

$$\tau \leq \bar{T} \text{ almost surely.}$$

Remark 8.7. *This result yields in particular that the support of ν "closes off", namely that it includes the support of μ. Also, interestingly the extinction time T does not depend on the size of the support of μ.*

Proof. Recall that τ has a barrier function s such that $\tau = \inf\{t |\ t \geq s(W_t)\}$. Consider the optimal target ν and the set E and F from Theorem 8.3 (a) (so in the (I) case) which satisfies that

$$\text{`}\tau > 0\text{' implies `}W_\tau \in E\text{' almost surely, and}$$
$$\text{`}\tau = 0\text{' implies `}W_0 \in F\text{' almost surely,}$$

and that for any measurable set S,

$$Prob[W_\tau \in S \ \&\ \tau > 0 \mid W_0 \sim \mu] = \nu[S \cap E] = \int_{S\cap E} df \ \geq \delta |S \cap E|$$

where the last inequality is from the assumption that $f \geq \delta$. Since $\nu \ll Leb$, we may assume, without loss of generality, that E^c consists of its Lebesgue points, by adding the non-Lebesgue density portion of E^c, which has zero Lebesgue measure, to E. Then, from Corollary 8.2, we see that

$$\text{if } \tau > 0, \text{ then almost surely } \tau \leq \tau_{E^c}. \tag{8.2}$$

Now consider for each $T > 0$, the set

$$Z_T := \{x \mid s(x) > T\}.$$

Notice that $\nu[Z_T] = \nu[Z_T \cap E]$. It suffices to show that there exists $\bar{T}$ such that $\nu[Z_{\bar{T}}] = 0$.

First, recall that from Lemma 3.9 we have $\tau = s(\tau)$ almost surely. Therefore, almost surely, $W_\tau \in Z_T$ if and only if $\tau > T$. Also note that $\tau > T$ implies $s(W_T) > T$ so $W_T \in Z_T$. Also, (8.2) implies $W_T \in E$ when $\tau > T$. Therefore, almost surely,

$$W_\tau \in Z_T \text{ implies } W_T \in Z_T \cap E \ \&\ \tau > T.$$

In particular, we have

$$Prob[W_\tau \in Z_T \mid W_0 \sim \mu] \leq Prob[W_T \in Z_T \cap E \ \&\ \tau > T \mid W_0 \sim \mu].$$

On the other hand, let ρ_T be the distribution of the heat flow in $\mathbb{R}^d$ at time T with the initial condition $\rho_0 = \mu$. Notice that $\rho_T \leq C_T$ for some constant C_T, depending only on $\|\mu\|_{L^1}$ and T, decaying exponentially to zero as $T \to \infty$. Consider

$$\begin{aligned}
C_T|Z_T \cap E| &\geq \int_{Z_T\cap E} d\rho_T = Prob[W_T \in Z_T \cap E \mid W_0 \sim \mu] \\
&\geq Prob[W_T \in Z_T \cap E \ \&\ \tau > T \mid W_0 \sim \mu\ ] \\
&\geq Prob[W_\tau \in Z_T \mid W_0 \sim \mu\ ] \\
&= \nu[Z_T] = \nu[Z_T \cap E] \\
&\geq \delta |Z_T \cap E|.
\end{aligned}$$

By examining the limit $T \to \infty$ we see that there exists $\bar{T} = \bar{T}(\delta, \|\mu\|_{L^1}) > 0$ such that $|Z_{\bar{T}} \cap E| = 0$. Thus we conclude that $\bar{\nu}[Z_{\bar{T}}] = 0$, completing the proof. □

Notice that a similar result does not hold in the (II) case as easily seen in the following example.

Example 8.8. *Consider the case under the (II) assumption where B_1, B_2 are the balls of radius* 1*,* 2*, respectively, centered at the origin in .*

$$\mu = 2^d \chi_{B_1}, \quad f = 1.$$

Then, we get $\bar{\nu} = 1\chi_{B_2}$ and the corresponding barrier function s is radially symmetric and can be given by a function of the form

$$s(x) = \begin{cases} 0 & \text{for } |x| \leq 1, \\ g(|x|) & \text{for } 1 \leq |x| < 2, \\ +\infty & \text{for } |x| \geq 2. \end{cases}$$

where $g : [1, 2) \to \mathbb{R}$ is an increasing function with $g(1) = 0$ and $g(t) \to \infty$ as $t \to 2$. Here the optimal stopping time τ which is the hitting time to such a barrier has not upper bound.

