Fall 2013 MATH 131A Lec 4 Final Exam

Please write clearly, and show your reasoning with mathematical rigor.

Name: Student ID:

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1. [17pt] (True or False. Please give a counterexample if false)

(a) If a set D is dense in $\mathbb{R}$, its complement $\mathbb{R} - D$ is also dense.

(b) If $S \subset T$ then $\inf T \leq \inf S$.

(c) If Σa_n converges and b_n is bounded, then $\Sigma(a_n b_n)$ converges.

(d) A sequence always has a convergence subsequence.

(e) There is a differentiable function $f : [0, 2] \to \mathbb{R}$ such that $\lim_{x\to 1} f'(x)$ exists but is not equal to $f'(1)$. Please explain your answer.

(f) If $f : (0, 1) \to \mathbb{R}$ is uniformly continuous in $(0, 1)$, then $\lim_{x\to 1^-} f(x)$ exists.

(g) If f is integrable in $[a, b]$, then it is either continuous or monotone.

2. [12pt] Suppose S and T is a subset of $\mathbb{R}$, and suppose $U = \{s + t : s \in S, t \in T\}$. Suppose $\sup S$ and $\sup T$ are finite. Show that $\sup(S + T) = \sup S + \sup T$ by proceeding as below:

(a) Show that $\sup(S + T) \leq \sup S + \sup T$;

(b) Show that if $t < \sup S + \sup T$, then t is not an upper bound of $S + T$;

(c) Using (b) conclude that $\sup S + \sup T \leq \sup(S + T)$.

3. [12pt] Let us define the sequence $\{x_n\} \subset \mathbb{R}$ as follows:

$$x_1 = 2 \text{ and } x_{n+1} = \frac{1}{2}(x_n + \frac{1}{x_n}).$$

Show that the sequence is monotone, and find its limit.

4. [12pt]

(a) State the definition of Cauchy sequence in $\mathbb{R}$.

(b) Using only the definition of the Cauchy sequence, show that if $\{s_n\}$ is a Cauchy sequence in $\mathbb{R}$ then $\limsup s_n = \liminf s_n$.

5. [12pt] Prove that if $f : [0, 1] \to \mathbb{R}$ be continuous then f is bounded.

6. [12pt] Let $f : [0,1] \to \mathbb{R}$ be continuous, $f(0) < f(1)$ and assume that f does not have a local maximum or minimum in $(0,1)$. Show that f must be monotone increasing. Note that f may not be differentiable.

7. [11pt] Let $f : (a, b) \to \mathbb{R}$ be continuous.

(a) Suppose f attains its maximum at $c \in (a, b)$, and suppose that f is differentiable at $x = c$. Use the definition of derivative to show that $f'(c) = 0$.

(b) Suppose $|f'(x)| \leq 1$ in (a, b). Using the Mean Value Theorem, show that f is uniformly continuous in (a, b).

8. [12pt] Let $f : [0,1] \to \mathbb{R}$ satisfy $f(x) = x$ if $0 \leq x \leq 1/2$ and $f(x) = 0$ if $1/2 < x \leq 1$.

(a) Explain why it is enough to show that, for each $\varepsilon > 0$, there exists $U(f,P)$ and $L(f,P)$ which differ at most by ε.

(b) Using (a), Show that f is integrable.