UCLA Geometry/Topology Qualifying Exam Solutions

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1 Spring 2014

Problem 1.
Let $\Gamma \subset \mathbb{R}^2$ be the graph of the function $y = |x|$.

(a) Construct a smooth function $f : \mathbb{R} \to \mathbb{R}^2$ whose image is Γ.

(b) Can f be an immersion?

Solution.

(a) Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth bump function so that $\operatorname{supp}\rho = \mathbb{R} \setminus [-1, 1]$ and $\rho(0) = 0$. Then define
$$f(x) = (\rho(x) \cdot x, \rho(x)|x|).$$
Then f is smooth, because it is smooth at 0 by construction and is piecewise smooth on $x < 0$ and $x > 0$. Further, $f(x)$ is always contained in Γ and is surjective, so this is the requisite function.

(b) Suppose we have such an immersion. Then locally it is the canonical immersion, i.e. there are local coordinates (x_1, x_2) around $0 \in \mathbb{R}^2$ so that Γ is equal to x_1. But this would require a diffeomorphism of a 'corner' in $\mathbb{R}^2$ to a line, which is not possible. That is, even though we could have a smooth function f, the 'corner' on Γ is a problem if we want f to be an immersion.

□

Problem 2.
Let W be a smooth manifold with boundary, and $f : \partial W \to \mathbb{R}^n$ a smooth map, for some $n \geq 1$. Show that there exists a smooth map $F : W \to \mathbb{R}^n$ such that $F|_{\partial W} = f$.

Solution.
We use the general ε-neighbourhood theorem. We know that W embeds in some Euclidean space $\mathbb{R}^N$, which gives an embedding of the boundaryless manifold ∂W into $\mathbb{R}^N$. Then there is a neighbourhood $\partial W^\varepsilon \subset \mathbb{R}^N$ of ∂W and a submersion $\pi : \partial W^\varepsilon \to \partial W$ that is the identify on ∂W. In particular, we have
$$\partial W^\varepsilon = \{x \in \mathbb{R}^N : |x - y| < \varepsilon(y) \text{ for some } y \in \partial W\}, \quad \varepsilon : \partial W \to \mathbb{R} \text{ smooth}.$$

Consider a bump function ρ that is supported on ∂W^ε and has $\rho(y) = 1$ for all $y \in \partial W$. Then define

$$F : \mathbb{R}^N \to \mathbb{R}^n, \quad F(x) = \begin{cases} 0 & x \notin \partial W^\varepsilon \\ \rho(x) \cdot f(\pi(x)) & x \in \partial W^\varepsilon \end{cases}$$

This is smooth by construction, and has $F|_{\partial W} = f$. The restriction of F to $W \subset \mathbb{R}^N$ is exactly the function we want. □

Problem 3.
Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Determine the values of $n \geq 0$ for which the antipodal map $S^n \to S^n$ is isotopic to the identity.

Solution.
This only occurs when n is odd. We know that the degree of the antipodal map is $(-1)^{n+1}$, therefore it is only possible when n is odd. Further, we can construct this isotopy explicitly for S^{2n-1}. If we view $\mathbb{R}^{2n}$ as the direct sum of n planes, i.e. $\mathbb{C}^n$, then the antipodal map is the same as rotating each copy of $\mathbb{C}$ through by π radians. Therefore the isotopy is given by rotating back. □

Problem 4.
Let $\omega_1, \ldots, \omega_k$ be 1-forms on a smooth n-dimensional manifold M. Show that $\{\omega_i\}$ are linearly dependent if and only if

$$\omega_1 \wedge \cdots \wedge \omega_k \neq 0.$$

Solution.
If the ω_i are linearly dependent, then without loss of generality we can write

$$\omega_k = \sum_{i=1}^{k-1} a_i \omega_i.$$

Then

$$\begin{aligned} \omega_1 \wedge \cdots \wedge \omega_k &= \omega_1 \wedge \cdots \wedge \left(\sum_{i=1}^{k-1} a_i \omega_i \right) \\ &= \sum_{i=1}^{k-1} a_i \, \omega_1 \wedge \cdots \wedge \omega_i \\ &= \sum_{i=1}^{k-1} (-1)^{k-(i+1)} a_i \, \omega_1 \wedge \cdots \widehat{\omega_i} \wedge \cdots \wedge \omega_i \wedge \omega_i \end{aligned}$$

Since for 1-forms we have $\omega \wedge \omega = 0$, we have

$$\sum_{i=1}^{k-1} (-1)^{k-(i+1)} a_i \, \omega_1 \wedge \cdots \widehat{\omega_i} \wedge \cdots \wedge 0 = 0.$$

Conversely, suppose that the ω_i are linearly independent. Then the dual vector fields associated to these, say V_i, are linearly independent as well. We can find local coordinates such that $\frac{\partial}{\partial x^i} = V_i$ for all i. Then clearly $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$ in these local coordinates. Since this works in any chart in M, the form cannot be identically zero. $\square$

Problem 5.
Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$, where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

Solution.
We know that $S \in H_1(M)$ is a cycle. We can describe $S \in H^1(M)^*$ in the following way: for $\omega \in H^1(M)$, there is (by Poincaré duality) a form η such that

$$\int_S \omega = \int_M \omega \wedge \eta.$$

We want to construct this (unique) η. Since $H^1(M) \cong \mathbb{R}^2$, we can let $\omega = a\theta_1 + b\theta_2$ for $a, b \in \mathbb{R}$, where θ_1 and θ_2 are 1-forms on the component copies of S^1 in M which have integral 1. Then

$$\int_S \omega = \int_S a\theta_1 + b\theta_2 = 3\int_{S^1} a\theta_1 + 7\int_{S^1} b\theta_2 = 3a + 7b.$$

The second equality holds since S is a 3-fold wrapping of one of the copies of S^1 and a 7-fold wrapping of the other. Hence it is fairly clear that $\eta = 7\theta_2 + 3\theta_1$. To see this,

$$\int_M \omega \wedge \eta = \int_M 3a\theta_1 \wedge \theta_2 + 7b\theta_1 \wedge \theta_2 = 3a + 7b.$$

$\square$

Problem 6.
Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, equipped with the round metric g_S (the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$). Consider also the hyperplane $H = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+!}$ equipped with the Euclidean metric g_H. Any line passing through the North Pole $p = (0, \ldots, 0, 1)$ and another point $A \in S^n$ will intersect this hyperplane in a point A'. The map

$$\Psi : S^n \setminus \{p\} \to H, \quad \Psi(A) = A'$$

is called the stereographic projection. Show that Ψ is conformal, i.e. for any $x \in S^n \setminus \{p\}$, the bilinear form $(g_S)_x$ is a multiple of the bilinear form $\Psi^*((g_H)_{\Psi(x)})$.

Solution.
The inclusion of a Riemannian geometry problem is very inconsiderate, so this problem will remained unanswered on principle. $\square$

Problem 7.
Let X be the wedge sum $S^1 \vee S^1$. Give an example of an irregular covering space $\tilde{X} \to X$.

Solution.
Recall that an irregular covering space corresponds to a subgroup of $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ which is not normal. Consider p.58 of Hatcher for actual pictures. Picture (13) there corresponds to the subgroup $\langle ab \rangle$ of $\mathbb{Z} * \mathbb{Z}$. This is not normal because it does not contain, in particular, the conjugate ba. □

Problem 8.
For $n \geq 2$, let X_n be the space obtain from a regular $(2n)$-gon by identifying the opposite sides with parallel orientations.

(a) Write down the cellular complex associated to this description.

(b) Show that X_n is a surface, and find its genus.

Solution.

(a) X_n has one 2-cell A, n 1-cells $e_1, \ldots, e_n$, and either 1 or 2 0-cells a and (possibly) b depending on the parity of n. We get 2 0-cells for odd n because different vertices can lie on the vertical line of symmetry, but this is not possible for even n. Therefore we split up into even and odd n:

For even n, each 1-cell is glued to the same vertex, so we have $\ker \partial_1 = \mathbb{Z}^n$. Since $\partial_2(A) = 0$, we have $H_1(X_n) = \mathbb{Z}^n$ and $H_2(X_n) = \mathbb{Z}$. Clearly $H_0(M) = \mathbb{Z}$ as well.

For odd n, we still have $H_2(M) = H_0(M) = \mathbb{Z}$. However, we have $\partial_1(e_i) = a - b$ or $b - a$ in differing amounts. Without loss of generality, there are $(n+1)/2$ which have $a - b$ and $(n-1)/2$ of $b - a$. Renumber the e_i in this order. We claim that $\dim \ker \partial_1 = n - 1$. To see this, we look at a new basis for C_i given by $e_i - e_{i+1}$. All of these successive differences have $\partial_1(e_i - e_{i+1}) = 0$ except for the one which has $e_{(n+1)/2}$ and $e_{(n+1)/2+1}$, because these have differing types. This shows $H_1(M) = \mathbb{Z}^{n-1}$.

(b) The above shows this is a surface since $H_2(M) = \mathbb{Z}$ in all cases. We know that, for an orientable genus g surface M_g, we have $\chi(M_g) = 2 - 2g$. Therefore we have the relation

$$\chi(X_n) = 1 - \operatorname{rk} H_1(X_n) + 1 = 2 - 2g \implies \operatorname{rk} H_1(X_n) = 2g.$$

The genus of these surfaces is $\frac{1}{2} \operatorname{rk} H_1(X_n)$, which is $n/2$ and $(n-1)/2$ depending on whether n is even or odd.

□

Problem 9.

(a) Consider the space Y obtained from $S^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also identifying $(x, 1)$ with $(-x, 1)$ for all $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.

(b) Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Solution.

(a) Since $\partial(S^2 \times [0,1]) = S^2 \sqcup S^2$, it is clear that $\partial Y = \mathbb{RP}^2 \sqcup \mathbb{RP}^2$. The antipodal identification of S^2 is precisely how $\mathbb{RP}^2$ is obtained. Further, we know that $\mathbb{RP}^3$ is obtained by gluing a 3-cell appropriately to $\mathbb{RP}^2$. Therefore $\mathbb{RP}^3$ less an open D^3, which is the first step in taking the connected sum, is homeomorphic to $\mathbb{RP}^2$ if we choose the appropriate disc. Therefore $\mathbb{RP}^3 \# \mathbb{RP}^3$ should look like two copies of $\mathbb{RP}^2$ connected by the 3-tube, which is $S^2 \times [0,1]$. This is exactly the case.

(b) Consider $S^1 = [0,1) = \{0\} \cup (0,1/2) \cup \{1/2\} \cup (1/2,1)$ for an appropriate parametrisation. Then define the covering map p in the following way: let $S^2 \times \{0\}$ 2-cover the $\{0\}$ copy of $\mathbb{RP}^2$; let $S^2 \times (0,1/2)$ 1-cover the tube $S^2 \times (0,1)$; let $S^2 \times \{1/2\}$ 2-cover the $\{1\}$ copy of $\mathbb{RP}^2$; and finally let $S^2 \times (1/2,1)$ 1-cover the tube a second time. This gives a smooth 2-cover of $\mathbb{RP}^3 \# \mathbb{RP}^3$.

□

Problem 10.
Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

Solution.
We can construct a nice Mayer-Vietoris sequence for this space. Consider $A = X \times [0,3/4)/\sim$ and $B = X \times (1/4,1]/\sim$. Then $A \cup B = S(X)$, and $A \cap B = X \times (1/4,3/4) \simeq X$. Further, since A and B are just cones, they deformation retract onto their vertex, i.e. the point $X \times \{0\}$ or $X \times \{1\}$. Hence we have a sequence

$$\cdots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(S(X)) \to H_{i-1}(A \cap B) \to \cdots$$

which, in terms of what we know, gives us

$$\cdots \to 0 \to H_i(S(X)) \to H_{i-1}(X) \to 0 \to \cdots$$

in each degree for $i > 1$. However when $i = 1$, $H_i(A) = H_i(B) = \mathbb{Z}$ (assuming $\mathbb{Z}$-coefficients), so we obtain

$$0 \to H_1(S(X)) \to H_0(X) \to \mathbb{Z}^2 \to H_0(S(X)) \to 0.$$

We know that $S(X)$ is connected even if X is not, because $S(X)$ is path connected. Therefore $H_0(S(X)) = \mathbb{Z}$. The surjection $\mathbb{Z}^2 \to H_0(S(X))$ means that this map has kernel isomorphic to $\mathbb{Z}$, whence $H_0(X) \to \mathbb{Z}^2$ has image isomorphic to $\mathbb{Z}$. Factoring through that image, we obtain the exact sequence

$$0 \to H_1(S(X)) \to H_0(X) \to \mathbb{Z} \to 0$$

which is, in particular, split. Therefore $H_0(X) = \mathbb{Z} \oplus H_1(S(X))$. This gives us

$$H_i(S(X)) = H_{i-1}(X) \text{ for } i > 1, \quad H_1(S(X)) \oplus \mathbb{Z} = H_0(X), \quad H_0(S(X)) = \mathbb{Z}.$$

□

2 Fall 2013

Problem 1.
Let $f : M \to N$ be a nonsingular smooth map between connected manifolds of the same dimension. Answer the following questions with a proof or counterexample.

(a) Is f necessarily injective or surjective?

(b) Is f necessarily a covering map when N is compact?

(c) Is f necessarily an open map?

(d) Is f necessarily a closed map?

Solution.
We know that f is nonsingular if and only if f is a local diffeomorphism for manifolds of the same dimension. We will use this criterion to answer the questions.

(a) f does not need to be surjective. The inclusion of the open unit ball in $\mathbb{R}^n$ into $\mathbb{R}^n$ itself is a local diffeomorphism but is not surjective.

f must be injective, however. Suppose $f(x) = f(y)$. We know that there is a neighbourhood U_x of x such that $f|_{U_x}$ is a diffeomorphism, and a U_y with the same property. We claim that $y \in U_x$. Indeed, because f is a local diffeomorphism on $U_x \cap U_y$, we know that $f^{-1}(f(U_x)) = U_x$. In particular, since $f(x) = f(y) \in f(U_x)$, we have

$$f^{-1}(f(y)) = y \in U_x = f^{-1}(f(U_x)).$$

Therefore since $f(x) = f(y)$ on $U_x \cap U_y$, on which f is a local diffeomorphism, we have $x = y$.

(b) No. Consider the map $f : S^1 \vee S^1 \to S^1$ which 'folds' the circles onto each other. Then if we let $x \in S^1 \vee S^1$ be the wedge point, we know that every point in $S^1 \vee S^1 \setminus \{x\}$ has a preimage with two points, and at at those points f is a two-sheeted covering map. However, this is not the case at x, where there is only one preimage point. Therefore we do not have a covering map.

(c) Yes. Because it is a local diffeomorphism, at any point $x \in M$, we have a neighbourhood U_x so that $f(U_x) \cong U_x$. Therefore every point $f(x)$ in the image has a neighbourhood $f(U_x)$ contained in the image, so the map is open.

(d) No. If we take the inclusion from part (a), the image of the open unit ball is not closed in $\mathbb{R}^n$. However, it is a closed set with respect to itself, so this map is not closed.

□

Problem 2.
Let M be a connected compact manifold with nonempty boundary ∂M. Show that M does not retract onto ∂M.

Solution.
See below, Spring 2013 #5. □

Problem 3.
Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds of dimensions m and n, respectively, such that $m+n=p$. Suppose that $M \cap N = \varnothing$. Consider the linking map

$$\lambda : M \times N \to S^p, \quad \lambda(x,y) = \frac{x-y}{\|x-y\|}.$$

The degree of λ is called the linking number $l(M,N)$.

(a) Show that $l(M,N) = (-1)^{(m+1)(n+1)} l(N,M)$.

(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from N, then $l(M,N) = 0$.

Solution.

(a) Let z be a regular value of λ, and let $\lambda^{-1}(z) = \{(x_i, y_i) : i = 1, \dots, k\}$. We know that $l(M,N)$ is the sum of the relative orientations at each (x_i, y_i). Then in the product $N \times M$, we know that $\lambda(y_i, x_i) = -z$, the antipodal point of z. We know that the relative orientation of (y_i, x_i) with respect to (x_i, y_i) is $(-1)^{n \cdot m}$. However, the relative orientation of z to $-z$ is $(-1)^{p+1} = (-1)^{m+n+1}$, and the relative orientation of (y_i, x_i) with respect to $-z$ is $(-1)^{n \cdot m+1}$. Therefore the total relative orienation is $(-1)^{n \cdot m+m+n+1}$. Therefore since we may applying the correcting term $(-1)^{(m+1)(n+1)}$ to each term in the sum giving us $l(N,M)$, we are done.

(b) We use the following proposition (as in Fall 2011, #4): suppose $f : X \to Y$ is a smooth map of compact oriented manifolds of the same dimension and $X = \partial W$ for a compact W. If f can be extended to all of W, then $\deg f = 0$. In our case, let $f = \lambda$. Then we see that λ is a map between smooth oriented manifolds of the same dimension (since the product of compact manifolds is compact). Further, if we have $M = \partial W$, then

$$\partial(W \times N) = (\partial W) \times N \sqcup W \times (\partial N) = M \times N$$

since $\partial N = \varnothing$. Therefore all we need to do is show that λ can be extended to $W \times N$ to finish the proof. But since we have assumed that $W \cap N = \varnothing$, we never have $\|x-y\| = 0$ for $(x,y) \in W \times N$, so λ is still well defined. Hence $\deg \lambda = 0$, so $l(M,N) = 0$.