9. Global-time existence of supercooled Stefan problem

Translating the saturation result, Theorem 8.3, into a PDE formulation via the consistency result, Theorem 5.4, we can derive global-time existence of the Stefan problem for both supercooled fluid (St_1) (Theorems 9.3 and 9.6) and melting ice (St_2) (Theorem 9.2). Our emphasis will be on the supercooled case which has not been well understood in the literature. We point out that, due to Theorem 8.3, $\nu = 1$ in the active region of the brownian particles. Thus ν satisfies the assumptions of Theorem 4.14 (b). For (St_1) we find the unique solution that vanishes in finite time (Theorem 9.3), for a certain class of initial data (μ, E), and the unique solution that vanishes outside of its support (Theorem 9.6). Remarkably, our choice of the initial domain E is necessary to have such solutions, and it is generated as the accumulated active region of the melting (St_2) problem (Theorem 9.10). This connects the two problems that have very different dynamics from each other.

Let us first translate positivity of the target in Lemma 8.1 in terms of the Eulerian flow η.

Lemma 9.1. *Let ν be the optimal target measure generated by* (1.1) *with μ for type (I) or (II) costs. Let η be the Eulerian variable associated with (μ, ν). Then we have*

$$\nu > 0 \text{ on the set } \{x : \eta(x,t) > 0 \text{ for some } t > 0\} \cap \{f > 0\}.$$

Proof. We can apply Lemma 8.1 to the set $G := \{\eta(x, t_1) > 0\}$ for each $t_1 > 0$. □

9.1. **Stable Stefan problem.** Let us first briefly discuss global well-posedness for the stable Stefan problem (St_2).

Due to the saturation theorem, Theorem 8.3, it follows that $\nu(x) = 1$ in the active region $\{\eta > 0\}$ if $f = 1$ and μ is larger than one in its support, thus from the consistency theorem, Theorem 5.4, we have the following:

Theorem 9.2 (Global-time existence for (St_2))**.** *Let $\mu = (1+\eta_0)\chi_\Sigma \in L^\infty(\mathbb{R}^d)$ with $\eta_0 > 0$ and a bounded Borel set Σ with positive measure. Then for costs of type (II), the corresponding η-variable for μ and the optimal target ν with $f \equiv 1$ is the unique weak solution of the stable Stefan problem (St_2),*

$$(\eta + \chi_{\{\eta>0\}})_t - \frac{1}{2}\Delta\eta = 0 \tag{St_2}$$

with initial data $(\eta_0\chi_\Sigma, \Sigma)$.

In this case the initial set for the active region coincides with the support Σ of the initial distribution μ. This is due to the nature for the type (II) costs whose barrier sets in the space-time for the optimal stopping are time-backward monotone, in contrast to the type (I) costs that gives time-forward barrier sets. For the supercooled (St_1) case corresponding type (I), not every initial set Σ has a global-time solution. This will be apparent in the theorems below for the supercooled case.

9.2. **Supercooled Stefan Problem.** From the same reasoning as for Theorem 9.2 applied to the type (I) case, we immediately get the following for the supercooled Stefan problem:

Theorem 9.3 (A global-time existence for (St_1))**.** *Suppose $\mu = (1+\eta_0)\chi_\Sigma \in L^\infty(\mathbb{R}^d)$ with $\eta_0 > 0$ and a bounded Borel set Σ with positive measure. Let $f \equiv 1$ and let $\nu = \chi_E$ be the corresponding optimal target for costs of type (I). Then the Eulerian variable η for (μ, ν) solves the supercooled Stefan problem*

$$(\eta - \chi_{\{\eta>0\}})_t - \frac{1}{2}\Delta\eta = 0 \tag{St_1}$$

with initial data (μ, E). Moreover we have $E = \{w > 0\}$, where $w \geq 0$ solves the obstacle problem

$$\chi_{\{w>0\}} - \Delta w = \mu. \tag{9.1}$$

In particular E contains Σ. Lastly, the set $\{\eta > 0\}$ vanishes in finite time.

Remark 9.4. *In the next subsection we will show that η in above theorem is indeed the unique solution with initial data μ that vanishes in finite time.*

From the point of view of Theorem 6.2), we have this result as (E, E) is subharmonically generated via Theorem 8.3 for $f = 1$. Note that (9.1) follows from (8.1) since we know that in this setting E includes the support of μ. In general case (9.1) no longer holds due to the instantly frozen part of μ in the region $\mu < f$. The finite time extinction of η is due to Theorem 8.6, and the characterization of E via the obstacle problem is due to Corollary 8.4.

