Problem 1. Find numbers $A, B, C$ such that

$$f(x) = \frac{2x^2 + x + 2}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}$$

Then compute $\int f(x) \, dx$.

Solution 1. The first part is pretty straightforward. Take the equality above and multiply through by the denominator:

$$2x^2 + x + 2 = A(x^2 + 4) + (Bx + C)(x - 1)$$

The first thing we should do is plug in $x = 1$ to isolate the $A$ variable.

$$2(1)^2 + 1 + 2 = A(1^2 + 4) + (Bx + C)(1 - 1) \implies 5 = A \cdot 5$$

so $A = 1$. Then we need to find $B$ and $C$, which we should do by multiplying it all out - there's not a better way to proceed. Filling in $A = 1$,

$$2x^2 + x + 2 = x^2 + 4 + Bx^2 - Bx + Cx - C$$

$$= (1 + B)x^2 + (C - B)x + (4 - C)$$

This gives us a system of equations

$$2 = 1 + B, \quad 1 = C - B, \quad 2 = 4 - C$$

from which it’s easy to see $B = 1$ and $C = 2$. That means we need to integrate

$$\int f(x) \, dx = \int \frac{1}{x - 1} + \frac{x + 2}{x^2 + 4} \, dx$$

We should split this into three separate integrals:

$$\int \frac{1}{x - 1} \, dx + \int \frac{x}{x^2 + 4} \, dx + \int \frac{2}{x^2 + 4} \, dx$$

The first integrates to $\ln |x - 1|$ by using the substitution $u = x - 1$. The second integrates to $\frac{1}{2} \ln |x^2 + 4|$ by using the substitution $u = x^2 + 4$. That’s why we separate the $Bx + C$ term into two pieces - each piece by itself is easier to integrate.
The third piece does not have a good $u$-substitution, because there’s no $x$ in the numerator. Instead, we have to remember that

$$\int \frac{1}{x^2 + a} \, dx = \frac{1}{\sqrt{a}} \arctan \left( \frac{x}{\sqrt{a}} \right)$$

That means

$$\int \frac{2}{x^2 + 4} \, dx = 2 \int \frac{1}{x^2 + 4} \, dx = 2 \left( \frac{1}{\sqrt{4}} \arctan \left( \frac{x}{\sqrt{4}} \right) \right) = \arctan \left( \frac{x}{2} \right).$$

Now we put all those bits together:

$$\int \frac{2x^2 + x + 2}{(x - 1)(x^2 + 4)} \, dx = \ln |x - 1| + \frac{1}{2} \ln |x^2 + 4| + \arctan \left( \frac{x}{2} \right) + C.$$
Problem 2.

(1) Compute the arc length of the graph of the function

\[ f(x) = \frac{1}{2}x^2 - \frac{1}{4}\ln x \]

over \([1, 2]\). Hint:

\[ a^2 + \frac{1}{2} + \frac{1}{16a^2} = \left(a + \frac{1}{4a}\right)^2 \]

for any \(a \neq 0\).

(2) Compute the area of the surface of revolution of the graph of the function

\[ g(x) = \sqrt{2} + x \]

over \([0, 1]\).

Solution 2.

(1) We need the derivative to compute arc length, so we might as well compute it first:

\[ f'(x) = x - \frac{1}{4x} \]

Then the formula gives us

\[
\int_1^2 \sqrt{1 + f'(x)^2} \, dx = \int_1^2 \sqrt{1 + \left(x - \frac{1}{4x}\right)^2} \, dx
\]

Let’s examine that quantity under the square root. We’d like it to be a perfect square, so that we can get rid of the square root altogether. But before we try a trig sub, let’s simplify the quantity.

\[
1 + \left(x - \frac{1}{4x}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2
\]

That last equality comes from the hint (or you can notice it yourself).

Then the integral we need to compute is

\[
\int_1^2 \sqrt{\left(x + \frac{1}{4x}\right)^2} \, dx = \int_1^2 x + \frac{1}{4x} \, dx = \frac{1}{2}x^2 + \frac{1}{4}\ln x \bigg|_1^2
\]

Plugging in gives us

\[
\left(\frac{1}{2} \cdot 2^2 + \frac{1}{4}\ln 2\right) - \left(\frac{1}{2} \cdot 1 + \frac{1}{4}\ln 1\right) = \frac{3}{2} + \frac{1}{4}\ln 2
\]
(2) Now, for the next one. This one is a bit harder. Again we need the derivative:

\[ g'(x) = \frac{1}{2}(2 + x)^{-1/2} \]

So plugging into the formula for surface area,

\[ 2\pi \int_0^1 g(x)\sqrt{1 + g'(x)^2} \, dx = 2\pi \int_0^1 \sqrt{2 + x} \cdot \sqrt{1 + \frac{1}{4}(2 + x)^{-1}} \, dx \]

Again we must ask ourselves, what is it we are trying to integrate? To simplify it a bit, we can combine the square roots:

\[ \sqrt{2 + x} \cdot \sqrt{1 + \frac{1}{4}(2 + x)^{-1}} = \sqrt{(2 + x) \left( 1 + \frac{1}{4}(2 + x)^{-1} \right)} = \sqrt{2 + x + \frac{1}{4}} \]

Remembering to distribute, the \(2 + x\) and \((2 + x)^{-1}\) just cancel out to leave the \(\frac{1}{4}\) on the right.