□

Problem 4.
Let ω be a 1-form on a connected manifold M. Show that ω is exact if and only if for all piecewise smooth closed curves $c : S^1 \to M$ it follows that $\int_c \omega = 0$.

Solution.
See Spring 2013, #2(b). □

Problem 5.
Let ω be a smooth, nowhere vanishing 1-form on a three-dimensional smooth manifold M^3.

(a) Show that $\ker \omega$ is an integrable distribution on M if and only if $\omega \wedge d\omega = 0$.

(b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.

Solution.

(a) We use that a distribution Δ is integrable if and only if it is closed under the Lie bracket if and only if the annihilator of Δ, which we call $I(\Delta)$, is closed under the exterior differential. $I(\Delta)$ is defined by

$$I(\Delta) = \{\eta \in \Omega(M) : i_v(\eta) = 0 \text{ for all } v \in \Delta\}.$$

This is an ideal in $\Omega(M)$. Suppose that $\omega \wedge d\omega = 0$. Suppose that $x, y \in \ker \omega$. Then we know that

$$d\omega(x, y) = \omega(x) - \omega(y) - \omega([x, y]) = -\omega([x, y])$$

Additionally,

$$\begin{aligned}(\omega \wedge d\omega)(x, y, v) &= (d\omega \wedge \omega)(x, y, v) \\ &= \omega(v) \cdot d\omega(x, y).\end{aligned}$$

Since this has to hold for any vector v in the third argument, this implies $d\omega(x, y) = 0$. As such, we must have had $\omega([x, y]) = 0$ above, so that $[x, y] \in \ker \omega$. This proves one direction.

For the converse, assume $I(\Delta)$ is closed under exterior differentiation. Then since $\omega \in I(\Delta)$, $d\omega \in I(\Delta)$. Therefore consider any linearly independent vectors x, y, z. Then we may assume that $x, y \in \ker \omega$, since up to a linear change this is true. Further, assume we can (locally) write $d\omega = \eta_1 \wedge \eta_2$ for ease of notation (linearity means this assumption is okay). Thus

$$\begin{aligned}(\omega \wedge d\omega)(x, y, z) &= \begin{vmatrix} \omega(x) & \omega(y) & \omega(z) \\ \eta_1(x) & \eta_1(y) & \eta_1(z) \\ \eta_2(x) & \eta_2(y) & \eta_2(z) \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & \omega(z) \\ \eta_1(x) & \eta_1(y) & \eta_1(z) \\ \eta_2(x) & \eta_2(y) & \eta_2(z) \end{vmatrix} \\ &= \omega(z) \cdot \begin{vmatrix} \eta_1(x) & \eta_1(y) \\ \eta_2(x) & \eta_2(y) \end{vmatrix} = \omega(z) \cdot d\omega(x, y) = 0.\end{aligned}$$

Therefore $\omega \wedge d\omega = 0$.

(b) We need to pick a nowhere vanishing 1-form on $\mathbb{R}^3$ that does not satisfy $\omega \wedge d\omega = 0$. We can express this globally in terms of dx, dy, dz. Let

$$\omega = -\frac{y}{2}\, dx + \frac{x}{2}\, dy + dz.$$

Then

$$\begin{aligned}\omega \wedge d\omega &= \left(-\frac{y}{2}\,dx + \frac{x}{2}\,dy + dz\right) \wedge (dx \wedge dy)\\ &= dx \wedge dy \wedge dz.\end{aligned}$$

This is a volume form on $\mathbb{R}^3$ and therefore not zero.

□

Problem 6.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

(a) Define the gradient ∇f as a vector field dual to the differential df.

(b) Define the Hessian $\operatorname{Hess} f(X, Y)$ as a symmetric $(0, 2)$-tensor.

(c) If the usual Euclidean inner product between tangent vectors in $T_p\mathbb{R}^n$ is denoted $g(X, Y) = X \cdot Y$, show that

$$\operatorname{Hess} f(X, Y) = \frac{1}{2}(\mathcal{L}_{\nabla f}\, g)(X, Y).$$

Solution.

(a) Classically, given local coordinates $x_1, \ldots, x_n$, we define

$$\nabla f = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)$$

The differential df is defined to be (in these coordinates)

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\, dx^i.$$

Because we have dual bases $\frac{\partial}{\partial x^i}$ and dx^i, the duality is clear.

(b) Since we are in $\mathbb{R}^n$, we can express vector fields by

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x^i}.$$

Then we define

$$\operatorname{Hess} f(X, Y) := \left(a_1 \frac{\partial}{\partial x^1}, \ldots, a_n \frac{\partial}{\partial x^n}\right) \cdot H_f \cdot \left(b_1 \frac{\partial}{\partial x^1}, \ldots, b_n \frac{\partial}{\partial x^n}\right)^t$$

where $H_f \in M_n(\mathbb{R})$ is defined by

$$(H_f)_{i,j} = \frac{\partial^2 f}{\partial x^i \partial y^i}.$$

Note that since mixed partials commute, this matrix is symmetric, so the tensor itself is symmetric.

(c) We can give g in local coordinates. Using the same notation as above,

$$g = \sum_{i=1}^{n} dx^i \otimes dx^i, \qquad g(X, Y) = \sum_{i=1}^{n} a_i \cdot b_i.$$

Then by the product rule,

$$\mathcal{L}_{\nabla f} g = \sum_{i=1}^{n} (\mathcal{L}_{\nabla f} dx^i) \cdot dx^i + dx^i \cdot (\mathcal{L}_{\nabla f} dx^i) = 2 \sum_{i=1}^{n} (\mathcal{L}_{\nabla f} dx^i) \cdot dx^i.$$

It is clear that $\mathcal{L}_{\nabla f} dx^i = d(\mathcal{L}_{\nabla f} x^i) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j$. Then all in all,

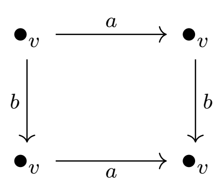
$$\sum_{i=1}^{n} (\mathcal{L}_{\nabla f} dx^i) \cdot dx^i = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dx^j.$$

But this was exactly our definition of the Hessian.

□

Problem 7.
Let $M = T^2 \setminus D^2$ be the complement of a disk inside the 2-torus. Determine all connected surfaces that can be described as 3-fold covers of M.

Solution.
These surfaces are determined, up to basepoint-preserving isomorphism, by index 3 subgroups of $\pi_1(M)$. Therefore it behooves us first to calculate $\pi_1(M)$. Take fundamental polygon of the torus:



If we remove a disk from this solid square, then we have a deformation retraction onto the hollow square. After the gluing, we are left with $S^1 \vee S^1$. Therefore 3-fold covers of M correspond to 3-fold covers of $S^1 \vee S^1$. These, in turn, correspond to subgroups of index 3 of $\mathbb{Z} * \mathbb{Z}$. We have the following formula for the number of subgroups of index k for a free group of rank r:

$$N(k, r) = k(k!)^{r-1} - \sum_{i=1}^{k-1} ((k-i)!)^{r-1} N(i, r).$$

In our case, this gives us $N(3, 2) = 4$. What these surfaces are is by no means clear from this description, unfortunately. □

Problem 8.
Let $n > 0$ be an integer and let A be an abelian group with a finite presentation by generators and relations. Show that there exists a topological space X with $H_n(X) \cong A$.

Solution.
We can construct a CW-complex satisfying what we want. Let $A = \mathbb{Z}^\ell/\langle r_1, \ldots, r_m\rangle$, i.e. it is generated by m elements with n relations. If $g_1, \ldots, g_\ell$ are the generators, we can write

$$r_j = g_{i_1}^{\varepsilon_1} \cdots g_{i_k}^{\varepsilon_k},$$

where each ε is ± 1. First, take the wedge of ℓ copies of S^n. This has $H_n(\bigvee^\ell S^n) = \mathbb{Z}^\ell$. Then, to this attach m $(n+1)$-cells according to the relations r_i. That is, the attaching maps $\partial D^{n+1} \to \bigvee^\ell S^n$ should be full covers of the appropriate copy of S^n in the order prescribed by the relation. Call this space X. By construction, we have precisely

$$H_n(X) = Z_n(X)/B_n(X) = \mathbb{Z}^m/\langle r_1, \ldots, r_m\rangle \cong A.$$

$\square$

Problem 9.
Let $H \subset S^3$ be the Hopf link, shown in the figure (which we cannot reproduce; it is two linked copies of S^1). Compute the fundamental group and the homology groups of the complement $S^3 \setminus H$.

Solution.
We will prove this using the Mayer-Vietoris sequence. Let A and B both be diffeomorphic to $S^3 \setminus S^1$, but with the removed S^1 in such a way that $A \cap B = S^3 \setminus H$. We know that $H_i(S^3) = \mathbb{Z}$ for $i = 0, 3$ and is trivial otherwise. Further, we know $H_i(S^3 \setminus S^1) = \mathbb{Z}$ of $i = 0, 1, 3$ and is trivial otherwise. We know that $H_i(S^3 \setminus S^1) = \mathbb{Z}$ because $\pi_1(S^3 \setminus S^1) = \mathbb{Z}$. To demonstrate this, take any loop in $S^3 \setminus S^1$. Then we can deform the loop into a neighbourhood of S^1 which locally looks like $\mathbb{R}^3 \setminus S^1$. Then any loop 'wrapping around' the removed S^1 is determined by the winding number of the loop around S^1, and any loop not wrapping around S^1 is nulhomotopic.

First using homology, we have $H_0(S^3 \setminus H) = \mathbb{Z}$. The rest of the sequence is

$$0 \to H_3(S^3 \setminus H) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to H_2(S^3 \setminus H) \to 0 \to 0 \to H_1(S^3 \setminus H) \to \mathbb{Z} \oplus \mathbb{Z} \to 0$$

This shows that $H_1(S^3 \setminus H) = \mathbb{Z}^2$ since that part of the sequence must be an isomorphism. Further, we see that $\mathbb{Z} \to H_2(S^3 \setminus H)$ is surjective. Finally, we know that $H_3(S^3 \setminus H) \cong \mathbb{Z}$ since $S^3 \setminus H$ is orientable with one component, and the map $H_3(S^3 \setminus H) \to \mathbb{Z} \oplus \mathbb{Z}$ is given by the diagonal map $1 \mapsto (1, 1)$. Since $\operatorname{im}(\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}) = (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}(1, 1) \cong \mathbb{Z}$, that map is surjective. Therefore the image of $\mathbb{Z} \to H_2(S^3 \setminus H)$ is zero, so $H_2(S^3 \setminus H) = 0$. Therefore

$$H_i(S^3 \setminus H) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}^2 & i = 1 \\ 0 & \text{else} \end{cases}$$

To find $\pi_1(S^3 \setminus H)$, we know that $\pi_1(S^3)$ is the pushout of the diagram

$$\begin{array}{ccc} \pi_1(S^3 \setminus H) & \longrightarrow & \pi_1(S^3 \setminus S^1) \\ \downarrow & & \\ \pi_1(S^3 \setminus S^1) & & \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \pi_1(S^3 \setminus H) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & 0 \end{array}$$

This shows that we must have $\pi_1(S^3 \setminus H) \cong \mathbb{Z} \times \mathbb{Z}$ because the universal property of the pushout in this case aligns with the universal property of the product. □

Problem 10.
Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the group of quaternions. The multiplicative group $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$ acts on $\mathbb{H}^n \setminus \{0\}$ by left multiplication. The quotient $\mathbb{HP}^{n-1} = (\mathbb{HP}^n \setminus \{0\})/\mathbb{H}^*$ is called quaternionic projective space. Calculate its homology groups.

Solution.
Believing in our hearts that this space should look like $\mathbb{CP}^n$, we try to define a cell structure recursively. The construction should follow in the same way as the construction of $\mathbb{CP}^n$ as in Spring 2009, #7. This would give us cells only in dimension $4k$ for $0 \leq k \leq n$. Then all of our boundary maps are zero, and the chain complex is actually exact. Thus

$$H_i(\mathbb{HP}^n) = C_i(\mathbb{HP}^n) = \begin{cases} \mathbb{Z} & i = 4k, 0 \leq k \leq n \\ 0 & \text{else} \end{cases}$$

□

3 Spring 2013

Problem 1.
Let $M_{m\times n}(\mathbb{R})$ be the space of $m \times n$ matrices with real-valued entries.

(a) Show that the subset $S \subset M_{m\times n}(\mathbb{R})$ of rank 1 matrices form a submanifold of dimension $m + n - 1$.

(b) Show that the subset $T \subset M_{m\times n}(\mathbb{R})$ of rank k matrices form a submanifold of dimension $k(m + n - k)$.

Solution.
(a) This will follow from part (b).

(b) See Guillemin and Pollack, Problem 1.14.13. Note that $k(m+n-k) = mn-(m-k)(n-k)$. Also note that for $k > 0$, having rank $> k$ is an open condition: perturbations do not affect linearly independent rows, but can affect linearly dependent rows, thereby increasing the rank. We can swap the rows of any $m \times n$ matrix with rank at least k to obtain the form

$$A = \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right),$$

where $B \in \mathrm{GL}_k(\mathbb{R})$ is invertible and the other dimensions follow from that. We may postmultiply by the matrix

$$\left(\begin{array}{c|c} I' & -B^{-1}C \\ \hline 0 & I' \end{array}\right)$$

where I' is the matrix with 1 in every diagonal entry and 0 elsewhere, depending on its dimensions, so that this matrix has full rank. The multiplication yields

$$\left(\begin{array}{c|c} B & 0 \\ \hline D & E - DB^{-1}C \end{array}\right) \overset{\text{row reduces to}}{\Longrightarrow} \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & E - DB^{-1}C \end{array}\right) = A'$$

This gives us a smooth map from matrices of rank at least k to $M_{(m-k)\times(n-k)}(\mathbb{R})$, where $A \mapsto E - DB^{-1}C$, which we call F. If A was a rank k matrix, then we must have $E - DB^{-1}C = 0$ since $\operatorname{rk} A = \operatorname{rk} A'$, so $T = F^{-1}(0)$. We need to show that F is a submersion at 0 to complete the proof. If we show this, then by the preimage theorem, T is a submanifold of codimension $(m-k)(n-k) = \dim M_{(m-k)\times(n-k)}(\mathbb{R})$.

Now, we need to show that dF_A has full rank for every A with rank exactly k. Further, we know that the tangent space to 0 in $M_{(m-k)\times(n-k)}(\mathbb{R})$ is isomorphic to $M_{(m-k)\times(n-k)}(\mathbb{R})$ itself. Therefore we need to find a path through a matrix of rank at least k that maps to every $X \in M_{(m-k)\times(n-k)}(\mathbb{R})$. But this is easy: we can define a path through A as above by

$$\gamma(t) = \left(\begin{array}{c|c} B & C \\ \hline D & E + tX \end{array}\right).$$

Then $F(\gamma(t)) = E + tX - DB^{-1}C$, hence the derivative of this path at $t = 0$ is just X.

□

Problem 2.
Let M be a smooth manifold and $\omega \in \Omega^1(M)$ a smooth 1-form.

(a) Define the line integral

$$\int_c \omega$$

along a piecewise smooth curve $c : [0,1] \to M$.

(b) Show that $\omega = df$ for a smooth function $M \to \mathbb{R}$ if and only if $\int_c \omega = 0$ for all closed curves $c : [0,1] \to M$.

Solution.
(a) We define

$$\int_c \omega := \int_0^1 \omega(c'(t))\, dt,$$

where this is the usual calculus integral.

(b) If $\omega = df$, then Stokes' theorem tells us that

$$\int_c \omega = \int_c df = \int_{\partial c} f = f(c(1)) - f(c(0)) = 0.$$

The last equality follows because c is a closed curve. Conversely, suppose that ω integrates to 0 along every c. Then we can construct f for this circumstance: without loss of generality, let M be connected, as this construction is independent on path components of M. Let p be some basepoint, and let $f(p) = 0$. Define

$$f(q) := \int_\gamma \omega,$$

where $\gamma : [0,1] \to M$ is a path between p and q. We claim that f does not actually depend on γ; suppose that γ' is another path. Then let $\bar{\gamma}'$ be the reverse path. Then $\bar{\gamma}' \circ \gamma$ is a closed loop at p, hence

$$0 = \int_{\bar{\gamma}' \circ \gamma} \omega = \int_\gamma \omega - \int_{\gamma'} \omega \implies \int_\gamma \omega = \int_{\gamma'} \omega.$$

Now we show that $df = \omega$. But this is true by the fundamental theorem of calculus (or otherwise totally obvious).

$\square$

Problem 3.
Let $S_1, S_2 \subset M$ be smooth embedded submanifolds.

(a) Define what it means for S_1, S_2 to be transversal.

(b) Show that if $S_1, S_2 \subset M$ are transversal, then $S_1 \cap S_2 \subset M$ is a smooth embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.

Solution.

(a) Two submanifolds are transversal if, for every $x \in S_1 \cap S_2$, we have

$$T_x S_1 + T_x S_2 = T_x M.$$

Note that this sum is not a direct sum, and in particular if $S_1 \cap S_2 = \varnothing$, then the manifolds are trivially transversal.

(b) This is not true if $S_1 \cap S_2 = \varnothing$, since $\dim \varnothing = -1$, but this case is unimportant.