Remark 9.5 (Relation between (St_1) and (St_2))**.** *Note that due to universality (Theorem 7.6) we have the same optimal target measure ν for type (I) and (II), for given μ and f. Therefore the solution of stable Stefan problem (St_2) in Theorem 9.2 determines the initial set E that gives global-time solution for (St_1). This aspect can also be understood as the obstacle problem (9.1) can be solved by the solution of (St_2) by taking $t \to \infty$.*

While our optimal target problem connects to the classical Stefan problem (St_1) with a restricted initial data of the form (μ, E), one can generate the solution of (St_1) for a wider class of initial data by making use of Theorem 8.3 and Theorem 5.4. For instance, to ensure that there is no instantly frozen set, one can choose $f = \chi_K$ where K and the support of μ are disjoint. Then the optimal stopping time τ must be $\tau > 0$ and, due to Theorem 8.3, we obtain a solution of (St_1) whose positive set instantly expands out to a part of K and then shrink over time, with its interface away from the support of μ. More precisely we have the following result which, like Theorem 9.3, is derived immediatly from Lemma 9.1, Theorem 8.3, and Theorem 5.4 as well as Corollary 8.4.

Theorem 9.6 (A global-time existence for (St_1) for $f \wedge \mu = 0$)**.** *Let μ be compactly supported, nonnegative function in $L^\infty(\mathbb{R}^d)$. Let us set $f = \chi_{K^c}$, where K is a compact set that includes the support of μ, and let $\nu := f\chi_E = \chi_{E\cap K^c}$ be the corresponding optimal target for costs of type (I). Then the Eulerian variable η for (μ,ν) solves (St_1) with initial data (μ, E). Here $E = \{w > 0\}$, where $w \geq 0$ solves the obstacle problem*

$$\Delta w = (f - \mu)\chi_{\{w>0\}} = \chi_{\{w>0\}\cap K^c} - \mu.$$

From the point of view of Theorem 6.2, $(E \cap K^c, E)$ is subharmonically generated via Theorem 8.3 for $f = \chi_{K^c}$.

As in the previous theorem, here again, ν can be determined by solving the type (II) problem with the same μ and f, or equivalently solving the corresponding (St_2) problem. In this case η does not vanish in finite time. It is because the Brownian motion will travel forever in the region Σ, where no stopping happens there due to the assumption $f|_\Sigma = 0$.

Remark 9.7. *Uniqueness is in general not true for (St_1) with initial data μ and E (see the example in Section 6). Thus the optimal target problem with f yields a cost (and energy) - oriented criteria for a unique solution of (St_1) with initial distribution μ.*

∘ **A discussion on the initial expansion**

An interesting feature of our solutions of (St_1) is the instantly expanded set E. It is well-understood that the local-in-time solutions to (St_1) develops jump discontinuity when the solution is "overloaded", namely when the average density goes beyond 1 (see for instance [14]). When $\mu > 1$ in its support, as given in Theorem 9.3, μ is entirely overloaded and thus it is natural that one must expand its support to activate the supercooling process. On the other hand when μ has small density as allowed in Theorem 9.6, such expansion follows from our choice of f, which describes the particular scenario where the freezing stimulus is only available outside of the initial supercooled region. Our approach by optimization, as well as imposing the "energy level" f, may provide an interesting interpretation to supercooling phenomena, where little is understood.

Our characterization of the initial domain E via (8.1) is also reminiscent of the classical paper by Dibenedetto and Friedman [21], which considers a time integrated version of (St_1). See the discussion in section 4.3.

Remark 9.8. *The form of μ in Theorems 9.2 and 9.3, and (μ, f) in Theorem 9.6 were made to ensure the stopping time $\tau > 0$, that is, there is no instant stopping. With more general μ, an instant stopping may occur as given in the saturation theorem, Thereom 8.3. Those general cases feature immediate phase transition, which is not present in the standard Stefan problem. Namely, the drain of energy μ occurs immediately on a certain part, say, the set F in the case (I) part of Theorem 8.3, or $f \wedge \mu$ in the (II) part of Theorem 8.3, for which the heat particles are not activated. In the freezing problem (I), this is where immediate freezing occurs (or perhaps one should understand it as the previously frozen region). In the melting problem (II) the energy $f \wedge \mu$ reduces the drain capacity f: only the remaining portion $\mu - f \wedge \mu$ of energy is used for the melting for the reduced drain capacity $f - f \wedge \mu$. The heat particles once activated move, and in the melting problem (II), the region (time-dependent) where the heat particles move is the region where melting occurs.*

9.3. **Existence of finite time vanishing solution characterizes the initial set.** In this section, we show that the choice of the initial set E in Theorem 9.3, where in particular the solution vanishes in finite time, is sharp. Namely, such a choice is *necessary* for a finite time vanishing solution to exist. First, let us connect finite time vanishing solutions to the notion of subharmonically generated sets as in Theorem 6.2.