But this is super easy to integrate. Recalling that \(2 + \frac{1}{4} = \frac{9}{4}\),

\[ \int_0^1 \sqrt{x + \frac{9}{4}} \, dx = \frac{(x + \frac{9}{4})^{3/2}}{3/2} \bigg|_0^1 = \frac{(13/4)^{3/2} - (9/4)^{3/2}}{3/2} \]

You could simplify that last quantity a bit, but why bother? However, do remember to multiply by \(2\pi\) at the end.
Problem 3. Compute the $n$th Taylor polynomial $T_{n,a}$ at the point $a$ of the function $f(x)$:

1. $T_{4,0}$ of $f(x) = \cosh 3x$.
2. $T_{2,1}$ of $f(x) = \sqrt{1 + 2x}$.

Solution 3.

1. We know we’re going to need up to the fourth derivative of $\cosh 3x$ to write this polynomial, so let’s summarise this all in a chart. In case you forgot, you would want to remember that the derivative of $\cosh x$ is $\sinh x$ and the derivative of $\sinh x$ is $\cosh x$ again. You would also want to remember that $\cosh(0) = 1$ and $\sinh(0) = 0$ (same as $\cos(0)$ and $\sin(0)$ if that helps you out).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(0)$</th>
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<tbody>
<tr>
<td>0</td>
<td>$\cosh 3x$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$3 \sinh 3x$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$9 \cosh 3x$</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>$27 \sinh 3x$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$81 \cosh 3x$</td>
<td>81</td>
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Recalling that the general term of our Taylor polynomial is

$$
\frac{f^{(n)}(a)(x-a)^n}{n!}
$$

(with $a = 0$ here) we obtain

$$
T_{4,0}(x) = 1 + 0 + \frac{9x^2}{2!} + 0 + \frac{81x^4}{4!} = 1 + \frac{9}{2}x^2 + \frac{27}{8}x^4
$$

(2) Same story here, but with up to only the second derivative and $a = 1$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(1)$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>$\sqrt{1 + 2x}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>1</td>
<td>$(1 + 2x)^{-1/2}$</td>
<td>$1/\sqrt{3}$</td>
</tr>
<tr>
<td>2</td>
<td>$-(1 + 2x)^{-3/2}$</td>
<td>$-1/\sqrt{27}$</td>
</tr>
</tbody>
</table>

And then we put it together

$$
T_{2,1}(x) = \sqrt{3} + \frac{x - 1}{\sqrt{3}} + \frac{-(x - 1)^2}{2 \cdot \sqrt{27}}
$$

It might be worth noting that $\sqrt{27} = 3\sqrt{3}$, but it doesn’t really matter for the purpose of this problem.
Problem 4. Determine whether the following improper integrals are convergent.

(1) \[ \int_1^{\infty} \frac{1}{1 + x + x^2} \, dx \]

(2) \[ \int_0^1 \frac{-\ln x}{x^2} \, dx \]

Solution 4.

(1) This is the easier of the two. You can tell why this is improper – there’s an \( \infty \) there. We can try our usual trick when we have an integral with 1 on top and a polynomial on bottom:

\[ 1 + x + x^2 \geq x^2 \implies \frac{1}{1 + x + x^2} \leq \frac{1}{x^2} \]

This works because \( x \geq 0 \). This implies

\[ \int_1^{\infty} \frac{1}{1 + x + x^2} \, dx \leq \int_1^{\infty} \frac{1}{x^2} \, dx \]

But the right side we know converges using the \( p \)-test! That means that the left side converges as well when we include the fact that

\[ 0 < \int_1^{\infty} \frac{1}{1 + x + x^2} \, dx \]

(because \( 1/(1 + x + x^2) \) is positive on \([1, \infty)\)).

(2) This one is harder. The problem we have is at \( x = 0 \) because we are both dividing by zero and trying to find \( \ln(0) \) which isn’t defined.

We would like to say that

\[ \frac{-\ln x}{x^2} \geq \frac{1}{x^2} \implies \int_0^1 \frac{-\ln x}{x^2} \, dx \geq \int_0^1 \frac{1}{x^2} \, dx \]

That’s the natural comparison, because the right side diverges (again by the \( p \)-test) which would mean that the left side diverges too. But this inequality isn’t actually true, because \( -\ln x \geq 1 \) is not true on the whole of \((0, 1]\), the interval over which we’re integrating. It is true on \((0, 1/e]\), but how do you turn that into the answer? You can’t, really.

So we look for another strategy. We instead will try to do the real computation.

\[ \int_0^1 \frac{-\ln x}{x^2} \, dx = \lim_{R \to 0^+} \int_R^1 \frac{-\ln x}{x^2} \, dx \]
This admits a pretty easy integration by parts, actually. There isn’t another good idea for integration immediately available, so we turn to LIATE and use \( u = \ln x \) (but not the substitution – that doesn’t work). That makes \( dv = -1/x^2 \, dx \) so \( v = 1/x \) and \( du = 1/x \, dx \). Hence

\[
\int_{1/R}^{1} \frac{-\ln x}{x^2} \, dx = \ln x \bigg|_{1/R}^{1} - \int_{1/R}^{1} \frac{1}{x^2} \, dx = \ln x + 1 \bigg|_{R}^{1} = 1 - \frac{\ln R + 1}{R}
\]

The last step involves combining the two fractions. So we now need to compute

\[
\lim_{R \to 0^+} \frac{\ln R + 1}{R}
\]

If this limit exists, so does the integral, so that the integral converges. But this limit doesn’t exist, not because of L’Hôpital’s rule (which doesn’t apply here) but because ‘plugging in’ gives us \(-\infty/0\) which is just \(-\infty\). Therefore this integral diverges.