Math 131AH Winter 2018: Homework 7, Due 3/7

1. For a given $k \in \mathbb{N}$, Consider two sequences (a_n) and $(b_{n,k})$ in $\mathbb{R}$. Suppose that

$$a_n \leq b_{n,k} \text{ for all } n, k \in N.$$

Show that $\limsup_{n\to\infty} a_n \leq B_k := \limsup_{n\to\infty} b_{n,k}$ for all $k \in N$. Consequently it follows that

$$\limsup_{n\to\infty} a_n \leq \liminf_{k\to\infty} B_k.$$

2. Rudin chapter 3 exercise problem 17.

3. (Construction of $\mathbb{R}$ by Cauchy) Let us first define some necessary notions in the field $\mathbb{Q}$. We denote $\mathbb{Q}^+$ by the set of positive rational numbers.

- For a sequence (q_n) in $\mathbb{Q}$, we say q_n *converges* to $r \in \mathbb{Q}$, or $q_n \to r$, if the following holds: for any $w \in \mathbb{Q}^+$, there exists $N \in \mathbb{N}$ such that $|q_n - r| \leq w$ if $n > N$.
- We say (q_n) is a *Cauchy sequence* in $\mathbb{Q}$ if the following holds: for any $w \in \mathbb{Q}^+$, there exists $N \in \mathbb{N}$ such that $|q_n - q_m| \leq w$ if $m, n > N$.

With these definitions, let us define $\mathbb{R}$ as follows:

$$\mathbb{R} := \{[P] : P = (p_n) \text{ is a Cauchy sequence in } \mathbb{Q}\},$$

where $[P]$ is the equivalence class of P defined by the following equivalence relation

$$P \sim \tilde{P} \quad \text{ if } |p_n - \tilde{p}_n| \to 0.$$

Let us next define a relation $<$ between $[P]$ and $[\tilde{P}]$ by the following:

$$[P] < [\tilde{P}] \text{ if there exists } w \in \mathbb{Q}^+ \text{ and } N \text{ such that } w < \tilde{p}_n - p_n \text{ for } n > N.$$

(1) Show that $(\mathbb{R}, <)$ is an ordered set. First show that the order is well-defined, i.e. that it does not depend on the choice of the sequence P in the same equivalence class.

(2) Show that $(\mathbb{R}, <)$ has the least upper bound property.

4-6. Rudin p99: problems 1,3,4.