Theorem 9.9. *Let η be a weak solution of (St_1) with the initial data (μ, E). Then η vanishes in finite time if and only if (E, E) is subharmonically generated by μ and η is the corresponding Eulerian variable to (μ, χ_E). In particular, there is at most one solution of (St_1) with the initial data (μ, E) that vanishes in finite time.*

Proof. The 'if' direction is a direct consequence of Theorem 6.2 and Theorem 8.6. The 'only if' direction follows from Theorem 6.2 and the fact that

$$s(x) := \sup\{t : \eta(x,t) > 0\} < T \text{ for some finite } T,$$

the vanishing time of η. Note that that 'at most one solution' follows from Theorem 6.2. □

Theorem 9.10 (Necessary condition for finite time vanishing solution)**.** *Let μ be a compactly supported measure on $\mathbb{R}^d$ with $\mu \ll Leb$. Suppose there is a vanishing in finite time solution to (St_1) with the initial data (μ, E). Then, the set E is determined by the optimal free target problem (1.1) of type (I) cost, for μ and $f \equiv 1$, that the optimal target is $\nu^* = \chi_E$.*

Before proving theorem, we point out that to have such E we must have $\tau > 0$ almost surely. This in particular shows that such a set E is determined by the obstacle problem (9.1). Hence we arrive at the following conclusion:

Corollary 9.11 (Unique Characterization)**.** *Let μ be as given above. Then there is a unique solution η of (St_1) with the initial data μ that vanishes in finite time. In this case, the initial domain E is given by (9.1), and η corresponds to the Eulerian variable given by μ and the optimal target $\nu^* = \chi_E$.*

We now prove Theorem 9.10.

Proof. For this result, we apply Theorem 9.9 and Theorem 7.1.

To have a vanishing in finite time solution to (St_1), from Theorem 9.9 it is necessary that (E, E) has to be subharmonically generated by μ. There is a corresponding type (I) optimal stopping time, say, τ, solving the optimal Skorokhod problem $\mathcal{P}(\mu, \nu)$. Here, $\nu = \chi_E$, moreover, $E = \limsup_{t\to 0^+}\{x \mid \eta(x,t) > 0\}$ is the initial set of the active region for the corresponding Eulerian flow (η, ρ).

On the other hand, we have the optimal stopping time τ^* for the free target problem (1.1) for type (I) cost, from the measure μ, with $f \equiv 1$, with the optimal target ν^* with its density $\nu^* \leq 1$ everywhere.

We claim that $\nu^* \leq \nu$. It suffices to show this claim, as it implies the desired equality $\nu^* = \nu$ because they have the same mass.

To prove the claim, first note that the monotonicity result, Theorem 7.1 part (1), does not require optimality of one of the target, therefore, we have $\tau^* \leq \tau$. Therefore, the Eulerian flow η^* for τ^* is supported inside the support of the Eulerian flow η of τ. In particular the initial domain for the Eulerian flow η^* is a subset of the initial domain for η, that is, E. Moreover, even when there is instant stopping $\tau^* = 0$ for a set S, since $\tau > 0$ almost surely, we have $S \subset E$. From these considerations and the fact that $\nu^* \leq 1$ and $\nu = \chi_E$, we have $\nu^* \leq \nu$ as desired. This completes the proof. ☐

From parallel reasoning we can uniquely characterize η from Theorem 9.6 as well:

Corollary 9.12 (Unique Characterization)**.** *Let μ, K, η and E be as given in Theorem 9.6. Then η is the unique weak solution of (St_1) such that $\cap_{\{t>0\}}\{\eta(\cdot, t) > 0\} = K$. In other words, E is the unique set for which $(E \cap K^c, E)$ is subharmonically generated.*

10. Monotonicity of the optimal barrier functions

We show in this section that the barrier function $s(x)$ for the optimal stopping time τ, enjoys a monotonicity property in the spirit of Theorem 7.1. Namely, the barrier functions s_i are ordered if the initial distributions μ_i are ordered (Theorem 10.1), and such order is strict (Theorem 10.2). The strict monotonicity we obtain generates a comparison principle, and we hope it may shed a light on understanding regularity of the barrier function $s(x)$, like comparison principles do for elliptic PDEs. Understanding regularity of $s(x)$, so regularity of the free boundaries of the Stefan problem, is a wide open problem in (St_1) where even well-posedness is poorly understood.