Otherwise, let $\dim S_1 = n_1$, $\dim S_2 = n_2$, and $\dim M = m$. Let $x \in S_1 \cap S_2$. Then near x we have an neighbourhood $U_1 \subset S_1$ so that U_1 is the zero set of some functions $f_1, \dots, f_{m-n_1}$. Similarly, we have a neighbourhood $U_2 \subset S_2$ which is the zero set of functions $g_1, \dots, g_{m-n_2}$. Therefore a neighbourhood of $x \in S_1 \cap S_2$, given by $U_1 \cap U_2$, is the vanishing set of $(m - n_1) + (m - n_2)$ functions. This means that a neighbourhood of x is a manifold, and since this holds for every $x \in S_1 \cap S_2$, gluing these together gives a manifold structure. Further, we have

$$\begin{aligned} \operatorname{codim}(S_1 \cap S_2) = (m - n_1) + (m - n_2) &\implies m - \dim(S_1 \cap S_2) = 2m - n_1 - n_2 \\ &\implies \dim(S_1 \cap S_2) = n_1 + n_2 - m, \end{aligned}$$

which is what we wanted to show.

□

Problem 4.
Let $C \subset M$ be given as $F^{-1}(c)$, where $F = (F^1, \dots, F^k) : M \to \mathbb{R}^k$ is smooth and $c \in \mathbb{R}^k$ is a regular value. If $f : M \to \mathbb{R}$ is smooth, show that its restriction $f|_C$ to $C \subset M$ has a critical point at $p \in C$ if and only if there exist constants $\lambda_1, \dots, \lambda_k$ such that

$$df_p = \sum \lambda_i dF_p^i$$

where $dg_p : T_pM \to \mathbb{R}$ denotes the differential at p of g.

Solution.
Suppose that $df_p = \sum \lambda_i dF_p^i$. Then since F is constant on C, we have $(dF^i|_C)_p = 0$ identically for any $p \in C$. Hence $(df|_C)_p = 0$ as well, so p is a critical point.

Conversely, suppose that $p \in C$ is a critical point of $f|_C$. Then we know that df_p can only be nonzero on directions normal to C. Because C is the preimage submanifold of a regular value, we know that the dF_p^i form a basis for these directions. Therefore there must exist constants λ expressing df_p in terms of dF_p^i. □

Problem 5.
Let M be a smooth, orientable, compact manifold with boundary ∂M. Show that there is no (smooth) retract $r : M \to \partial M$.

Solution.
Without loss of generality, we may assume M and ∂M are connected. Consider the long exact sequence of the good pair $(M, \partial M)$:

$$0 \to H_n(\partial M) \to H_n(M) \to H_n(M, \partial M) \xrightarrow{\delta} H_{n-1}(\partial M) \to H_{n-1}(M) \to H_{n-1}(M, \partial M) \to \cdots$$

We claim that the map $H_{n-1}(\partial M) \to H_{n-1}(M)$ is trivial, which holds if and only if the connecting homomorphism δ is surjective. Because M is a compact orientable manifold with boundary, we have $H_n(M) = 0$. Since ∂M is a compact orientable manifold without boundary, we have $H_{n-1}(\partial M) = \mathbb{Z}$. Further $H_n(\partial M) = 0$ since $\dim \partial M = n - 1$ This gives

$$0 \to H_n(M, \partial M) \xrightarrow{\delta} H_{n-1}(\partial M) \to H_{n-1}(M) \to H_{n-1}(M, \partial M) \to \cdots$$

Since identifying the boundary of M gives a compact orientable manifold without boundary, we have $H_n(M, \partial M) = \mathbb{Z}$. Hence δ is surjective (since it is a priori injective). This shows that

$$\ker(H_{n-1}(\partial M) \to H_{n-1}(M)) = \operatorname{im} \delta = H_{n-1}(\partial M).$$

Since $H_{n-1}(\partial M) = \mathbb{Z}$, this map is not injective. But this is exactly the map that a retraction $r : M \to \partial M$ would induce, so no such retraction can exist. □

Problem 6.
Let $A \in \mathrm{GL}_{n+1}(\mathbb{C})$.

(a) Show that A defines a smooth map $A : \mathbb{CP}^n \to \mathbb{CP}^n$.

(b) Show that the fixed points of $A : \mathbb{CP}^n \to \mathbb{CP}^n$ correspond to eigenvectors for the original matrix.

(c) Show that $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is a Lefschetz map if the eigenvalues of A all have multiplicity 1.

(d) Show that the Lefschetz number of $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is $n + 1$. Hint: you are allowed to use that $\mathrm{GL}_{n+1}(\mathbb{C})$ is connected.

Solution.

(a) Since A is defined on $\mathbb{C}^{n+1}$, we need to show it behaves well via the identification of $x \sim \lambda x$ for $\lambda \in \mathbb{C}^\times$. First, since $\ker A = 0$, A descends to a map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$. Second, suppose that $A(x) = y$. Then $A(\lambda x) = \lambda^{n+1} y$. Since $\lambda^{n+1} y \in \mathbb{C}^\times$, we have $A(\lambda x) \sim A(x)$, which is necessary. Put another way, linear transformations preserve $\mathbb{C}$-linear subspaces, the identification of which is how we obtain $\mathbb{CP}^n$ from $\mathbb{C}^{n+1}$. Smoothness is obvious.

(b) Let A' be the original matrix now. Suppose that $A(x) = x$ in $\mathbb{CP}^n$. Then since $x \sim \lambda x$, we have that $A'(x') = \mu x'$, where x' is a preimage of x and $\mu \in \mathbb{C}^\times$. Therefore x' is an eigenvalue of A'. Conversely, if we have $y' \in \mathbb{C}^{n+1}$ so that $A'(y') = \mu y'$, then $A(y) = y$ for y the image of y'.

(c) Recall that a map $f : X \to X$ between compact manifolds is Lefschetz if graph f and the diagonal Δ are transversal in $X \times X$. Unpacked, this means that at every fixed point $f(x) = x$, $\mathrm{graph}(df_x) \cap \Delta_x = 0$, since these two submanifolds have complementary dimension. This means that df_x has no eigenvector of eigenvalue $+1$.

Therefore suppose that A' has eigenvalues of only multiplicity 1. Then A' is diagonalisable, and so up to a change of basis we have

$$A' = \mathrm{diag}(\lambda_0, \ldots, \lambda_n).$$

We need to look at all the fixed points of A, which are all of the form $[0 : \cdots : 1 : \cdots : 0]$. It suffices to examine $[1 : 0 : \cdots : 0]$. In a neighbourhood of this point, take local coordinates x. At some point $z = [1 : x_1 : \cdots : x_n]$, we have

$$A[1 : x_1 : \cdots : x_n] = [\lambda_0 : \lambda_1 x_1 : \cdots : \lambda_n x_i] = \left[1 : \frac{\lambda_1}{\lambda_0} x_1 : \cdots : \frac{\lambda_n}{\lambda_0} x_n\right].$$

Therefore

$$dA_z = \mathrm{diag}\left(\frac{\lambda_1}{\lambda_0}, \ldots, \frac{\lambda_n}{\lambda_0}\right).$$

This has no eigenvalue 1 if and only if $\lambda_0 \neq \lambda_i$ for all i. But this is exactly the case we are in, so the map is Lefschetz.

(d) We could write down the definition of the Lefschetz number, but we needn't. Recall that $\Lambda_{\text{id}} = \chi(X)$ for any space X. Further recall that $\chi(\mathbb{CP}^n) = n+1$, which is calculable (e.g.) by the sum of the Betti numbers:

$$\beta_i(\mathbb{CP}^n) = \begin{cases} 1 & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Finally, Lefschetz number is a homotopy invariant. Since $\mathrm{GL}_{n+1}(\mathbb{C})$ is connected, there is a path from A to id for any map A. If A is not Lefschetz, then since this property is generic, we do not have a problem. This path defines a smooth homotopy $A \sim \text{id}$. Hence $\Lambda_A = \chi(\mathbb{CP}^n) = n+1$. Put otherwise, since $\deg A' = +1$, we know there is a homotopy $A' \sim \text{id}_{S^{2n+1}}$. This homotopy descends to $\mathbb{CP}^n$.

□

Problem 7.
Let $F : S^n \to S^n$ be a continuous map.

(a) Define the degree $\deg F$ of F and show that when F is smooth,

$$\deg F \int_{S^n} \omega = \int_{S^n} F^*\omega$$

for all $\omega \in \Omega^n(S^n)$.

(b) Show that if F has no fixed points, then $\deg F = (-1)^{n+1}$.

Solution.
(a) We define $\deg F$ in the following way (which we have done elsewhere): F induces a map F_* on homology of S^n. Since $H_n(S^n) = \mathbb{Z}$, let $1 \in H_n(S^n)$ be a generator. Then we define $\deg F$ to be the image of 1 under F_*, i.e. $F_*(1) \in \mathbb{Z}$. We choose the generator so that $\deg \text{id} = 1$, so there is no ambiguity. Now,

$$\int_{S^n} F^*\omega = \int_{F_* S^n} \omega,$$

where we view S^n is an n-cycle in the domain S^n. Since $F_*(S^n)$ is an $(\deg F)$-fold cover of S^n, we have

$$\int_{S^n} F^*\omega = \int_{F_*(S_n)} \omega = \deg F \int_{S^n} \omega.$$

(b) We will show that $F \sim A$, the antipodal map. We know that $\deg A = (-1)^{n+1}$, since A is comprised of $n+1$ transpositions which each have degree -1. Consider a the map $h : [0,1] \times S^n \to \mathbb{R}^{n+1}$

$$h(t,x) = (1-t)F(x) + tA(x) = (1-t)F(x) - tx.$$

For each fixed x, $h(t,x)$ defines a line through S^n that avoids the origin, because we never have $F(x) = x$. Therefore if we let

$$H : [0,1] \times S^n \to S^n, \quad H(t,x) = \frac{h(t,x)}{\|h(t,x)\|}.$$

This is a homotopy of A with F, which completes the proof.

□

Problem 8.
Let $f : S^{n-1} \to S^{n-1}$ be a continuous map and D^n the disk with $\partial D^n = S^{n-1}$.

(a) Define the adjunction space $D^n \cup_f D^n$.

(b) Let $\deg f = k$ and compute the homology groups $H_p(D^n \cup_f D^n, \mathbb{Z})$ for $p = 0, 1, \ldots$.

(c) Assume that f is a homeomorphism. Show that $D^n \cup_f D^n$ is homeomorphic to S^n.

Solution.

(a) We define $D^n \cup_f D^n$ by $D^n \sqcup D^n / \sim$, where $\sim$ is a relation between the disjoint copies of D^n given by $x \sim f(x)$. Essentially, we glue the boundaries of the disks together as prescribed by f.

(b) We have a very easy cellular structure on this space. We form a copy of S^{n-1} by gluing a $(n-1)$-cell S to a 0-cell v. We then glue one n-cell e_1 to S^{n-1} with the identity attaching map and the another one e_2 by the k-fold covering map, which we know is homotopic to f by the Hopf degree theorem. This space X which we have constructed has the same homology as $D^n \cup_f D^n$. It has no homology outside of (possibly) degrees 0, $n-1$, and n. Since this space is connected, $H_0(X) = \mathbb{Z}$. Now, in the chain complex we have at the highest degrees

$$0 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \to \cdots$$

We need to understand the map above to compute the homology. This map is given by $(a, b) \mapsto a + kb$, because the degrees of the attaching maps are 1 and k, and it is clearly surjective. Therefore $H_{n-1}(X) = 0$, This map has kernel generated by $(k, -1)$. Therefore $H_n(X) = \mathbb{Z}$. This completes the description.

(c) This is so obvious as to confuse the reader. Given S^n, we can construct a homeomorphism from its closed upper hemisphere to D^n, as well as from its closed lower hemisphere. This gives a homeomorphism from the equator of S^n, which is homeomorphic to S^{n-1}, to both copies of ∂D^n. This would be a problem if ∂D^n was not glued from one to the other by a homeomorphism, but it is, so there is no confusion in gluing these two homeomorphisms together to all of S^n to all of $D^n \cup_f D^n$.

□

Problem 9.
Let $F : M \to N$ be a finite covering map between closed manifolds. Either prove or find counterexamples to the following questions:

(a) Do M and N have the same fundamental groups?

(b) Do M and N have the same de Rham cohomology groups?

(c) When M is simply connected, do M and N have the same singular homology groups?

Solution.

(a) No. Consider the 2-sheeted covering of $\mathbb{RP}^2$ by S^2. We have $\pi_1(S^2) = 0$. However, we have $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$. Since H_1 is the abelianisation of π_1, in particular we could not have $\pi_1(\mathbb{RP}^2) = 0$.

(b) No. Consider the same example. We know that $H^2(S^2) = \mathbb{Z}$. However, $H^2(\mathbb{RP}^2) = H_2(\mathbb{RP}^2) \oplus T_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$.

(c) No. Consider the same example. We know that $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, but $H_1(S^2) = 0$ by simple connectivity.

□

Problem 10.
Let $A \subset X$ be a subspace of a topological space. Define the relative singular homology groups $H_p(X, A)$ and show that there is a long exact sequence

$$\cdots \to H_p(A) \to H_p(X) \to H_p(X, A) \to H_{p-1}(A) \to \cdots$$

Solution.
If $i : A \to X$ is the inclusion, then it induces an inclusion $C_p(A) \to C_p(X)$. Since we are working with abelian groups, we can take the quotient $C_p(X)/C_p(A)$. We can define a chain complex on these groups:

$$\cdots \to C_{p+1}(X)/C_{p+1}(A) \to C_p(X)/C_p(A) \to C_{p-1}(X)/C_{p-1}(A) \to \cdots$$

using the boundary map on $C_\bullet(X)$. Since the boundary map on $C_\bullet(X)$ restricted to $C_\bullet(A)$ is the boundary map on $C_\bullet(A)$, we see that $\partial(C_p(A)) \subset C_{p-1}(A)$, so these maps are well defined. We define $H_\bullet(X, A)$ as the homology of this sequence. The fact that

$$0 \to C_p(A) \to C_p(X) \to C_p(X)/C_p(A) \to 0$$

is a short exact sequence means that we have the same exact explanation of the long exact sequence as there always is, e.g. Spring 2012, #5. □

4 Fall 2012

Problem 1.

(a) Show that the Lie group $\mathrm{SL}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) : \det A = 1\}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(b) Show that the Lie group $\mathrm{SL}_2(\mathbb{C}) = \{A \in M_2(\mathbb{C}) : \det A = 1\}$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.

Solution.

(a) We know that $\mathrm{SL}_2(\mathbb{R})$ is the group of isotopic transformations of $\mathbb{R}^2$. These consist of rotations and translations. Therefore we have a diffeomorphism if we consider the group of translations as diffeomorphic to $\mathbb{R}^2$ and the group of rotations diffeomorphic to S^1. Note that this is not a Lie group isomorphism if we take the usual Lie group structure on $S^1 \times \mathbb{R}^2$, but the diffeomorphism is there.

(b) We can use a polar decomposition to show this. We know that S^3 corresponds to the rotations of $\mathbb{C}^2$. Then we can write every

$$A = UP,$$

where U is a unitary (rotation) matrix and P is a positive-semidefinite Hermitian matrix. This is determined by three real numbers: the top right entry can be any real number, and the top left entry can be any complex number. The bottom left entry is the inverse of the top right and the bottom left is the complex conjugate of the top left. Therefore we obtain $S^3 \times \mathbb{R}^3$.

$\square$

Problem 2.
For $n \geq 1$, construct an everywhere non-vanishing smooth vector field on the odd-dimensional real projective space $\mathbb{RP}^{2n-1}$.

Solution.
We can take a vector field on S^{2n-1} and push it down to $\mathbb{RP}^{2n-1}$ if it is A-invariant, where A is the antipodal map. This makes the form well-defined on $\mathbb{RP}^{2n-1} = S^{2n-1}/\sim$ where $x \sim y$ if $x = A(y)$. Now, let ω be the form on S^{2n-1} given by, for $x = (x_1, \ldots, x_{2n})$,

$$V(x) = (x_2, -x_1, \ldots, x_{2n}, -x_{2n-1}).$$

Then we have $V(A(x)) = -V(x) \sim V(x)$, so indeed V is compatible with A. Therefore the vector field as written is a nowhere-vanishing vector field in $\mathbb{RP}^{2n-1}$. $\square$

Problem 3.
Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

Solution.
We will use transversality. Let $x, y \in \mathbb{R}^n$, and let γ be a path connecting them in $\mathbb{R}^n$. Then we claim that γ is homotopic to a path avoiding M. Indeed, since transversality is generic, we have some $\gamma' \sim \gamma$ which is transversal to M. Suppose γ' intersects M at a point p. Since $\gamma : S^1 \to \mathbb{R}^n$ has dimension 1 and M has dimension $< n-2$, then sum of dimensions of their tangent spaces at p is strictly less than n. Therefore γ' cannot intersect M, so γ' is a valid path in $\mathbb{R}^n \setminus M$. Since $\mathbb{R}^n$ is path connected, it is connected, because these notions are equivalent in a manifold.

To show it is simply connected we use the same point. Suppose we have a loop γ at the origin and a homotopy $H : \gamma \times I \to \mathbb{R}^n$ taking γ to $\{0\}$. We may assume H is transversal to M by the above. Then the sum of dimensions of the tangent spaces at any point of intersection is less than n, since the image of H is 2-dimensional and M is $(< n-2)$-dimensional. Therefore the image of H is disjoint from M, so the homotopy still works in $\mathbb{R}^n \setminus M$. □

Problem 4.

(a) Show that for any $n \geq 1$ and $k \in \mathbb{Z}$, there exists a continuous map $f : S^n \to S^n$ of degree k.

(b) Let X be a compact, oriented n-manifold. Show that for any $k \in \mathbb{Z}$, there exists a continuous $f : X \to S^n$ of degree k.

