Spring 2017 Math 134: Homework 3, Due April 28th

Note: At least one of the homework problems, including the suggested exercises, will appear on each exam.

1. In parts (a)-(c), let $V(x)$ be the potential, in the sense that $\dot{x} = -\frac{dV}{dx}$. Sketch the potential as a function of x. Be sure to show all the qualitatively different cases, including bifurcation values of r.

(a) (Saddle-node) $\dot{x} = r + x^2$

(b) (Transcritical) $\dot{x} = rx + x^2$

(c) (Subcritical pitchfork) $\dot{x} = rx + x^3$

2. Consider the second order system (modeling an elastic spring with friction.)

$$\ddot{x} = -b\dot{x} - \kappa x.$$

where $\kappa > 0$ is the spring constant and b represents a friction coefficient. Find under what limiting conditions for b and κ (for instance $b/\kappa \to 0$?) can we approximate this system with a one dimensional equation.

Hint: Introduce a new time scale $t = Ts$, for a T depending on b and κ (as we did to analyze the overdamped bead on a hoop). Solve the equation explicitly in the new time scale and identify the limit.

3. Consider the system $\dot{x} = h + rx - x^2$. When $h = 0$, this system undergoes a transcritical bifurcation at $r = 0$. Our goal is to see how the bifurcation diagram of x^* vs r is affected by the imperfection parameter h.

(a) Plot the bifurcation diagram for $\dot{x} = h + rx - x^2$, for $h < 0, h = 0$, and $h > 0$.

(b) Sketch the regions in the (r, h) plane that correspond to qualitatively different vector fields, and identify the bifurcations that occur on the boundaries of those regions.

(c) Plot the potential $V(x)$ corresponding to all the different regions in the (r, h) plane.

(d) What happens if you add a small imperfection to a system that has a saddle-node bifurcation?

4. Consider the system

$$\dot{x} = f(x, r),$$

where f is a smooth function.

(a) Suppose that the phase point x_0 and the parameter r_0 are such that $v(x_0, r_0) = 0$ and $\frac{\partial}{\partial r} v(x_0, r_0) \neq 0$. Could this system have a pitchfork bifurcation at r_0?.

(b) In the same situation as in the previous question, add the assumption $\frac{\partial}{\partial x} v(x_0, r_0) \neq 0$ and that there aren't any other fixed points besides x_0. With this hypothesis, can there be a bifurcation *at all* at the parameter r_0?

Suggested exercises that does not need to be turned in: Strogatz 3.4.11, 3.4.12, 3.5.5, 3.6.1, 4.1.5.