Our result in Theorem 4.14 shows that its corresponding space-time barrier set R is closed, so s is lower semicontinuous for type (I) and upper semicontinuous for type (II). To our knowledge even this very mild regularity result for s is new for (St_1). However, for such semicontinuity we only use the optimality of the stopping time for the optimal Skorokhod problem $\mathcal{P}(\mu, \nu)$. Our monotonicity of s below, which is a result of optimality of the target measure, may lead to a nicer regularity result.

Theorem 10.1 (Monotonicity for optimal barrier functions)**.** *Assume $\mu_1 \le \mu_2 \ll Leb$ and let f be a bounded measurable function on $\mathbb{R}^d$. Let ν_i, $i = 1, 2$, be the optimal solutions of the Problem* (1.1) *with μ_i, $i = 1, 2$, respectively. Let s_i be the barrier functions associated with (μ_i, ν_i) as in Definition 4.17. Then we have*

$$s_1 \le s_2 \quad \nu_1\text{-a.e. for type (I) cost, and}$$
$$s_1 \ge s_2 \quad \nu_1\text{-a.e. for type (II) cost.}$$

Here, we are able to show only a.e inequality for (I) due to technical reasons, and we conjecture it holds everywhere; for example, we immediately get everywhere inequality if s_i's are continuous. To the best of our knowledge the continuity of the barrier function s is not known in general.

Proof. We utilize the potential flow formulation in Section 4 as well as the monotonicity result for the optimal target problem (Theorem 7.1). Let us define τ_i^U,R_i^U and s_i^U as in Definition 4.13 generated by the stopping time τ_i. We aim to show the order between s_1^U and s_2^U, from which and (4.4) we can conclude.

Let $\tilde\tau_2$ denote the restriction of τ_2 from the initial distribution μ_1, which is the common mass $\mu_1 \wedge \mu_2$ for $\mu_1 \le \mu_2$. Then $\tau_1 \le \tilde\tau_2$ due to Theorem 7.1. Let $\tilde\nu_2$ denote its distribution, namely $W_{\tilde\tau_2} \sim \tilde\nu_2$. Recall that τ_1 and τ_2 are optimal stopping times between μ_i and ν_i, $i = 1, 2$. It is also easy to see that $\tilde\tau_2$ is an optimal stopping time between μ_1 and $\tilde\nu_2$.

The monotonicity (Theorem 7.1) gives $\tau_1 \le \tilde\tau_2$. Let μ_t^1 be the distribution of $W_{\tau_1\wedge t}$, that is, $W_{\tau_1\wedge t} \sim \mu_t^1$. Similarly, let $W_{\tau_1\wedge t} \sim \mu_t^1$, $W_{\tilde\tau_2\wedge t} \sim \tilde\mu_t^2$. Since $\tau_1 \wedge t \le \tilde\tau_2 \wedge t$, from Corollary 4.4 we have the potentials satisfy

$$U_{\mu_t^1} \le U_{\tilde\mu_t^2}.$$

Also, restricting τ_2 to the initial distribution $\mu_2 - \mu_1 \ge 0$ and applying Corollary 4.7, we have

$$U_{\mu_2-\mu_1} \le U_{\mu_t^2-\tilde\mu_t^2} \le U_{\nu_2-\tilde\nu_2}.$$

Below we treat type (I) and (II) separately.