Solution.

(a) See Spring 2010, #8(b). The process is the same, albeit with more dimensions

(b) We can perform the same process: take $|k|$ disjoint n-balls on X. Map them homeomorphically, preserving orientation, onto $S^n \setminus \{n\}$, and map the rest of X onto $\{n\}$, the north pole. This will have degree $|k|$. To obtain degree $-|k|$, map them orientation reversing instead.

□

Problem 5.

Assume that $\Delta = \{X_1, \ldots, X_k\}$ is a k-dimensional distribution spanned by vector fields on an open set $\Omega \subset M^n$. For each open subset $V \subset \Omega$, define

$$Z_V = \{u \in C^\infty(V) : X_1 u = X_2 u = \cdots = X_k u = 0\}.$$

Show that the following two statements are equivalent:

(a) The distribution Δ is integrable.

(b) For each $x \in \Omega$ there exists an open neighbourhood $x \in V \subset \Omega$ and $n-k$ functions $u_1, \ldots, u_{n-k} \in Z_V$ such that the differential $du_1, \ldots, du_{n-k}$ are linearly independent at each point in V.

Solution.

For (a) implies (b), if Δ is integrable then there exists a submanifold $x \in N \subset M$ so that $T_x N \cong \Delta_x$ for each $x \in N$. Therefore we can choose coordinates in a neighbourhood $V \subset \Omega$ corresponding to this tangent space, k of them corresponding to Δ_x and $n-k$ of them not. If we take the differentials of the latter coordinates, they give us a basis for the complement of $T_x N$ in $T_x M$, so they must be linearly independent.

For the converse, we take the following integrability criterion: Δ is integrable if and only if its annihilator ideal $I(\Delta)$ in the algebra of all forms is closed under exterior differentiation. If (b) holds, then we know that $df_1, \ldots, df_{n-k} \in I(\Delta)$, and we would like to show that these span $I(\Delta)$ (as an ideal in the algebra of forms). Equivalently, we can show that the f_i generate Z_V. Indeed, suppose that $g \in Z_V$ that is not in the span of the f_i, so that dg is not in the span of the df_i. Then in local coordinates, $dg = \sum_{i=1}^n a_i \, dx_i$ must have a component in Δ by dimensionality. This implies that $X_j g \neq 0$ for some j, a contradiction.

Since $I(\Delta)$ is generated by df_i, it is clear that $I(\Delta)$ is closed under exterior differentiation, which proves (b) implies (a). □

Problem 6.
On $\mathbb{R}^n \setminus \{0\}$ define the $(n-1)$-forms

$$\sigma = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \quad \omega = \frac{\sigma}{|x|^n}.$$

(a) Show that $\omega = r^* \circ i^*(\sigma)$, where $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is the natural inclusion of the unit sphere and $r(x) = \frac{x}{|x|} : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the natural retraction.

(b) Show that σ is not a closed form.

(c) Show that ω is a closed form that is not exact.

Solution.
(a) We know that $r^* \circ i^*(\sigma)(x) = \sigma(r(i(x)))$. Because i^* is just inclusion, $i(x) = x$, so we can write σ in the same way without causing any problems. We have

$$\sigma(r(x)) = \sum_{i=1}^{n} (-1)^{i-1} \frac{x}{|x|} d(r(x^1)) \wedge \cdots \wedge \widehat{d(r(x^i))} \wedge \cdots \wedge d(r(x^n))$$

We know that $d(r(x^i)) = dx^i/|x|$, so this gives us

$$\begin{aligned} &= \sum_{i=1}^{n} (-1)^{i-1} \frac{x}{|x|} \left(\frac{dx^1}{|x|} \wedge \cdots \wedge \frac{\widehat{dx^i}}{|x|} \wedge \cdots \wedge \frac{dx^n}{|x|} \right) \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n}{|x|^n} = \frac{\sigma(x)}{|x|^n}. \end{aligned}$$

(b) We have

$$d\sigma = \sum_{i=1}^{n} (-1)^{i-1} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

To move the dx^i over to where it is 'missing' requires multiplying by -1 for each transposition, for a total of $(-1)^{i-1}$. Thus

$$= \sum_{i=1}^{n} dx^1 \wedge \cdots \wedge dx^n \neq 0.$$

This is clearly nonzero because it is (a multiple of) the volume form on S^{n-1}. Therefore σ is not closed.

(c) Since d commutes with pullbacks, we have

$$\begin{aligned} d\omega &= d(r^* \circ i^*(\sigma)) = r^* \circ i^*(n \cdot dx^1 \wedge \cdots \wedge dx^n) \\ &= r^*(n \cdot dx^1 \wedge \cdots \wedge dx^n) \\ &= n \cdot \frac{dx^1 \wedge \cdots \wedge dx^n}{|x|^n} = 0. \end{aligned}$$

This is because $d\omega$ is an n-form on S^{n-1}, which is $(n-1)$-dimensional, so there is no n-form. To show the form is not exact, recall that a closed form on the sphere is exact if and only if it has zero integral. We have

$$\int_{S^{n-1}} \omega = \int_{D^n} d\omega,$$

by Stokes' theorem, where D^n is the unit ball in $\mathbb{R}^n$. Hence

$$\int_{S^{n-1}} \omega = n \int_{D^n} \frac{dx^1 \wedge \cdots \wedge dx^n}{|x|^n} \neq 0$$

because the integrand is a multiple of the volume form on D^n.

□

Problem 7.
Let $n \geq 0$ be an integer. Let M be a compact, orientable smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

Solution.
The statement of the problem implies strongly that we will use Poincaré duality. Let $\dim H^{2n+1}(M;\mathbb{R}) = k$. We have a map

$$F : H^{2n+1}(M;\mathbb{R}) \times H^{2n+1}(M;\mathbb{R}) \to H^{4n+2}(M;\mathbb{R}), \qquad F(\omega,\eta) = \omega \wedge \eta.$$

Because $\omega \wedge \omega = 0$, $\omega \wedge \eta = (-1)^{2n+1}\eta \wedge \omega = -\eta \wedge \omega$, and $H^{4n+2}(M;\mathbb{R}) \cong \mathbb{R}$, we get an antisymmetric bilinear form on $\mathbb{R}^k$. This is representable by a matrix $A \in M_k(\mathbb{R})$ such that $A^t = -A$. Therefore

$$(-1)^k \det(A) = \det(-A) = \det(A^t) = \det(A),$$

whence k is even. □

Problem 8.
Show that there is no compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{RP}^2$.

Solution.
Suppose that such an M exists. Let N be the space obtained by gluing two copies of M together along ∂M. Then N is a compact connected three-dimensional manifold without boundary, so we have $\chi(N) = 0$ by Poincaré duality, where χ is the Euler characteristic. Since $M \cup M$ is a partition of N into open sets, we can apply the Meyer-Vietoris sequence to obtain a long exact sequence

$$\cdots \to H_n(\partial M) \to H_n(M) \oplus H_n(M) \to H_n(N) \to \cdots$$

This implies that

$$\chi(N) = 2\chi(M) - \chi(\partial M).$$

Therefore $\chi(\partial M) = \chi(\mathbb{RP}^2)$ is even. But $\chi(\mathbb{RP}^2) = 1$, so we have a contradiction. □

Problem 9.
Consider the coordinate axes in $\mathbb{R}^n$:

$$L_i = \{(x_1, \ldots, x_n) : x_j = 0 \text{ for all } j \neq i\}.$$

Calculate the homology groups of the complement $\mathbb{R}^n \setminus (L_1 \cup \cdots \cup L_n)$.

Solution.
First, let $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ be the usual deformation retraction, i.e. $f(x) = x/|x|$. If we restrict f to $\mathbb{R}^n \setminus (\bigcup L_i)$, then we obtain a deformation retraction from that domain to S^{n-1} with $2n$ points removed, namely the points $(0, \ldots, \pm 1, \ldots, 0)$. Write $\{k\}$ for a set of k points. Removal of the first point gives a homotopy $S^{n-1} \setminus \{2n\} \simeq \mathbb{R}^{n-1} \setminus \{2n-1\}$.

From here, each neighbourhood of the removed points is homotopy equivalent to S^{n-2}. With a proper deformation, we can isolate these neighbourhood and retract $\mathbb{R}^{n-1} \setminus \{2n-1\}$ onto a wedge of $2n-1$ copies of S^{n-2}. This gives

$$H_i\left(\mathbb{R}^n \setminus \left(\bigcup L_i\right)\right) = H_i\left(\bigvee^{2n-1} S^{n-2}\right) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{2n-1} & i = n-1 \\ 0 & \text{else} \end{cases}$$

□

Problem 10.

(a) Let X be a finite CW complex. Explain how the homology groups of X are related to the homology groups of $X \times S^1$.

(b) For each integer $n \geq 0$, give an example of a compact smooth manifold of dimension $2n+1$ such that $H_i(X) = \mathbb{Z}$ for all $i = 0, \ldots, 2n+1$.

Solution.

(a) We will use cellular homology, since the problem suggests it. We view S^1 as a 1-cell a attached to a 0-cell v. We have $\partial_1(a) = v - v = 0$ and $\partial_0(v) = 0$. We know that a cell decomposition of $X \times S^1$ is composed of products of cells. In particular, if e_i is an i-cell of X, then we have

$$\begin{aligned} \partial_{i+1}(e_i, a) &= (\partial_i(e_i), a) + (-1)^i(e_i, \partial_1(a)) = (\partial_i(e_i), a) \\ \partial_i(e_i, v) &= (\partial_i(e_i), v) + (-1)^i(e_i, \partial_0(v)) = (\partial_i(e_i), v). \end{aligned}$$

Thus the kernel of ∂_i on $X \times S^1$ is generated by pairs (e_{i-1}, a) and (e_i, v) where e_{i-1} is a $(i-1)$-cycle in X and e_i is an i-chain in X. Similarly, we see that the image of ∂_{i+1} is generated by (e_i, a) and (e_{i+1}, v) where e_i is an i-boundary and e_{i+1} is an $(i+1)$-boundary. Therefore

$$\begin{aligned} H_i(X \times S^1) &= Z_i(X \times S^1)/B_i(X \times S^1) \\ &\cong (Z_i(X) \oplus Z_{i-1}(X))/(B_i(X) \oplus B_{i-1}(X)) \\ &= H_i(X) \oplus H_{i-1}(X). \end{aligned}$$

(b) Recall the homology of $\mathbb{CP}^n$: because it is comprised of only one cell in each even dimension, we have

$$H_i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Further, $\mathbb{CP}^n$ is compact because it is a quotient of S^{2n+1}, which is compact. Therefore $\mathbb{CP}^n \times S^1$, which has dimension $\dim \mathbb{CP}^n + \dim S^1 = 2n+1$, is what we are looking for:

$$H_i(\mathbb{CP}^n \times S^1) = H_i(\mathbb{CP}^n) \oplus H_{i-1}(\mathbb{CP}^n) = \mathbb{Z}$$

because exactly one of i and $i-1$ is even.

□

5 Spring 2012

Problem 1.
Explain in detail, from the viewpoint of transversality theory, why the sum of the indices of a vector field with isolated zeroes on a compact orientable manifold M is independent of what vector field we choose.

Solution.
This is the Poincaré-Hopf Index Theorem. Let V be a vector field with isolated zeroes. Let ϕ_t be the flow of V for some small t, which we may make global because M is compact. The fixed points of ϕ_t are just the zeroes of V, and in particular they are isolated. Therefore ϕ_t is a Lefschetz map. In particular, notice that

$$\sum_{p:V_p=0} \text{ind}_p(V) = \sum_{p:\phi_t(p)=p} L_p(\phi_t) = \Lambda_{\phi_t}.$$

The Lefschetz number of ϕ_t is the sum of the local Lefschetz numbers, which are identically the indices of the zeroes of V. Now because ϕ_t is a diffeomorphism for small enough t, we have $\phi_t \sim \text{id}_M$, the homotopy given by the backwards flow ϕ_{-t}. Therefore $\Lambda_{\phi_t} = \Lambda_{\text{id}_M}$ because Lefschetz number is a homotopy invariant. This shows that no matter the choice of V, we always obtain the same value Λ_{id_M}. □

Problem 2.
Let the Euler characteristic $\chi(M)$ be the sum from above. Explain why the Euler characteristic of a genus g surface is $2-2g$. Do this explicitly, not using the homological description of $\chi(M)$.

Solution.
See Guillemin and Pollack, p.125 for the appropriate picture. The typical example of the genus g surface is the g-holed 2-torus, M_g. Flip it 'vertical', so that we have (effectively) a stack of donuts with outward facing holes. Consider the paths of a drop of water from the top to the bottom. This defines a vector field on M_g. It is zero at the top, zero at the bottom, and zero at the top and bottom of every hole. The index at the top and the bottom is $+1$, since these are a 'source' and a 'sink'. At the top and bottom of the holes, however, we have saddle points, which have index -1. Hence $\chi(M_g) = 2-2g$. □

Problem 3.
Suppose that M is a triangulated compact orientable manifold.

(a) Show that the alternating sum of the Betti numbers $b_0 - b_1 + b_2 - \cdots$ is equal to the alternating sum (# vertices)−(# edges)+(# faces)− ⋯ .

(b) Show that there is a vector field with its sum of its indices equal to the number described in part (a).

Solution.
(a) We know that $b_i = \operatorname{rk} H_i(M)$, and further since $H_i(M) = Z_i(M)/B_i(M)$, we have $\operatorname{rk} H_i(M) = \operatorname{rk} Z_i(M) - \operatorname{rk} B_i(M)$. We need to rephrase this in terms of the number of k-chains. Let $\partial_i : C_i(M) \to C_{i-1}(M)$ be the boundary operator. We know that $\operatorname{im} \partial_i \cong C_i(M)/\ker \partial_i$. If we just need to count rank, we have $\operatorname{rk} C_i(M) = \operatorname{rk} \operatorname{im} \partial_i + \operatorname{rk} \ker \partial_i$. But $\ker \partial_i = Z_i$ and $\operatorname{im} \partial_i = B_{i-1}$. This gives the alternating sum of the number of subsimplices is

$$\sum_{i=0}^{n} (-1)^i \operatorname{rk} C_i(M) = \sum_{i=0}^{n} (-1)^i (\operatorname{rk} Z_i(M) + \operatorname{rk} B_{i-1}(M))$$

Since $B_{-1} = 0 = B_n$, we can replace $\operatorname{rk} B_{i-1}$ by $-\operatorname{rk} B_i$ and not affect anything, whence

$$= \sum_{i=0}^{n} (-1)^i (\operatorname{rk} Z_i - \operatorname{rk} B_i) = \sum_{i=0}^{n} (-1)^i \operatorname{rk} H_i(M).$$

(b) We will describe how this vector field looks on n-simplices, and from there it will apply smoothly to a triangulated manifold M. Let X be an n-simplex. For each k-face of X, define a zero at the centre of the face. In the interior of even k-faces, define flow lines directly inward towards that zero, so that we have a sink. On odd k-faces, this then should define a saddle point at the centre. Then the index of the zeros is the appropriate -1 or 1 depending on the parity of k. Thus the sum of the indices of this vector field is just the alternating sum of the number of each k-face, weighted appropriately.

□

Problem 4.
Suppose V is a smooth vector field on $\mathbb{R}^3$ that is nonzero at $(0,0,0)$. The vector field is said to be *gradient-like* at $(0,0,0)$ if there is a neighbourhood of $(0,0,0)$ and a nowhere zero smooth function $\lambda(x,y,z)$ on that neighbourhood such that λV is the gradient of some smooth function some neighbourhood of $(0,0,0)$.

(a) Write $V = (P, Q, R)$. Show by example that there are functions P, Q, R for which V is not gradient-like in a neighbourhood of $(0,0,0)$. (Suggestion: the orthogonal complement of V taken at each point would have to be an integrable 2-plane distribution.)

(b) Derive a general differential condition on (P, Q, R) which is necessary and sufficient for V to be gradient-like in a neighbourhood of $(0, 0, 0)$.

Solution.

(a) Let $V = (-y/2, x/2, 1)$. Taking the hint, we know that $V^\perp$ is the kernel of the dual of the generator of V, i.e.

$$V^\perp = \ker\left(-\frac{y}{2}dx + \frac{x}{2}dy + dz\right).$$

We know (from Fall 2013, #5(b)) that $\ker\omega$ is integrable if and only if $\omega \wedge d\omega = 0$. We have shown above that, for this particular example, $\omega \wedge d\omega = dx \wedge dy \wedge dz$. Therefore V cannot be gradient-like since $V^\perp$ is not integrable.

Now we want to show V is gradient-like if and only if $V^\perp$ is an integrable distribution. Therefore suppose that $\lambda V = \nabla f$. Then at any point $p \in U$, the vector V_p is perpendicular to ∇f_p. Therefore any vector perpendicular to V is perpendicular to ∇f, which shows $V^\perp = \ker(df)$. By one of our integrability criteria, $df \wedge d(df) = 0$ implies $\ker(df)$ is an integrable distribution.

For the converse, suppose $V^\perp$ is integrable. Let M be the 2-submanifold of $\mathbb{R}^3$ so that $T_pM = V_p^\perp$ for every $p \in M$. In local coordinates, we can express M as the zero set of the third coordinate function x^3 on $\mathbb{R}^3$. We know that the normal bundle to M at p is 1-dimensional and contains both V_p and ∇f_p, so these are scalar multiples of each other at every point. Therefore we can construct λ to account for this scaling at every point so that $\lambda V = \nabla f$.

