Math 131AH Winter 2018: Homework 2, Due Wed. 1/24

1. Show that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for all $a \in \mathbb{R}$.

2. Show that if $x, y \in \mathbb{R}$ with $x^2 > y^2$ and $x > 0$ then $x > y$.

3. This exercise proves that $x^2 = 2$ can be solved in $\mathbb{R}$.

(a) Show that $E := \{x \in \mathbb{R} : x^2 < 2\}$ has a supremum in $\mathbb{R}$. Let us call $\alpha := \sup E$. Show that $0 < \alpha < 2$.

(b) Show that $(\alpha + 1/n)^2 < 2$ for some $n \in \mathbb{N}$ if $\alpha^2 < 2$.

(c) Show that $(\alpha - 1/n)^2 > 2$ for some $n \in \mathbb{N}$ if $\alpha^2 > 2$.

(d) Conclude that $\alpha^2 = 2$.

4. Show that irrationals are dense in $\mathbb{R}$: i.e. prove that for any $x, y \in \mathbb{R}$ with $x < y$, there exists an irrational number γ such that

$$x < \gamma < y.$$

5-6. Rudin p22 (chapter 1) Exercise 6, 15.