Type (I). For type (I), define the functions

$$U_1(x,t) := U_{\nu_1}(x) - U_{\mu_t^1}(x),\ \tilde U_2(x,t) := U_{\tilde\nu_2}(x) - U_{\tilde\mu_t^2}(x), \text{ and } U_2(x,t) := U_{\nu_2}(x) - U_{\mu_t^2}(x).$$

Notice that they are all nonnegative, continuous, and monotone decreasing in time, due to Corollary 4.9) and Corollary 4.7. Since $U_{\mu\pm\nu} = U_\mu \pm U_\nu$ in general, we have

$$U_2 = \tilde U_2 + \left[U_{\nu_2-\tilde\nu_2} - U_{\mu_t^2-\tilde\mu_t^2}\right] \ge \tilde U_2.$$

Recall that $R_i^U = \{U_i = 0\}$. Thus to show that $s_1^U \le s_2^U$ a.e. it suffices to prove that for each t and for a.e x, $\tilde U_2(x,t) = 0$ implies $U_1(x,t) = 0$, or

$$|\{U_1(\cdot,t) > 0\} \cap \{\tilde U_2(\cdot,t) = 0\}| = 0. \tag{10.1}$$

From now on let us fix t. Suppose for contradiction that $|\{U_1(\cdot,t) > 0\} \cap \{\tilde U_2(\cdot,t) = 0\}| > 0$. This implies that $U_1(\cdot,t) > 0$ and $\tilde U_2(\cdot,t) = 0$ in $E \subset \mathbb{R}^d$ for some positive measure set E that only consists of Lebesgue density points (of density 1). Lemma 4.3 then yields that for

Brownian motions starting from μ_1 there exists a monotone decreasing sequence of open sets $O_n \supseteq E$, with $\lim_{n\to\infty} |O_n \setminus E| = 0$, such that

$$0 < \mathbb{E}\left[\int_{\tau_1 \wedge t}^{\tau_1} \lim_{n\to\infty} \chi_{O_n}(W_{\bar{t}}) d\bar{t}\right].$$

In particular, this implies

$$0 < Prob\left[\ \exists\ \bar{t} \text{ such that } \lim_{n\to\infty} \chi_{O_n}(W_{\bar{t}}) = 1 \ \&\ t < \bar{t} < \tau_1\right]. \tag{10.2}$$

Note the strict inequality $t < \bar{t} < \tau_1$ in the above expression.

Now, if $\lim_{n\to\infty} \chi_{O_n}(W_{\bar{t}}) = 1$ for some $\bar{t}$ for a Brownian path, then because E consists only of its Lebesgue density point, we have that for each $\epsilon > 0$, there exists with positive probability, $\bar{t} \le t' \le \bar{t} + \epsilon$ such that $W_{t'} \in E$; here we have used the fact that Brownian motion that visits an open neighbourhood of E sufficiently close to E while E has Lebesgue density 1, has to visit E with comparable probability. Since such $\bar{t}$ exists with $t < \bar{t} < \tau_1$ with positive probability, this means that

with positive probability there exists $\tilde{t}$ with $t < \tilde{t} < \tau_1$ such that $W_{\tilde{t}} \in E$.

However, for those $\tilde{t}$ and corresponding Brownian paths we have $\tilde{U}_2(W_{\tilde{t}}, t) = 0$ due to the fact $W_{\tilde{t}} \in E \subset \{\tilde{U}_2(\cdot, t) = 0\}$. Since $\tilde{t} > t$ and $\tilde{U}_2$ is monotonically decreasing in time we have $\tilde{U}_2(W_{\tilde{t}}, \tilde{t}) = 0$. Therefore, $(W_{\tilde{t}}, \tilde{t})$ belongs to the barrier $\{\tilde{U}_2 = 0\}$ of $\tilde{\tau}_2$, and thus $\tilde{t} \ge \tilde{\tau}_2$ for those Brownian paths. On the other hand, since $\tau_1 \le \tilde{\tau}_2$ almost surely, we can assume that along such Brownian paths we have that $\tilde{t}$ satisfies $\tilde{t} \ge \tau_1$, contradicting the fact $\tilde{t} < \tau_1$. This proves (10.1), which yields $s_1^U \le s_2^U$ a.e.

Type (II). For type (II), define the functions

$$U_1(x,t) := U_{\mu_t^1}(x) - U_{\mu_1}(x),\ \tilde{U}_2(x,t) := U_{\tilde{\mu}_t^2}(x) - U_{\mu_1}(x), \text{ and } U_2(x,t) := U_{\mu_t^2}(x) - U_{\mu_2}(x).$$

Notice that they are all nonnegative. From Theorem 4.14, these give the barrier sets for $\tau_1, \tilde{\tau}_2$, and τ_2, respectively. Note that from the definition

$$s_i^U(x) := \sup\{t \mid U_1(x,t) = 0\}.$$

Observe that $U_1 \le \tilde{U}_2$ becasue $U_{\mu_t^1} \le U_{\tilde{\mu}_t^2}$. Moreover, we have

$$U_2 = \tilde{U}_2 + \left[U_{\mu_t^2 - \tilde{\mu}_t^2} - U_{\mu_2 - \mu_1}\right] \ge \tilde{U}_2.$$

So $U_1 \le U_2$, which implies $s_1^U \ge s_2^U$. □

We now prove strict monotonicity for s, which is derived by combining the above theorem with the monotonicity of the stopping time (Theorem 7.1) and the saturation result (Theorem 8.3).