(b) For specific functions, we claim that $V^\perp$ is integrable if and only if $\operatorname{curl} V \perp V$. If $V = (P, Q, R)$, we have

$$\operatorname{curl} V = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}.$$

Let $\omega = P\,dx + Q\,dy + R\,dz$. Then we claim that $\langle \operatorname{curl} V, V\rangle = 0$ is equivalent to $\omega \wedge d\omega = 0$. Writing it out literally, it is obvious. Since $V = \ker\omega$, by that integrability criterion, this proves the claim.

□

Problem 5.

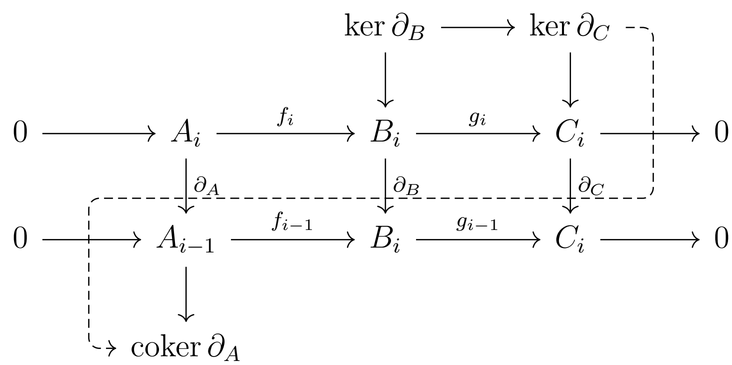
(a) Define carefully the 'boundary map' which defines the H_n to H_{n-1} mapping that arises in the long exact sequence arising from a short exact sequence of chain complexes.

(b) Prove that the kernel of the boundary map is equal to the image of the map into the H_n.

Solution.

(a) This has been done in various places over this document, e.g. Spring 2011, #6(b).

(b) We have not had to show it is exact yet. We appeal to our snake diagram again, with the previous map added in:



Let δ be the connecting homomorphism. We need to show we have exactness at $\ker \partial_C$. Suppose that $\beta \in \ker \partial_B$ maps to $\gamma \in \ker \partial_C$. Then $\beta = f_i(\alpha)$ for some $\alpha \in A_i$. Then we know that $\partial_B(f_i(\alpha)) = f_{i-1}(\partial_A(\alpha)) = 0$, so since f_{i-1} is injective, $\partial_A(\alpha) = 0$. Therefore by the commutativity of everything, $\delta(\gamma) = 0$ in $\operatorname{coker} \partial_A$.

Let $\delta(\gamma) = 0$. Then we know that we can change this by the image of elements of A_i with impunity. But the above shows that γ would have to have come from an element of $\ker \partial_B$. This shows exactness. □

Problem 6.
Compute the homology of $\mathbb{RP}^n$ for $n > 1$.

Solution.
We have done this below for $\mathbb{RP}^2$, but we might as well do it in general. We can construct $\mathbb{RP}^n$ using cellular homology in the following way: $\mathbb{RP}^n$ is the quotient of S^{n+1} by the antipodal map. If we just look at one hemisphere, we can view $\mathbb{RP}^n$ as an n-cell with its boundary S^n identified via the antipodal map. But this in turn is the description of $\mathbb{RP}^{n-1}$.

Therefore we build up $\mathbb{RP}^n$ inductively: begin with $\mathbb{RP}^1$ by gluing the boundary of a 1-cell to a 0-cell. Then glue a 2-cell e_2 onto $\mathbb{RP}^1$ by a double cover of the 1-skeleton of $\mathbb{RP}^1$ by ∂e_2. Repeat this process. Our chain complex is

$$0 \to \mathbb{Z} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

since we have one cell in each dimension. We know that ∂_1 is the zero map. Further, in general, the attaching map ∂_i is first the identity and second the antipodal map on ∂e_i. Hence ∂_i is multiplication by the value $\deg(\mathrm{id}) + \deg(A) = 1 + (-1)^i$. This gives

$$\cdots \xrightarrow{\partial_4 = 2\cdot} \mathbb{Z} \xrightarrow{\partial_3 = 0} \mathbb{Z} \xrightarrow{\partial_2 = 2\cdot} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z} \longrightarrow 0$$

Therefore at even i, we have $\ker \partial_i = 0$, so $H_i(\mathbb{RP}^n) = 0$. At odd i, $\ker \partial_i = \mathbb{Z}$ and $\operatorname{im} \partial_{i+1} = 2\mathbb{Z}$, so $H_i(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$. The only other case to check is $i = n$. if n is even, then we have the same case as before. However, if n is odd, then we have $\ker \partial_n = \mathbb{Z}$ but

$\operatorname{im} \partial_{n+1} = 0$. Therefore $H_n(\mathbb{RP}^n) = \mathbb{Z}$ in the final case. Summarised,

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0,\ i = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 < i < n \text{ odd} \\ 0 & \text{else} \end{cases}$$

□

Problem 7.

(a) Define $\mathbb{CP}^n$.

(b) Show that $\mathbb{CP}^n$ is compact for all n.

(c) Show that $\mathbb{CP}^n$ has a cell decomposition with one cell in each even dimension and no other cells. Include a careful description of the attaching maps.

Solution.
See Spring 2009, #7 for all answers. □

Problem 8.
Suppose a compact (real) manifold M has a (finite) cell decomposition with only even dimensional cells. Is M necessarily orientable?

Solution.
Yes. We know that M is orientable if it has nontrivial rank in its top homology, which implies it has nontrivial rank in its top cohomology. Therefore examine the (cellular) chain complex

$$0 \to C_n \overset{\partial_n}{\to} C_{n-1} \to \cdots$$

with $n = \dim M$ even. We know that $C_{n-1} = 0$ since there are no $(n-1)$-cells. Further, $\operatorname{im} \partial_{n+1} = 0$ since $C_{n+1} = 0$. Therefore

$$H_n(M) = \ker \partial_n / \operatorname{im} \partial_{n+1} \cong C_n.$$

Since C_n is free on at least one element, is has positive rank. Therefore M is orientable. □

Problem 9.
Suppose that a finite group G acts smoothly on a compact manifold M and that the action is free.

(a) Show that M/G is a manifold.

(b) Show that $M \to M/G$ is a covering space.

(c) If $H^k_{dR}(M) = 0$ for some $k > 0$, then does $H^k_{dR}(M/G) = 0$ necessarily? Prove your answer.

Solution.

(a) Recall that an action is free if $g \cdot x = x \implies g = e$. We claim that this action is properly discontinuous, which will allow us to put a manifold structure on M/G. Recall that an action is properly discontinuous if for every $x, y \in M$, there exist neighbourhoods U_x and U_y such that only finitely many $g \in G$ give $g \cdot U_x \cap U_y \neq \varnothing$. This is always true in a finite group. Properly discontinuous also tells us (in the case of compact manifolds) that every point has a neighbourhood U so that the set $\{g \cdot U\}$ is pairwise disjoint.

Therefore we can define charts on M/G: we know that every point $x \in M/G$ has a neighbourhood U so that the orbit $\{g \cdot U\}$ is disjoint. Therefore we can take charts on any particular $g \cdot U$ we'd like in M and, because $U \cong g \cdot U$, use them canonically on U. This makes M/G a smooth manifold.

(b) We have basically shown this. The above showed that $p : M \to M/G$ is a proper, surjective smooth map, and every $x \in M/G$ has a neighbourhood U so that its preimage $\{g \cdot U\}$ is disjoint. Since g is a homeomorphism of M, $g \cdot U \cong U$ for every g. In particular, this is a finitely-sheeted ($|G|$-sheeted) covering space.

(c) For a finite cover, every form on M/G can be viewed as a G-invariant form on M, which gives an injection $H^k(M/G) \to H^k(M)$. Any form on $U \subset M/G$ can be viewed as a form on G by gluing together its action on finitely many open subsets of M. Therefore if $H^k(M) = 0$, then $H^k(M/G) = 0$ as well.

□

Problem 10.
Let $M = \mathbb{RP}^2 \times \mathbb{RP}^2$. In a general product manifold, homology elements can arise by taking in effect the product of a cycle in each factor. Show that in the case, there is an element in degree 3 homology with $\mathbb{Z}$ coefficients that does not arise this way. Exhibit this element explicitly, e.g. in terms of a cell decomposition.

Solution.
See Spring 2011, #8. □

6 Fall 2011

Problem 1.
Let M be a compact smooth manifold. Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be smoothly embedded in the Euclidean space $\mathbb{R}^n$.

Solution.
For compact manifolds, an embedding is an injective immersion. Let $\dim M = m$. Cover M by its charts $\phi_i : U_i \to \mathbb{R}^m$. By compactness, we may pick a finite subcover $\phi_1, \ldots, \phi_N$. We can construct smooth bump functions $\lambda_i : M \to [0, 1]$ with respect to the U_i , i.e. $\operatorname{supp} \lambda_i = U_i$.

Define $F : M \to \mathbb{R}^{N(m+1)}$ by

$$F(x) = (\lambda_1(x)\phi_1(x), \ldots, \lambda_n(x)\phi_n(x), \lambda_1(x), \ldots, \lambda_n(x))$$

We will show that this is an injective immersion. Suppose that $F(x) = F(y)$. Then in particular, $\lambda_i(x) = \lambda_i(y)$ for all i. Since some $\lambda_i(x) \neq 0$, this shows that $\phi_i(x) = \phi_i(y)$. Since ϕ_i was a was a coordinate function, it was a diffeomorphism, hence injective. Thus $x = y$.

Now suppose that $v, w \in T_xM$ and $dF_x(v) = dF_x(w)$. Then we must have $d(\lambda_i)_x(v) = d(\lambda_i)_x(w)$ for all i. Further, we must have

$$d(\lambda_i \cdot \phi_i)_x = d(\lambda_i)_x \cdot \phi_i(x) + \lambda_i(x) \cdot d(\phi_i)_x.$$

At some j we have $\lambda_j(x) \neq 0$, so this gives

$$d(\phi_j)_x(v) = d(\phi_j)_x(w).$$

Since ϕ_j was a diffeomorphism, $d(\phi_j)_x$ is injective at all x, so $v = w$.

□

Problem 2.
Prove that the real projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n.

Solution.
We begin with its construction: consider the sphere S^{n+1}, which has dimension n, and consider the antipodal map $A : S^{n+1} \to S^{n+1}$. Then $\mathbb{RP}^n$ is homeomorphic to the quotient $S^{n+1}/\sim$, where $x \sim y$ if and only if $A(x) = y$.

We see that $\mathbb{RP}^n$ inherits the smooth structure and charts from S^{n+1}, since locally on S^{n+1} the action of A is trivial. This also shows that $\mathbb{RP}^n$ is of the same dimension. Put another way, S^{n+1} is a 2-to-1 covering space of $\mathbb{RP}^n$, so they must have the same dimension. □

Problem 3.
Let M be a compact, simply connected smooth manifold of dimension n. Prove that there is no smooth immersion $f : M \to T^n$, where T^n is the n-torus.

Solution.
Suppose we have such an immersion. Then $f(M)$ is a compact, hence closed submanifold of T^n. Since $\dim M = \dim T^n = n$, we know that at every point df_x is actually an isomorphism, so f is a submersion. Therefore $f(M)$ is open as well, so $f(M) = T^n$. f is a local diffeomorphism. Therefore we have a surjective proper local diffeomorphism, which means that $f : M \to T^n$ is a universal covering map. But we know that $\mathbb{R}^n$ is the universal cover of T^n, and $\mathbb{R}^n$ is not compact. This gives us a contradiction, so no such f can exist. □

Problem 4.
Give a topological proof of the fundamental theorem of algebra.

Solution.
Let $f(x) : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of order $m > 0$ (the assumption of monic is of no consequence). We will show that it has a root by contradiction. We use the following proposition, found on p.110 of Guillemin and Pollack:

Lemma. Suppose that $f : X \to Y$ is a smooth map of closed manifolds of the same dimension and that $X = \partial W$ (W compact). If f can be extended to all of W, then $\deg f = 0$.

This is a consequence of the so-called 'extendability lemma': if X is the boundary of a compact manifold, f is transversal to a submanifold $Z \subset Y$, and f can be extended to a map $f : W \to Y$, then $I(f, Z) = 0$.

In our case, suppose S is the circle of radius r in $\mathbb{C}$, and f has no zeroes on S. Then define two functions $S \to S^1$

$$\frac{f(x)}{|f(x)|} \quad \text{and} \quad \frac{x^m}{|x^m|} = \left(\frac{x}{r}\right)^m.$$

These two maps are homotopic when r is large enough, when the lower degree terms of $f(x)$ are of no consequence. Therefore they have the same degree m. We know that S is the boundary of a compact disc D_r in $\mathbb{C}$. If f has no zero in D_r, then we can extend $f/|f|$ to all of D_r. This implies that $\deg(f/|f|) = 0$ by the lemma above, but we know that $\deg(f/|f|) = m > 0$ by the above, a contradiction. Therefore f must have a zero in D_r for r sufficiently large. □

Problem 5.
Let $f : M \to N$ be a smooth map between two manifolds M and N. Let α be a p-form on N. Show that $d(f^*\alpha) = f^*(d\alpha)$.

Solution.
See Spring 2008, #1. □

Problem 6.

(a) What are the de Rham cohomology groups of a smooth manifold?

(b) State de Rham's theorem.

Solution.

(a) We construct de Rham cohomology in the following way: let $\Omega^k(M)$ be the algebra of k-forms on M, which should need no definition. Let d be the exterior differential on forms. d defines a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ in the obvious way for all k. In particular since $d^2 = 0$, we have a chain complex:

$$0 \xrightarrow{d_{-1}} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0,$$

where $n = \dim M$. Then we define $H^i_{dR}(M)$, the ith de Rham cohomology group of M, to be $\ker d_i / \operatorname{im} d_{i-1}$.

(b) de Rham's theorem relates de Rham cohomology groups with cohomology groups obtained via singular homology. Let $H^i(M)$ be the ith cohomology group with $\mathbb{R}$ coefficients, given by $H^i(M; \mathbb{R}) = \operatorname{Hom}(H_i(M; \mathbb{R}), \mathbb{R})$. Then consider the map $I : H^i_{dR}(M) \to H^i(M; \mathbb{R})$ given by integration in the following way: for a cohomology class $\omega \in H^i_{dR}(M)$, let $I(\omega)$ be the element corresponding to the map

$$H_i(M) \ni \alpha \mapsto \int_\alpha \omega,$$

where we consider α as an i-cycle in the integral. de Rham's theorem says that this map is an isomorphism.

□

Problem 7.
Consider the form

$$\omega = (x^2 + x + y)dy \wedge dz.$$

on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and let $i : S^2 \to \mathbb{R}^3$ by the inclusion.

(a) Calculate $\int_{S^2} \omega$.

(b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

Solution.

(a) We can use Stokes' theorem here. We know that $S^2 = \partial D_3$, the unit ball in $\mathbb{R}^3$. Therefore

$$\int_{S^2} \omega = \int_{D^3} d\omega.$$

We see that

$$d\omega = (2x + 1)dx \wedge dy \wedge dz.$$

Then what we are seeking is

$$\int_{D^3} (2x + 1)dx \wedge dy \wedge dz = \int_{D^3} 2x\, dx \wedge dy \wedge dz + \int_{D^3} 1\, dx \wedge dy \wedge dz.$$

We could convert this into the usual sort of three dimensional integral. If we did so, then we would see that the lefthand integral above is zero based on the symmetry of the unit ball. Therefore

$$\int_{S^2} \omega = \int_{D^3} dx \wedge dy \wedge dz = \mathrm{Vol}(D^3) = \frac{4\pi}{3}.$$

(b) Suppose we could construct such an α. Because α is closed on $\mathbb{R}^3$, $i^*\alpha = i^*\omega$ is closed on S^2. Further, by the Poincaré lemma, α is exact, so let $\alpha = d\eta$. Then

$$i^*\omega = i^*\alpha = i^*(d\eta) = d(i^*\eta).$$

Therefore $i^*\omega$ is an exact form on S^2. By a well-known result, a closed form on S^2 is exact if and only if its integral is zero. But we showed that the integral of ω is not zero, so no such α exists.

□

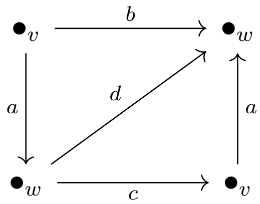
Problem 8.

(a) Let M be a Möbius band. Using homology, show that there is no retraction from M to ∂M.

(b) Let K be a Klein bottle. Show that there exist homotopically nontrivial simple closed curves γ_1 and γ_2 on K such that K retracts to γ_1 but does not retract to γ_2.

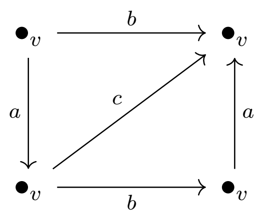
Solution.

(a) Recall that if $r : M \to A$ is a retraction, then $r^* : H_1(A) \to H_1(M)$ is injective. We claim that this is not the case. Consider the below simplicial structure on M:

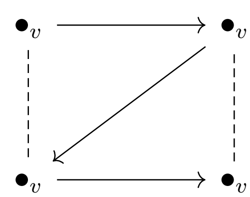


Let the top 2-simplex be A and the bottom be B. The simplicial structure on ∂M is just given by the 1-simplices b, c. The generating element of $H_1(\partial M)$ is $a+c+a-b = 2a-b+c$. We claim that this is a boundary in M, so r^* is the zero map identically, making the map (in particular) noninjective. We see that $\partial_2(A) = d-b+a$ and $\partial_2(B) = a-d+c$. We have $\partial_2(A+B) = 2a-b+c$. This is exactly what we wanted to prove.

