Spring 2017 MATH 134: Homework 5, Due May 19th

Note: At least one of the homework problems, including the suggested exercises, will appear on each exam.

1. This exercise leads you through the solution of a linear system where the eigenvalues are complex. The system is $\dot{x} = x + y$, $\dot{y} = y - x$.

(a) Find A , the eigenvalues λ_1, λ_2 and the corresponding eigenvectors v_1 and v_2. (Note that the eigenvalues are complex conjugates and so are the eigenvectors–this is always the case for real A with complex eigenvalues.)

(b) The general solution is $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$. So in one sense, we're done! But this way of writing $x(t)$ involves complex coefficients and looks unfamiliar. Express $x(t)$ purely in terms of real valued functions. (Hint: use $e^{i\theta} = \cos\theta + i\sin\theta$ to rewrite $x(t)$ in term of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.)

2. Consider the system

$$\dot{x} = 1 + y - e^{-x}, \quad \dot{y} = x^3 - y.$$

Sketch the phase portrait by completing the following steps.

(a) Find and classify the fixed points.

(b) Sketch the nullclines.

(c) Fill in representative trajectories using the classification of the fixed points.

3. Find and classify the fixed points for the following system

$$\dot{x} = \cos(y), \quad \dot{y} = x - x^3.$$

Sketch a phase portrait that demonstrates the local behavior around each fixed point. (You do not need to draw the entire phase portrait - just describe what happens near the fixed points.)

4. Strogatz 6.3.9.

5. In this question, we return to the problem of a bead on a rotating hoop. Recall that the bead's motion is governed by

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi.$$

Previously, we could only treat the overdamped limit. Now, we will consider the undamped case $b = 0$.

(a) Show that the equation can be nondimensionalized to $\phi'' = \sin\phi(\cos\phi - \gamma^{-1})$ where $\gamma = r\omega^2/g$ as before, where prime denotes differentiation with respect to dimensionless time $\tau = \omega t$.

(b) Find all fixed points and classify them for all qualitatively different values of γ.

(c) What does the classification of the fixed points imply about the physical motion of the bead?

6. Still considering the bead's motion,

(a) Show that for fixed m, g, ω and r there is a threshold b_0 such that if $b > b_0$ then the system is not conservative. *Hint: When b is large enough look for behavior near 0 which is uncharacteristic of conservative systems.*

(b) Now fix all the parameters but ω and ask that $b > 0$. Show that if ω is small enough then he initial configuration where the bead is at rest at the bottom of the hoop the system behaves like a stable spiral. What does this tell us about the behavior of the damped bead when the hoop is spinning slow?

7. Given a take the second order system $\ddot{x} = -x - ax^3$. Show that for any number a the origin is a nonlinear center, so in particular all trajectories near the origin are periodic. However, when $a < 0$ then there are some trajectories far away from the origin that are not periodic: explain why.

8. Consider the system $(\dot{x}, \dot{y}) = (\sin y, \sin x)$. Show that it is reversible, find all fixed points and determine their stability. Show that although the lines $y = \pm x$ are not traced by a single trajectory, any initial data on that line remains on that line for all times. Sketch the phase portrait.

Other suggested exercises not to be turned in : Strogatz 5.2.9, 5.2.13, 6.1.2, 6.2.2, 6.3.14, 6.5.12, 6.5.14, 6.7.3.