As a preparation of the statement of the result, let us recall that the potential barrier sets R_i^U, $i = 1, 2$ for the stopping times in Theorem 10.1 are closed, therefore their complements are open. Define for $i = 1, 2$,

$$\text{in type (I)}, \quad E_i := \limsup_{t\to 0^+}\{x \mid \eta_i(x,t) > 0\}, \qquad (10.3)$$
$$\text{in type (II)}, \quad E_i := \limsup_{t\to\infty}\{x \mid \eta_i(x,t) > 0\}.$$

Note that due to the monotonicity of R_i, $E_i = R_i^C = \cup_{t>0}\{\eta_i(\cdot,t) > 0\}$, and thus they are open. Note also that as E_i's are connected to the active regions, the Brownian paths that start from E_i's have $\tau_i > 0$ almost surely, except those in type (II) that may immediately stop; see Theorem 8.3, where the mass that immediately stop in type (II) is characterized by the initial data as $\mu \wedge f$. Thus by taking the initial data as $\mu - \mu \wedge f$ we may assume $\tau_i > 0$ in $E_i's$. The complement E_2^c is the set where there is immediate and complete stopping occurs for τ_2, so τ_2 is zero and so is $\tau_1 = 0$ from monotonicity (Theorem 7.1). There the corresponding s_i's are 0 in type (I) case, ∞ in type (II) case. Moreover, in type (I) case, E_2^c is exactly the set where the immediate stopping occurs. This justifies that for comparison between s_i we can without loss of generality assume that $\tau_i > 0$ in both type (I) and (II), and compare s_i's only over E_2. Also assuming $f > 0$ everywhere does not cost much generality for our purpose of comparing s_i's, because in the region $f = 0$ no stopping to accumulate mass occurs, so the value s_i can only be either ∞ or 0. With these considerations we see that the following theorem essentially covers the general case and the whole domain for the strict monotonicity of s.

Theorem 10.2 (Strict monotonicity for optimal barrier functions). *Let f, μ_i, τ_i and s_i be as given in Theorem 10.1, in particular, with $\mu_1 \le \mu_2$ and s_1, s_2 satisfying the monotonicity. Assume that $f > 0$ everywhere and that $\tau_i > 0$, $i = 1, 2$, almost surely. Let O be a path connected component of E_2 given in* (10.3). *Suppose $\mu_2 > \mu_1$ on a subset $G \subset O$ with $|G| > 0$. Then $s_1^U(x) \neq s_2^U(x)$ for a.e. $x \in O$.*

Proof. From the continuity of Brownian paths, almost surely any Brownian path starting from O should stay inside the corresponding path-connected component of the active region, say $A \subset \mathbb{R}^d \times \mathbb{R}_{\ge 0}$, that is connected to O, that is,

$$A = \{(x,t) \mid x \in O, \quad t < s_2^U(x)\} \text{ in type (I)}, \; A = \{(x,t) \mid x \in O, \quad t > s_2^U(x)\} \text{ in type (II)}.$$

From this and the condition $\mu_1 \le \mu_2$, we can without loss of generality assume that $O = E_2$, that means, we consider only those mass, Brownian motion, and the barriers, associated to O; for example, we assume $\mu_i|_E = \mu_i|_O$, $\nu_i|_E = \nu_i|_O$. Notice that A is a connected open set and that the Eulerian flow $\eta_2 > 0$ on A. From this we also see that

$$\nu_2 > 0 \text{ on } O.$$

(To see this use for example, Lemma 8.1, which uses optimality of ν_2 with respect to Problem 1.1.) Therefore, it suffices to show that $s_1^U(x) \neq s_2^U(x)$ for ν_2-a.e. x.

Recall that $\tau_1 \le \tau_2$ and $\nu_1 \le \nu_2$ from Theorem 7.1. Because of the Markov property of Brownian motion, the inequality can be applied to all Brownian paths from the initial distribution μ_2 not only from μ_1 (see Remark 7.3).