(b) We have the following simplicial structure on K:



where we again name the top 2-simplex A and the bottom B. Let r_1, r_2 be the supposed retractions onto γ_1, γ_2. Consider the closed loop γ_1 given by the solid line below

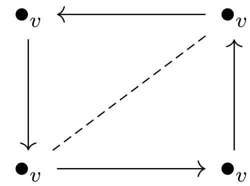


This is a loop since it goes from v to v, and it is nontrivial. This loop corresponds to $b-c+b = 2b-c$ from the above. We claim that $r_1^*(2b-c) \neq 0$, so r_1^* is an injection. Indeed, we have $\partial_2(A) = c-b+a$ and $\partial_2(B) = a-c+b$, so for a general element $xA+yB$, for the coefficient of c to be -1 we must have $x-1 = y$. But this gives

$$\partial(xA + (x-1)B) = 2x \cdot a - b - c.$$

Hence we must have had $x = 0$, but then we do not have the required element. Therefore $r_1^*(2b-c) \notin \operatorname{im} \partial_2$, so it represents a nontrivial element of $H_1(K)$, and the map is injective.

We can use this approach to construct a trivial element, however. Consider the loop



This corresponds to the element $a + b + a - b = 2a$. From the above, we see that $\partial_2(A + B) = 2a$, so $r_2^*(2a) = 0 \in H_1(K)$. Therefore the 'retraction' r_2 cannot exist.

□

Problem 9.
Let X be the topological space identified from a pentagon by identifying its edges cyclically. Calculate the homology and cohomology groups of X with integer coefficients.

Solution.
We will use a cell decomposition: let v be a vertex, a an edge, and A a 2-cell. Then we construct X in the following way: glue ∂a to v, and glue ∂A to a by a 5-fold cover. We know that $H_0(X) = \mathbb{Z}$, since our space is connected. We can calculate $H_1(X)$ by looking at the degree of the attaching map, which the above shows is 5. This means that the map $C_2 \to C_1$ is given by $1 \mapsto 5$. Further, the map $C_1 \to C_0$ is the zero map, since we glue ∂a in two places with opposite orientations. Therefore

$$H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}/5\mathbb{Z}.$$

Since the map $C_2 \to C_1$ is injective, $\ker \partial_2 = 0$, so $H_2(X) = 0$.

Now we use the universal coefficient theorem to calculate cohomology. We see that

$$H^0(X) = \mathbb{Z}, \quad H^1(X) = 0, \quad H^2(X) = \mathbb{Z}/5\mathbb{Z},$$

since we shift the torsion part 'up' one degree. □

Problem 10.
Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $Y \sqcup (X \times [0, 1])$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. So there is a long exact sequence of the form

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \cdots$$

Solution.
This problem is straight from Hatcher, p.151. We begin with the following long exact sequence on relative homology:

$$\cdots \to H_{n+1}(X \times I, X \times \partial I) \xrightarrow{\partial} H_n(X \times \partial I) \xrightarrow{i_*} H_n(X \times I) \to \cdots$$

The pair $(X \times I, X \times \partial I)$ is a good pair, which we will use later. i_* is surjective since $H_n(X \times \partial I)$ is two copies of $H_n(X)$ and $X \times I$ deformation retracts onto $X \times \{0\}$ or $X \times \{1\}$. This is how we know that the end maps are 0, which implies that the map ∂ is injective. This means that

$$0 \to H_{n+1}(X \times I, X \times \partial I) \xrightarrow{\partial} H_n(X \times \partial I) \xrightarrow{i_*} H_n(X \times I) \to 0$$

is actually exact. We know that $\ker i_* = (\alpha, -\alpha)$ for $\alpha \in H_n(X)$. Therefore $H_{n+1}(X \times I, X \times \partial I) \cong H_n(X)$.

We have a quotient map $q : X \times I \to Z$ which is a restriction of the above construction of Z. We actually have $q : (X \times I, X \times \partial I) \to (Z, Y)$, so we have q_* on the relative homology sequences:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{0} & H_{n+1}(X \times I, X \times \partial I) & \xrightarrow{\partial} & H_n(X \times \partial I) & \xrightarrow{i_*} & H_n(X \times I) & \xrightarrow{0} & \cdots \\
 & & \downarrow q_* & & \downarrow q_* & & \downarrow q_* & & \\
\cdots & \longrightarrow & H_{n+1}(Z, Y) & \xrightarrow{\partial} & H_n(Y) & \xrightarrow{i_*} & H_n(Z) & \longrightarrow & \cdots
\end{array}$$

Since q is a map on good pairs and $(X \times I)/(X \times \partial I) \cong Z/Y$, the lefthand q_* is actually an isomorphism. Therefore $H_{n+1}(Z, Y) \cong H_n(X)$, so the bottom long exact sequence is actually the one we were looking for all along.

□

7 Spring 2011

Problem 1.
Show that if V is a smooth vector field on a smooth manifold of dimension n and if $V(p)$ is nonzero for some point of p, then there is a coordinate system defined in a neighbourhood of p, say $(x_1, \ldots, x_n)$, such that $V = \frac{\partial}{\partial x_1}$.

Solution.
See Spring 2008 #2.

□

Problem 2.

(a) Demonstrate the formula $\mathcal{L} = d \circ i_X + i_X \circ d$, where $\mathcal{L}$ is the Lie derivative and i is the interior product.

(b) Use this formula to show that a vector field X on $\mathbb{R}^3$ has a local flow that preserves volume if and only if $\operatorname{div} X = 0$ everywhere.

Solution.

(a) We know that, if ϕ is a flow of X, that

$$\mathcal{L}_X \omega = \lim_{h \to 0} \frac{1}{h} \left[\phi_h^*(\omega) - \omega \right].$$

Therefore $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega)$ since the exterior derivative commutes with pullbacks. Suppose we just have a function f. Then

$$\mathcal{L}_X f = Xf = \lim_{h\to 0}\frac{1}{h}\left[f\circ\phi_h - f\right] = df(X).$$

Also,

$$d(i_X f) + i_X(df) = d(0) + df(X),$$

so this holds. We now need only show $\mathcal{L}_X$ behaves nicely with respect to the wedge product, and we are done since we have proven the identity on arbitrary forms in local coordinates and we can use linearity. We know that for a k-form ω and an l-form η,

$$i_X(\omega\wedge\eta) = (i_X\omega)\wedge\eta + (-1)^k\omega\wedge(i_X\eta).$$

We also know that

$$\mathcal{L}_X(\omega\wedge\eta) = (\mathcal{L}_X\omega)\wedge\eta + \omega\wedge(\mathcal{L}_X\eta).$$

Working this all out directly,

$$\begin{aligned}
d(i_X(\omega\wedge\eta)) &+ i_X(d(\omega\wedge\eta)) \\
&= d((i_X\omega)\wedge\eta) + (-1)^k d(\omega\wedge(i_X\eta)) + i_X(d\omega\wedge\eta) + (-1)^k i_X(\omega\wedge d\eta) \\
&= d(i_X\omega)\wedge\eta + (-1)^{k-1}(i_X\omega)\wedge(d\eta) + (-1)^k d\omega\wedge(i_X\eta) \\
&\quad + (-1)^{2k}\omega\wedge d(i_X\eta) + i_X(d\omega)\wedge\eta + (-1)^{k+1}d\omega\wedge(i_X\eta) \\
&\quad + (-1)^k(i_X\omega)\wedge d\eta + (-1)^{2k}\omega\wedge(i_X(d\eta)) \\
&= (d(i_X\omega) + i_X(d\omega))\wedge\eta + \omega\wedge(d(i_X\eta) + i_X(d\eta)) \\
&= \mathcal{L}_X(\omega\wedge\eta).
\end{aligned}$$

Four of the terms cancel in the second line, and the remainder reduce to the third line. The fourth line follows by the identity for the Lie derivative of a wedge product. This completes the identity.

(b) Write X in local coordinates:

$$X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}.$$

X has a local flow that preserves volume if and only if $\mathcal{L}_X(dx\wedge dy\wedge dz) = 0$. That is,

$$\mathcal{L}_X(dx\wedge dy\wedge dz) = d\circ i_X(dx\wedge dy\wedge dz) + i_X\circ d(dx\wedge dy\wedge dz).$$

The rightmost term is zero, since there are no 4-forms on $\mathbb{R}^2$. Further, we see

$$i_X(dx\wedge dy\wedge dz) = (dx\wedge dy\wedge dz)(X,-,-) = f\,dy\wedge dz - g\,dx\wedge dz + h\,dx\wedge dy.$$

Therefore

$$d\circ i_X(dx\wedge dy\wedge dz) = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)dx\wedge dy\wedge dz.$$

Thus this is zero everywhere if and only if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \operatorname{div} X$ is zero everywhere.

□

Problem 3.

(a) Explain some systematic reason why there is a closed 2-form on $\mathbb{R}^3 \setminus \{0\}$ that is not exact.

(b) With ω such a form as in (a), discuss why, for any smooth map F of S^2 to itself, the number

$$\frac{\int_{S^2} F^*\omega}{\int_{S^2} \omega}$$

is $\deg F$. This includes explaining why the denominator integral cannot be 0.

Solution.

(a) Note that $\mathbb{R}^3 \setminus \{0\} \simeq S^2$, so they have isomorphic homology groups. We claim that $H^2(S^2) \neq 0$. Since $H^2(\mathbb{R}^3 \setminus \{0\})$ is the abelian group of (cohomology classes of) closed 2-forms on this space, this would prove the claim. We know that $H_0(S^2) = \mathbb{Z}$ since the 2-sphere is connected. By Poincaré duality, since S^2 is a closed oriented manifold, $H_0(S^2) \cong H^2(S^2)$. This completes the proof.

(b) For the second part of (b), see Problem 4 below. For the first part, see Spring 2013, #7.

□

Problem 4.

Show without using de Rham's Theorem (but you may use the Poincaré Lemma without proof) that a 2-form ω on the 2-sphere S^2 that has integral 0 is exact.

Solution.

Consider $A = S^2 \setminus \{n\}$ and $B = S^2 \setminus \{s\}$, where n and s are the north and south poles of the sphere. We can take the restrictions $i_A^*(\omega)$ and $i_B^*(\omega)$, and we have

$$\int_A i_A^*(\omega) = \int_{S^2} \omega = 0,$$

and similar for B, since A and S^2 differ by a set of measure zero. Since $A, B \simeq \mathbb{R}^2$, we know that $i_A^*(\omega)$ and $i_B^*(\omega)$ are in fact exact. Say that $\omega|_A = d\eta_A$ and $\omega|_B = d\eta_B$. Then we would like to modify these η so that we get a 1-form on all of S^2. Look at $\eta_A - \eta_B$ on $A \cap B$. Because d commutes with pullbacks,

$$d(i_{A\cap B}^*(\eta_A - \eta_B)) = i_{A\cap B}^*(i_A^*\omega - i_B^*\omega) = i_{A\cap B}^*(\omega - \omega) = 0.$$

Therefore the 1-forms η_A and η_B differ by an exact 1-form df. Hence we can define η globally by means of a bump function so that at a small neighbourhood of the south pole, we let $\eta = \eta_A + df$ and let $\eta = \eta_B$ elsewhere. This is a satisfactory gluing which satisfies $d\eta = \omega$ everywhere, so we are done. □

Problem 5.
Suppose that $V : U \to S^2$ is a smooth map, considered as a vector field of unit vectors, where $U = \mathbb{R}^3 \setminus \{p_1, \dots, p_n\}$, where all p_i lie strictly inside $S^2 \subset \mathbb{R}^3$. Explain carefully, from basic facts about critical values and critical points and the like, why the degree of $V|_{S^2} : S^2 \to S^2$ is equal to the sum of the indices of the vector field V at the points $p_1, \dots, p_n$.

Solution.
First, because these p_i must be isolated, they have neighbourhoods U_i which are pairwise disjoint. These may also be chosen so that $\partial U_i \cap S^2 = \varnothing$ for all i. Then consider the region $X = \mathbb{R}^3 \setminus (\bigcup U_i)$, whose boundary is $\partial X = S^2 \sqcup (\bigcup \partial U_i)$. The map $V : \partial X \to S^2$ is a map between two manifolds of the same dimension, the codomain connected, and V extends to all of X. Therefore $\deg V = 0$ on ∂X.

Because $\partial X = S^2 - (\bigcup \partial U_i)$ (taking the orientation of the boundary into account), we have

$$\deg V|_{S^2} - \deg V|_{\bigcup \partial U_i} = 0 \implies \deg V|_{S^2} = \deg V|_{\bigcup \partial U_i} = \sum \deg V|_{\partial U_i}.$$

But $\deg V|_{\partial U_i}$ is exactly the index of V at p_i by construction, which solves the problem. □

Problem 6.

(a) Explain what a short exact sequence of chain complexes is.

(b) Describe how a short exact sequence of chain complexes gives rise to a long exact sequence in homology. Include how the connecting homomorphism arises. You do not need to prove exactness of the sequence.

Solution.

(a) Let $A_\bullet, B_\bullet, C_\bullet$ be three chain complexes with boundary map ∂ with degree -1. Then given chain maps $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to C_\bullet$, we say that the sequence

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

is a short exact sequence of complexes if at every degree $i \in \mathbb{Z}$ we have

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

is a short exact sequence in the abelian category $\mathcal{A}$ we are working over. To be perfectly clear, this means that $\operatorname{im} f_i = \ker g_i$, where these notions are well understood in the appropriate category.

(b) First, we see that chain maps descend to the level of cycles: suppose that $x \in A_i$ is a cycle, i.e. that $\partial x = 0$. Then we have

$$\partial \circ f_i(x) = f_{i-1} \circ \partial(x) = 0,$$

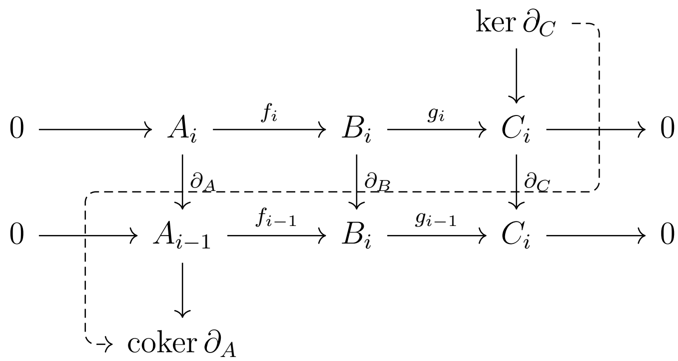
so $f_i(x)$ is a cycle in B_i. The same reasoning shows that exact forms map to exact forms: if $x = \partial y$ for $y \in A_{i+1}$,

$$f_i(x) = f_i \circ \partial(y) = \partial \circ f_{i+1}(y).$$

Therefore chain maps descend to homology. This allows us to construct a sequence

$$H_i(A_\bullet) \xrightarrow{f_{i*}} H_i(B_\bullet) \xrightarrow{g_{i*}} H_i(C_\bullet)$$

which we do not need to show is exact, but is. However, we would like to construct a map $H_i(C_\bullet) \to H_{i-1}(A_\bullet)$. To see how (i.e. to demonstrate the snake lemma), we will need a picture. We know that $H_i(C_\bullet)$ is a quotient of the boundaries of C_i, which is the kernel of $\partial : C_i \to C_{i-1}$. Further, $H_{i-1}(A_\bullet)$ is a quotient of the cokernel of $\partial : A_i \to A_{i-1}$. Therefore we want to construct the following dashed map, which we will call δ:



Let $x \in \ker \partial_C$. Since g_i is surjective, we can find $y \in B_i$ so that $g_i(y) = x$. Therefore we have $\partial_C(g_i(y)) = 0$. Since $g_\bullet$ was a chain map, $\partial_C(g_i(y)) = g_{i-1}(\partial_B(y)) = 0$. Therefore $\partial_B(y) \in \ker g_{i-1} = \operatorname{im} f_{i-1}$ by exactness. Therefore let $z \in A_{i-1}$ so that $f_{i-1}(z) = \partial_B(y)$. This choice is unique since f_{i-1} is injective. This is our choice for $\delta(x)$.

Suppose that we had chosen a different $y \in B_i$ in the first step, say y'. Then because $C_i \cong B_i/A_i$, $y - y'$ must be in the image of f_i. We may represent it by $w \in A_i$. As such, the choice z' we get from y' differs from z by $\partial_A(w)$, which is an boundary in A_{i-1}. Therefore on the induced map on $\operatorname{coker} \partial_A$, there is no difference. This means our connecting homomorphism is well defined on homology, so we have the long exact sequence

$$\cdots \to H_i(A_\bullet) \xrightarrow{f_{i*}} H_i(B_\bullet) \xrightarrow{g_{i*}} H_i(C_\bullet) \xrightarrow{\delta} H_{i-1}(A_\bullet) \to \cdots$$

□

Problem 7.

(a) Define complex projective space $\mathbb{CP}^n$, $n = 1, 2, 3, \ldots$.

(b) Compute the homology and cohomology of $\mathbb{CP}^n$ with $\mathbb{Z}$ coefficients.

Solution.

(a) See Spring 2009, #7.