If $s_1(W_{\tau_1}) = s_2(W_{\tau_1})$, then from the hitting time characterization of τ_i's (Proposition 3.9) we have that $\tau_1 = \tau_2$. Therefore,

$$\text{almost surely, “} s_1(W_{\tau_1}) = s_2(W_{\tau_1}) \text{” implies “} \tau_2 = \tau_1 \text{”.} \tag{10.4}$$

Also, recall from the saturation result (Theorem 8.3) that $\nu_i \neq f$ only where the Brownian paths (from μ_i) stop immediately. From our assumption $\tau_i > 0$ almost surely, so we have $\nu_i = f$ on its support. Therefore to prove the theorem it suffices to show that

$$\nu_2[S] = 0,$$

where

$$S := \{x \mid s_1(x) = s_2(x) < \infty\} \cap \{x \mid \nu_1(x) = \nu_2(x) = f(x)\} \cap O.$$

We have from $W_0 \sim \mu_i|_O$, $W_{\tau_i} \sim \nu_i|_O$ that

$$\nu_i[S] = \int_O Prob[W_{\tau_i} \in S | W_0 = x] d\mu_i(x) \quad \text{ for } i = 1, 2.$$

Let G be as given in the theorem, and observe that $\mu_2[G] > 0$. Then, for $i = 1, 2$,

$$\nu_i[S] = \int_G Prob[W_{\tau_i} \in S | W_0 = x] d\mu_i(x) + \int_{O \setminus G} Prob[W_{\tau_i} \in S | W_0 = x] d\mu_i(x). \tag{10.5}$$

On the other hand, for μ_2-a.e. x,

$$\begin{aligned} &Prob[W_{\tau_2} \in S \mid W_0 = x] \\ &= Prob[W_{\tau_2} \in S \ \& \ \tau_2 = \tau_1 \mid W_0 = x] + Prob[W_{\tau_2} \in S \ \& \ \tau_2 > \tau_1 | \ W_0 = x \ ] \\ &\geq Prob[W_{\tau_2} \in S \ \& \ \tau_2 = \tau_1 \mid W_0 = x] \\ &= Prob[W_{\tau_1} \in S \ \& \ \tau_2 = \tau_1 \mid W_0 = x]. \end{aligned}$$

From (10.4) it holds that for μ_2-a.e. x,

$$Prob[W_{\tau_1} \in S \ \& \ \tau_2 = \tau_1 \mid W_0 = x] = Prob[W_{\tau_1} \in S \mid W_0 = x].$$

Therefore, from the previous inequality we have for μ_2-a.e. x,

$$Prob[W_{\tau_2} \in S \mid W_0 = x] \geq Prob[W_{\tau_1} \in S \mid W_0 = x].$$

Apply this to the above integrals (10.5) and get

$$\begin{aligned} \nu_2[S] - \nu_1[S] &\geq \int_G Prob[W_{\tau_2} \in S | W_0 = x] d\mu_2(x) - \int_G Prob[W_{\tau_1} \in S | W_0 = x] d\mu_1(x) \\ &\geq 0. \end{aligned}$$

On the other hand, due to the definition of S, $\nu_1(S) = \nu_2(S)$. This and the condition $\mu_2 > \mu_1$ on G yields

$$Prob[W_{\tau_2} \in S \mid W_0 = x] = Prob[W_{\tau_1} \in S \mid W_0 = x] = 0 \text{ for } \mu_2 \text{ a.e. } x \text{ in } G. \tag{10.6}$$

Let $\tilde{\tau}_2$ be the restriction of τ_2 to the to the initial distribution $\mu_2|_G$ and consider its Eulerian flow $\tilde{\eta}_2$. Since the active set A is a connected open set, we have $\tilde{\eta}_2 > 0$ everywhere in A. Recall from above that $\nu_2 > 0$ on O and that $\eta_2 > 0$ on A for the Eulerian flow η_2 of τ_2 with μ_2. In other words, $\tilde{\eta}_2 > 0$ wherever $\eta_2 > 0$. Therefore, from the Markov property of the Brownian motion, the resulting target distribution $\tilde{\nu}_2$ of $\tilde{\tau}_2$ has $\tilde{\nu}_2 > 0$ on O, because it has to be > 0 wherever $\nu_2 > 0$. Now, (10.6) implies that $\tilde{\nu}_2[S] = 0$, so $\nu_2[S] = 0$ as desired. This completes the proof.

□

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