(b) See the same for the calculation of homology. For the calculation of cohomology, we use Poincaré duality. $\mathbb{CP}^n$ is a closed manifold without boundary, as we can view it as a quotient of S^{2n-1} which is certainly closed. Further, it is orientable since it has a volume form (i.e. $H_{2n}(\mathbb{CP}^n) \neq 0$). Thus , for $0 \leq i \leq 2n$,

$$H^i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

□

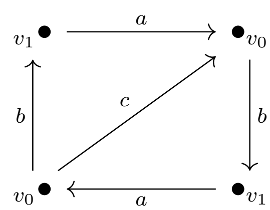
Problem 8.

(a) Find the $\mathbb{Z}$ coefficient homology of $\mathbb{RP}^2$ by any systematic method.

(b) Explain explicitly (not using the Künneth Theorem) how a nonzero element of the 3-homology with $\mathbb{Z}$ coefficents of $\mathbb{RP}^2 \times \mathbb{RP}^2$ arises.

Solution.

(a) The systematic method we choose is simplicial homology. We can represent $\mathbb{RP}^2$ with the following (bad) picture of its fundamental polygon:



We label the top 2-simplex A and the bottom 2-simplex B. We have $\partial_2(A) = c - a + b$ and $\partial_2(B) = a - (-c) + b$. We have $\partial_1(a) = v_0 - v_1$, $\partial_1(b) = v_1 - v_0$, and $\partial_1(c) = 0$. We see that $H_0(\mathbb{RP}^2) = \mathbb{Z}$ since this space is connected. We have $H_1(\mathbb{RP}^2) = \ker \partial_1 / \operatorname{im} \partial_2$. Since $\ker \partial_1$ is generated by $a - b$ and c and $\operatorname{im} \partial_2$ is generated by $c - a + b$ and $a + b + c$, we have

$$\mathbb{Z}(a-b) \oplus \mathbb{Z}(c) \big/ (c = a - b, a + b + c = 0) \cong \mathbb{Z}(a-b) \big/ (a + b = a - b) \cong \mathbb{Z}/2\mathbb{Z},$$

since under this equivalence the only choices are $a - b$ or 0. For $H_2(\mathbb{RP}^2) = \ker \partial_2$, we see that ∂_2 is injective so $H_2(\mathbb{RP}^2) = 0$.

(b) We know that the product manifold has a cell decomposition with the products of cells. Let A_1, A_2 be the 2-cells, a_1, a_2 the 1-cells, and v_1, v_2 the 0-cells. The 3-cells of the product are given by (A_1, a_2) and (a_1, A_2), and the (only) 4-cell given by (A_1, A_2). By the product rule, we know that

$$\begin{aligned} \partial_4(A_1, A_2) &= (\partial_2 A_1, A_2) + (A_1, \partial_2 A_2) = (2a_1, A_2) + (A_1, 2a_2) \\ \partial_3(A_1, a_2) &= (\partial_2 A_1, a_2) + (A_1, \partial_1 a_2) = (2a_1, a_2) + (A_1, 0) \\ \partial_3(a_1, A_2) &= (\partial_1 a_1, A_2) - (a_1, \partial_2 A_2) = (0, A_2) - (a_1, 2a_2). \end{aligned}$$

Now, note that $(2a_1 \times A_2) = 2(a_1 \times A_2)$, and similarly $(0, A_2) = 0$. We can see this directly, and it holds because generally $C_\bullet(X \times Y) \cong C_\bullet(X) \otimes C_\bullet(Y)$. Therefore the above gives us

$$\partial_4(A_1, A_2) = 2((a_1, A_2) + (A_1, a_2)),\ \partial_3(A_1, a_2) = 2(a_1, a_2),\ \partial_3(a_1, A_2) = -2(a_1, a_2).$$

We see that $\ker \partial_3$ is generated by $(a_1, A_2) + (A_1, a_2)$, but this element is not in the image of ∂_4 because we are working with $\mathbb{Z}$ coefficients. This is the required element.

□

Problem 9.

(a) State the Lefschetz Fixed Point Theorem.

(b) Show that the Lefschetz number of any map from $\mathbb{CP}^{2n}$ to itself is nonzero and hence every map from $\mathbb{CP}^{2n}$ to itself has a fixed point. Suggestion: the action of the map on cohomology with $\mathbb{Z}$ coefficients is determined by what happens to the degree 2 cohomology, since the whole cohomology ring is generated by H^2.

Solution.

(a) The theorem states: let $f : X \to X$ be a smooth map on a compact orientable manifold. Then if $\Lambda_f \neq 0$, then f has a fixed point. We define the Lefschetz number of a map f by the intersection number of two submanifolds of the product manifold $X \times X$:

$$\Lambda_f = I(\Delta, \operatorname{graph} f),$$

where $\Delta = \{(x, x)\}$ and $\operatorname{graph} f = \{(x, f(x))\}$.

(b) Let $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ be a smooth map. We know that $H^2(\mathbb{CP}^{2n}) = \mathbb{Z}$, so let ω be a generator of this cohomology ring. We know that $f^* : H^2(\mathbb{CP}^{2n}) \to H^2(\mathbb{CP}^{2n})$ is representable by an integer m so that $f^*(\omega) = m \cdot \omega$. Now, we know that $\omega \wedge \omega$ is a 4-form on $\mathbb{CP}^{2n}$, and in fact generates it. The same is true for the k-fold wedge of ω and $H^{2k}(\mathbb{CP}^{2n})$. Therefore the action of f^* on $H^{2k}(\mathbb{CP}^{2n})$ is multiplication by m^k.

Lefschetz number has another definition in terms of homology, which is verifiable from the above definition: assuming X can be written as a finite CW-complex, we have the Lefschetz number of a map $f : X \to X$ is

$$\Lambda_f = \sum_{k \geq 0} (-1)^k \operatorname{tr}(f_* | H_k(X; \mathbb{Q})).$$

In our case, Poincaré duality applies, so we have

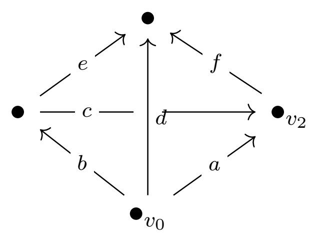
$$\Lambda_f = \sum_{k \geq 0} (-1)^k \operatorname{tr}(f^* | H^k(\mathbb{CP}^{2n}; \mathbb{R})) = \sum_{k \geq 0} m^k = \frac{m^{n+1} - 1}{m - 1}.$$

If $m = 1$, then we know that f^* is the identity map on cohomology, so $f \simeq \mathrm{id}_{\mathbb{CP}^{2n}}$. As such, $\Lambda_f = n + 1 = \chi(\mathbb{CP}^{2n})$. If $m \neq 1$, then $\Lambda_f = 0$ if and only if $m^{n+1} - 1 = 0$, i.e. $m^{n+1} = 1$. The only way this is possible is if $m = -1$ and $n + 1$ is even. But by assumption, n is even, so this is impossible. Therefore $\Lambda_f \neq 0$ always, so it has a fixed point by the Lefschetz fixed point theorem.

□

Problem 10.
Compute explicitly the simplicial homology of the surface of a tetrahedron, thus obtaining the homology of the 2-sphere.

Solution.
Let Δ_3 be a 3-simplex and let X be a (hollow) tetrahedron. Since homology in degree i depends only on the i- and $(i+1)$- skeletons we know that $H_0(\Delta_3) = H_0(X) = \mathbb{Z}$ and $H_1(\Delta_3) = H_1(X) = 0$. We need only to calculate the homology in degree 2 for X. We have the following (bad) picture:



To calculate $H_2(X)$, we need to determine $\ker \partial_2$ and $\operatorname{im} \partial_3$. Since $C_3(X) = 0$, we only need to calculate $\ker \partial_2$. We have four 2-simplices: $[v_0, v_1, v_2], [v_0, v_1, v_3], [v_0, v_2, v_3]$, and $[v_1, v_2, v_3]$. Name these a, b, c, d (in that order). Let the 1-simplices be defined by

$$\alpha = [v_0, v_1], \quad \beta = [v_0, v_2], \quad \gamma = [v_0, v_3], \quad \delta = [v_1, v_2], \quad \epsilon = [v_1, v_3], \quad \zeta = [v_2, v_3].$$

Then

$$\partial_2(a) = \delta - \beta + \alpha, \quad \partial_2(b) = \epsilon - \gamma + \alpha, \quad \partial_2(c) = \zeta - \gamma + \beta, \quad \partial_2(d) = \zeta - \epsilon + \delta.$$

Each 1-simplex appears exactly twice. We claim the kernel is a rank 1 subgroup of C_2. To illustrate: let $x = (n_a, n_b, n_c, n_d) \in \ker \partial_2$. Then we must have $n_a = -n_b$, so that the α terms cancel. We must have $n_a = n_c$ so the β terms cancel. We must have $n_b = -n_c$ so that the γ terms cancel (which we already know). We must have $n_a = -n_d$ so that the δ terms cancel. We must have $n_b = n_d$ so that the ϵ terms cancel (which we already know). Finally, we must have $n_c = -n_d$ so that the ζ terms cancel (which we already know). This means that $x = n_a(1, -1, 1, -1)$. Therefore $a - b + c - d$ generates $\ker \partial_2$, so we have $H_2(X) = \mathbb{Z}$. $\square$

8 Fall 2010

Problem 1.
Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors v_1 at x_1 and v_2 at x_2, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.

Solution.
We have shown below (e.g. Fall 2008 #3) how to construct ϕ so that $\phi(x_1) = x_2$. The added requirement on tangent vectors is not any more stringent. We can construct another diffeomorphism $\psi : M \to M$ which is the identity on x_2 but will change a neighbourhood U of x_2 to obtain what we want. We know we can construct a linear automorphism F of

$T_{x_2}M$ so that $F(d\phi(v_1)) = v_2$. The matrix associated to this map tells us how a local basis $\frac{\partial}{\partial x_i}$ is affected, and thus we can see how a system of local coordinates dx_i for U is affected and define a ψ accordingly with $d\psi = F$. So long as we parametrise so that x_2 is identified with $0 \in \mathbb{R}^n$, it will not be moved. Therefore the composition $f = \psi \circ \phi$ will have

$$df(v_1) = d(\psi \circ \phi)(v_1) = d\psi(d\phi(v_1)) = v_2$$

as well as $f(x_1) = x_2$. □

Problem 2.
Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversally.

Solution.
This is a consequence of the transversality theorem (e.g. Guillemin and Pollack p.68). Its statement for our purposes is, given a smooth map $F : X \times S \to Z$ such that F is transversal to a submanifold $Y \subset Z$, then at almost every point $s \in S$ we have $F(-, s) = f_s$ transversal to Y. For our purposes, we let $F : X \times \mathbb{R}^n \to \mathbb{R}^n$ be given by $F(x, a) = x + a$. First, we see that F is transversal to Y. Because the action of F on $\mathbb{R}^n$ is given by translation, dF has full rank. Therefore we must have

$$\operatorname{im} dF_{(x,a)} + T_y(Y) = T_y(\mathbb{R}^n)$$

for any (x, a) so that $F(x, a) = y \in Y$. This shows that almost all f_a are transversal to Y. Since $f_a(X) = X + a$, this shows that $X + a$ and Y intersect transversally for almost every a. □

Problem 3.
Let $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ be the space of $n \times n$ matrices with real coefficients.

(a) Show that

$$\mathrm{SL}(n) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$$

is a smooth submanifold of $M_n(\mathbb{R})$.

(b) Identify the tangent space to $\mathrm{SL}(n)$ at the identity matrix I_n.

(c) Show that $\mathrm{SL}(n)$ has trivial Euler characteristic.

Solution.

(a) See Spring 2009 #6.

(b) Ibid.

(c) We use the Poincaré-Hopf Index Theorem (e.g. Guillemin and Pollack p.134). It states that, if X is a smooth vector field on a compact, oriented manifold M with only finitely many zeroes, then the global sum of the indices of X equals $\chi(X)$.

In our case, we know that $\mathrm{SL}(n)$ is a Lie group, with multiplication inherited from $M_n(\mathbb{R})$. As such, we can define a nowhere-vanishing vector field on $\mathrm{SL}(n)$ by translating a nonzero vector v at I_n by left multiplication. Therefore the global sum of the indices of X is empty, so its sum is $0 = \chi(\mathrm{SL}(n))$.

□

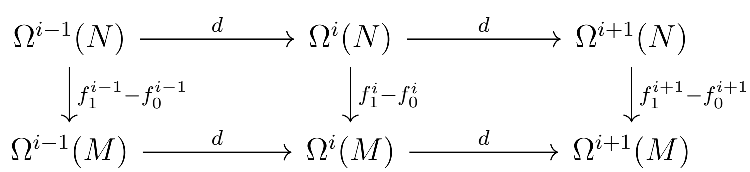
Problem 4.

(a) Let $f_i : M \to N$, $i = 0, 1$, be two smooth maps between smooth manifolds M and N, and $f_i^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$, $i = 0, 1$, be the induced chain maps between the respective de Rham complexes. Define the notion of chain homotopy between f_0^* and f_1^*. Here the coboundary operators on the de Rham complex are the exterior derivatives.

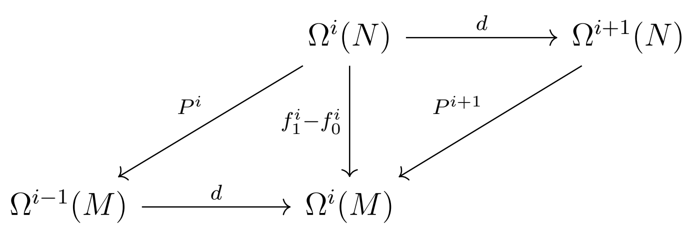
(b) Let X be a smooth vector field on a compact smooth manifold M, and let $\phi_t : M \to M$ be the flow generated by X at time t. Find an explicit chain homotopy between the chain maps ϕ_0^* and ϕ_1^* from $\Omega^\bullet(M)$ to itself. Hint: use Cartan's magic formula $\mathcal{L}_X\omega = d \circ i_x\omega + i_X \circ d\omega$.

Solution.

(a) We define a chain homotopy P between f_0^* and f_1^* in the following way: let f_0^i be the induced map on chains at degree i. Since $\mathrm{Hom}(\Omega^i(N), \Omega^i(M))$ is an abelian group, $f_1^i - f_0^i$ is well defined, which yields a picture.



Then we say that $f_0^* \sim f_1^*$, i.e. that the maps are chain homotopic, if there exists a map $P : \Omega^\bullet(N) \to \Omega^\bullet(M)$ of degree -1, i.e. $P^i : \Omega^i(N) \to \Omega^{i-1}(M)$, so that $f_1^i - f_0^i = d \circ P^i + P^{i+1} \circ d$, illustrated in the diagram below (which does *not* commute):



(b) Note that ϕ_0^* is nothing but the identity, since $\phi_0 = \mathrm{id}$. We claim that we that the chain homotopy we want is i_X, the interior product by the vector field X. Cartan's magic formula tells us

$$\mathcal{L}_X\omega = \lim_{h\to 0} \frac{1}{h}[\phi_h^*\omega - \omega] = i_X(d\omega) + d(i_X\omega).$$

Now, we know by the Fundamental Theorem of Calculus that, for a form ω and a point p, we have

$$(\phi_1^* - \phi_0^*)\omega(p) = \int_0^1 \frac{\partial \phi_t^*(\omega(p))}{\partial t}\, dt.$$

But this righthand integrand is just the Lie derivative of ω with respect to X at the point $\phi_t(p)$. Hence

$$(\phi_1^* - \phi_0^*)\omega(p) = \int_0^1 \mathcal{L}_X \omega(\phi_t(p))\, dt.$$

By Cartan's magic formula,

$$\begin{aligned} &= \int_0^1 di_X(\omega(\phi_t(p))) + i_X(d\omega(\phi_t(p)))\, dt \\ &= \int_0^1 di_X(\omega(\phi_t(p)))\, dt + \int_0^1 i_X(d\omega(\phi_t(p)))\, dt. \end{aligned}$$

Since the d on the left integral has to do with the integral itself, we can move it to the outside:

$$= d\int_0^1 i_X(\omega(\phi_t(p)))\, dt + \int_0^1 i_X(d\omega(\phi_t(p)))\, dt.$$

Therefore our requisite chain homotopy is the map

$$\omega(p) \mapsto \int_0^1 i_X \omega(\phi_t(p))\, dt,$$

which indeed is degree -1. Hence the two maps are chain homotopic.

□

Problem 5.
Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$ be a 2-form on $\mathbb{R}^{2n}$, where $(x_1, \ldots, x_{2n})$ are the standard coordinates. Define an S^1-action on $\mathbb{R}^{2n}$ as follows: for each $t \in S^1$, define $g_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by considering $\mathbb{R}^{2n}$ as the direct sum of n copies of $\mathbb{R}^2$ and rotating each $\mathbb{R}^2$ summand an angle t. Let X be the vector field on $\mathbb{R}^{2n}$ defined by

$$X(x) = \frac{dg_t(x)}{dt}\Big|_{t=0}$$

for any $x \in \mathbb{R}^{2n}$.

(a) Find the Lie derivative $\mathcal{L}_X \omega$ and a function f on $\mathbb{R}^{2n}$ such that $df = i_X \omega$.

(b) The S^1 action above induces an action on S^{2n-1}. Let $\mathbb{P}^{n-1}$ be the quotient space of S^{2n-1} by this action. Show that the quotient space $\mathbb{P}^{n-1}$ has a natural smooth structure and that the tangent space of $\mathbb{P}^{n-1}$ at any point $\underline{x}$ can be identified with the quotient of the tangent space $T_x S^{2n-1}$ by the line spanned by $X(x)$ for any $x \in \underline{x}$. Here $\underline{x}$ is the orbit of x under the S^1-action.

(c) Show that ω descends to a well-defined 2-form on the quotient space $\mathbb{P}^{n-1}$ and that the 2-form so defined is closed.

(d) Is the closed form in (c) exact?

Solution.

(a) The vector field is easily shown to be

$$X(x_1, \ldots, x_{2n}) = (x_2, -x_1, \ldots, x_{2n}, -x_{2n-1}).$$

We will use Cartan's magic formula, stated in the above problem. We have

$$\mathcal{L}_X \omega = d \circ i_X \omega + i_X \circ d\omega.$$

We have

$$i_X \omega = \omega(X, -) = \left(\sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i} \right) (X, -).$$

We only need to understand what is going on at each of the summands. At the point $x = (x_1, \ldots, x_{2n})$, we have

$$X = \sum_{i=1}^{n} -x_{2i} \frac{\partial}{\partial x^{2i-1}} + x_{2i-1} \frac{\partial}{\partial x^{2i}},$$

so if $Y = \sum_{j=1}^{2n} y_j \frac{\partial}{\partial y^j}$ is an arbitrary vector field, we have

$$dx_{2i-1} \wedge dx_{2i}(X, Y) = \det \begin{pmatrix} -x_{2i} & y_{2i-1} \\ x_{2i-1} & y_{2i} \end{pmatrix} = -(x_{2i-1} y_{2i-1} + x_{2i} y_{2i}).$$

Therefore we have

$$i_X \omega = -\sum_{i=1}^{2n} x_i \, dx_i.$$

We can see that the choice

$$f(x_1, \ldots, x_{2n}) = -\frac{1}{2} \left(x_1^2 + \cdots + x_{2n}^2 \right)$$

gives us $i_X \omega = df$. Since it is clear that $d\omega = 0$, and we have shown that $i_X \omega$ is exact, we have

$$\mathcal{L}_X \omega = d(df) + i_X(0) = 0.$$

(b) We claim we are working with $\mathbb{CP}^{n-1}$, though we aren't saying it out loud. Consider $\mathbb{R}^{2n} = \mathbb{C}^n$, where each plane from the premise is identified with a copy of $\mathbb{C}$. Then the rotation by S^1 in each plane corresponds to complex multiplication by $\lambda \in \mathbb{C}^\times$ with $|\lambda| = 1$. This is exactly the identification on $\mathbb{C}^n$ that gives us $\mathbb{CP}^{n-1}$. We will now refer to $\mathbb{P}^{n-1}$ by its proper name. $\mathbb{CP}^{n-1}$ has a natural smooth structure as a quotient manifold.

For the second part, we know that $X(x)$ is the tangent vector to $T_x S^{2n-1}$ that corresponds to the rotation in the appropriate complex plane that we are identifying to a point. Since $T_{\underline{x}} \mathbb{CP}^{n-1}$ has dimension $n-1$, what remains after the quotient by the rotation tangent vector must be the remainder.

(c) ω descends to $\mathbb{CP}^{n-1}$ exactly if ω is invariant with respect to the quotient action. Since this identification is given by the flows of vector field X, we need to show $\mathcal{L}_X\omega = 0$. But that was part (a). The form is closed because, pulling back ω from $\mathbb{CP}^{n-1}$ gives the original form on $\mathbb{RP}^{2n}$. But this form was originally closed upstairs, so it must be closed downstairs as well.

(d) ω is not exact. Though the form upstairs is exact, and we can write $\omega = d\eta$ for

$$\eta = \sum_{i=1}^{n} x_{2i-1}\, dx_{2i}.$$

But η is not invariant under the group action, so cannot descend to $\mathbb{CP}^{n-1}$.

□

Problem 6.
Suppose that $f : S^n \to S^n$ is a smooth map of degree not equal to $(-1)^{n+1}$. Show that f has a fixed point.

Solution.
This is the contrapositive of Spring 2013, #7(b). □

Problem 7.

(a) Let G be a finitely presented group. Show that there is a topological space X with fundamental group $\pi_1(X) \cong G$.

(b) Give an example of X in the case $G = \mathbb{Z} * \mathbb{Z}$, the free group on two generators.

(c) How many connected, 2-sheeted covering spaces does the space X from (b) have?

Solution.

(a) If G is finitely presented, then let $g_1, \ldots, g_m$ be a minimal generating set for G and let $r_1, \ldots, r_n$ be a minimal set of relations so that $G \cong F^m/\langle\langle r_1, \ldots, r_n\rangle\rangle$, where F is the free group of rank m. We construct the topological space in the following way: take a 0-cell and glue to it the boundaries of m 1-cells, where we label each copy with one generator g_i. This space is homeomorphic to $\bigvee^m S^1$ and will have fundamental group F^m, so we need only construct the relations.

Each relation r_j can be realised as a finite string of generators or their inverses, e.g.

$$r_j = g_{i_1}^{\varepsilon_1} g_{i_2}^{\varepsilon_2} \cdots g_{i_k}^{\varepsilon_k},$$

where $\varepsilon = \pm 1$. Let that finite string give instructions for gluing n 2-cells to the wedge of circles we have constructed, where $\varepsilon = -1$ means glue the boundary while reversing orientation, and call this new space X. This gives us a map $F^m \to \pi_1(X)$ whose kernel is generated by the relation words r_j. This means that $\pi_1(X) \cong G$, as required.

(b) We may take X to be the wedge of 2 circles, since $G \cong F^2$ and hence we need no relations.

(c) The (basepoint-preserving isomorphism classes of) connected, 2-sheeted covering spaces of X correspond to the subgroups of $\pi_1(X)$ of index 2. Suppose $H \subset F^2$ is a subgroup of index 2. Let a, b be the generators of F^2, and consider the cosets aH, bH, H. Two of these must be the same. Further, we cannot have $aH = bH$, because then we cannot construct a coset other than H, which is a contradiction.

Therefore suppose we have $aH = H$. Then every word in H must start with an a. Further, since no words in bH start with a and $H \cup bH = F^2$, we must have $H = \{x \in F^2 : x = a \cdots\}$. If $bH = H$ then we see that $H = \{x \in F^2 : x = b \cdots\}$. This exhausts the possibilities for H, so we have two connected 2-sheeted covering spaces of X.

□

Problem 8.
Let G be a connected topological group. Show that $\pi_1(G)$ is a commutative group.

Solution.
See Fall 2008 #8. □

Problem 9.
Show that if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.

Solution.
We will use simplicial homology to show this. First, if $\mathbb{R}^m \cong \mathbb{R}^n$, then removing a point from each does not change the homeomorphism. Assume that $m, n \geq 2$, because if we have $m = 1$ and $n \geq 2$ (without loss of generality) then removing a point from $\mathbb{R}^m$ makes it disconnected, but not so for $\mathbb{R}^n$, so we have a contradiction.

Now, because $\mathbb{R}^k \setminus \{pt\}$ deformation retracts onto S^{k-1}, this implies that we have an isomorphism of $H_i(S^{n-1})$ and $H_i(S^{m-1})$ at each i. Because a k-sphere is comprised of a vertex and an k-cell with boundary glued onto that one vertex, we can calculate directly that

$$H_i(S^k) = \begin{cases} \mathbb{Z} & i = 0, k \\ 0 & \text{else} \end{cases}$$

Therefore spheres are uniquely determined by where their homology is supported outside of degree 0. Therefore since we see that $H_{n-1}(S^{n-1}) = \mathbb{Z} = H_{m-1}(S^{m-1})$, we must have $n - 1 = m - 1$, which implies that $n = m$ as required. □

Problem 10.
Let N_g be the nonorientable surface of genus g, that is, the connected sum of g copies of $\mathbb{RP}^2$. Calculate the fundamental group and homology groups of N_g.

Solution.
We can put a simplical structure on N_g in the following way: take one 0-cell v, g 1-cells $\{e_1, \ldots, e_g\}$ all glued to the 0-cell, and one 2-cell A glued according to $e_1^2 \cdots e_g^2$. Therefore we have

$$\partial(p) = 0, \qquad \partial(e_i) = 0, \qquad \partial(A) = 2e_1 + \cdots + 2e_g,$$

where ∂ is the boundary operator. This shows that $H_0(N_g) = \mathbb{Z}$ (since it is connected) and $H_2(N_g) = 0$. We also have $H_1(N_g) \cong \mathbb{Z}^g/\langle(2,\dots,2)\rangle$. We may rewrite an orthogonal basis for $\mathbb{Z}^g$ such that $(1,\dots,1)$ is a basis element, which gives us

$$\begin{aligned} H_1(N_g) &= \left(\langle(1,\dots,1)\rangle \oplus \mathbb{Z}^{g-1}\right)/\langle(2,\dots,2)\rangle \\ &\cong \langle(1,\dots,1)\rangle/\langle(2,\dots,2)\rangle \oplus \mathbb{Z}^{g-1} \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{g-1}. \end{aligned}$$

For $\pi_1(N_g)$, we use the same method we did for Problem 7 above. The fundamental group is a quotient of F^g by the attaching map for A, which gives us

$$\pi_1(N_g) = F^g/\langle e_1^2 \cdots e_g^2\rangle.$$

We will not give a better description of this group. □

9 Spring 2010

Problem 1.
Let M_n be the space of all $n \times n$ matrices with real entries and let S_n be the subset consisting of all symmetric matrices. Consider the map $F : M_n \to S_n$ defined by $F(A) = AA^t - I_n$.

(a) Show that 0_n is a regular value of F.

(b) Deduce that $O(n)$, the set of all $n \times n$ matrices such that $A^{-1} = A^t$, is a submanifold of M_n.

(c) Find the dimension of $O(n)$ and determine the tangent space of $O(n)$ at the identity matrix as a subspace of $TM_n \cong M_n$.

Solution.
(a) We need to show that F is a submersion at 0_n. That is, given any $A \in F^{-1}(0)$, we need to show $dF_A : T_A M_n \to T_0 S_n$ surjective. Because M_n and S_n are Euclidean spaces, we may identify their tangent spaces with the space itself. Therefore, we need to show if, given $C \in S_n$, that there is a $B \in M_n$ so that $dF_A(B) = C$. By definition,

$$dF_A(B) = \lim_{t\to 0} \frac{F(A+tB) - F(A)}{t} = \lim_{t\to 0} \frac{(A+tB)(A^t + tB^t) - AA^t}{t} = AB^t + BA^t.$$

We claim $B = \frac{1}{2}CA$ works. To see this,

$$AB^t + BA^t = \frac{1}{2}(AA^tC + CAA^t) = \frac{1}{2}(2C) = C,$$

since by assumption $AA^t = I_n$. Therefore 0_n is a regular value.

(b) By the preimage theorem, we know that the preimage of a regular value is a submanifold. We see that $F^{-1}(0_n) = O(n)$, since

$$AA^t - I_n = 0_n \iff AA^t = I_n \iff A^t = A^{-1}.$$

(c) By the same theorem, we have $\dim O(n) = \dim M_n - \dim S_n$. We have $\dim M_n = n^2$, as we have one degree per matrix entry. By the same reasoning, there is one degree per matrix entry for S_n, but a symmetric matrix is defined by its entires on the upper triangle. This gives $\dim S_n = n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}$. Hence

$$\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{2n^2 - n^2 - n}{2} = \frac{n(n-1)}{2}.$$

Now, we need to calculate the tangent space. We know that $T_{I_n}O(n)$ is the space of derivations of lines through I_n, each of which is representable by a matrix in M_n. Let $A \in M_n$ and consider the line $I_n + tA$ for $t \in \mathbb{R}$. The derivative of this path is just A, so we need to see which A satisfies that $I_n + tA \in O(n)$ for small enough t. Hence we need

$$\begin{aligned} I_n + tA^t = (I_n + tA)^{-1} &\implies (I_n + tA^t)(I_n + tA) = I_n \\ &\implies I_n + tA^t + tA + t^2 A^t A = I_n. \end{aligned}$$

We may ignore the t^2 term as it goes to 0 much faster than the t term, so this means we need $t(A + A^t) = 0$ for small t, so this implies that $-A = A^t$, i.e. A is antisymmetric. Hence the tangent space to I_n is the space of antisymmetric matrices.

□

Problem 2.
Show that $T^2 \times S^n$ is parallelisable, where S^n is the n-sphere, T^2 is the 2-torus, and a manifold of dimension k is parallelisable if there exist k vector fields $V_1, \dots, V_k$ such that $V_1(p), \dots, V_k(p)$ are linearly independent everywhere.

Solution.
In fact, we can show that $S^1 \times S^n$ is parallelisable. If that is the case, then the product of two parallelisable manifolds S^1 and $S^1 \times S^n$ is still parallelisable.

Consider TS^n. If we take $TS^n \times \mathbb{R}$, we obtain $S^n \times \mathbb{R}^{n+1}$, since this $\mathbb{R}$ term can be realised as the normal bundle to the sphere as a submanifold of $\mathbb{R}^{n+1}$. We know that $TS^1 = S^1 \times \mathbb{R}$, because taking the unit tangent vector in $\mathbb{R}^2$ to S^1 at every point is a nowhere-vanishing vector field. Let $\pi_1 : S^1 \times S^n \to S^1$ be projection onto the first coordinate and $\pi_2 : S^1 \times S^n \to S^n$ be projection onto the second coordinate. Then

$$T(S^1 \times S^n) = \pi_1^*(TS^1) \times \pi_2^*(TS^n) = \pi_1^*(S^1 \times \mathbb{R} \times TS^n) = S^1 \times S^n \times \mathbb{R}^{n+1},$$

so this is parallelisable. □

Problem 3.
Let $\pi : M_1 \to M_2$ be a smooth map of compact differentiable manifolds. Suppose that for each $p \in M_1$, the differential $d\pi : T_pM_1 \to T_{\pi(p)}M_2$ is a vector space isomorphism.

(a) Show that if M_1 is connected, then π is a covering space projection.

(b) Give an example where M_2 is compact but $\pi : M_1 \to M_2$ is not a covering space (but has the $d\pi$ isomorphism property).

Solution.

(a) We have to assume that M_2 is connected, I believe, otherwise π may not be surjective. Then the problem is just Spring 2009, #2.

(b) Ibid.

□

Problem 4.
Let $\mathcal{F}^k(M)$ denote the differentiable smooth k-forms on a manifold M. Suppose U and V are open subsets of M.

(a) Explain carefully how the usual exact sequence

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$$

arises.

(b) Write down the long exact sequence in de Rham cohomology associated to the above short exact sequence and describe explicitly how the map

$$H^k(U \cap V) \to H^{k+1}(U \cup V)$$

arises.

Solution.

(a) First, it suffices to explain this exact sequence in each degree k, since the maps between these chains are going to have degree 0. First, suppose ω is a k-form on $U \cup V$. Then we can restrict ω to $U \subset U \cup V$ or to $V \subset U \cup V$. The first map f^k is the one defined by

$$\omega \mapsto (\omega|_U, \omega|_V).$$

For the second map, we can restrict a form ω on U to $U \cap V \subset U$ and a form η on V to $U \cap V \subset V$. The second map g^k is defined by

$$(\omega, \eta) = \omega|_{U\cap V} - \eta|_{U\cap V}.$$

To see this sequence is exact, first we see that for a form $\omega \in \mathcal{F}^k(U \cup V)$, we have

$$\omega \mapsto (\omega|_U, \omega|_V) \mapsto \omega|_{U\cap V} - \omega|_{U\cap V} = 0,$$

so $\operatorname{im} f^k \subset \ker g^k$. Now, suppose $(\omega, \eta) \in \ker g^k$. Then $\omega|_{U\cap V} = \eta|_{U\cap V}$, so we may glue the two forms together with no contradiction on the overlap $U \cap V$, i.e.

$$\theta(x) = \begin{cases} \omega(x) & x \in U \\ \eta(x) & x \in V \setminus U \end{cases}.$$

This is a form on $U \cup V$, so we have $\ker g^k \subset \operatorname{im} f^k$. Therefore the sequence is exact.

(b) The long exact sequence we obtain is

$$\cdots \to H^{k-1}(U\cap V) \xrightarrow{\delta^{k-1}} H^k(U\cup V) \xrightarrow{f_*^k} H^k(U)\oplus H^k(V) \xrightarrow{g_*^k} H^k(U\cap V) \xrightarrow{\delta^k} \cdots$$

We need to define the missing maps, however. Let $\omega \in H^k(U\cap V)$, which we can represent by $\eta_U - \eta_V$ from the short exact sequence, where $\eta_U \in \mathcal{F}^k(U)$ and $\eta_V \in \mathcal{F}^k(V)$ are closed forms. Then $d\omega = d\eta_U - d\eta_V = 0$, so $d\eta_U = d\eta_V$ on $U\cap V$. Therefore we can define $\delta^k(\omega)$ by gluing together $d\eta_U$ and $d\eta_V$ exactly as we did above to obtain a $k+1$ form on $U\cup V$, and the sequence is seen to be exact at this point in exactly the same way.

□

Problem 5.
Explain carefully why the following holds: if $\pi : S^N \to M$, $N > 1$ is a covering space with M orientable, then every closed k-form on M, $1 \le k < N$, is exact.

Solution.
Suppose that ω is a closed k-form on M. We know that π^* induces a map on cohomology $\pi^* : \mathcal{F}^k(M) \to \mathcal{F}^k(S^N)$, and in particular π^* maps closed forms to closed forms. Therefore $\pi^*(\omega)$ is a closed k-form on S^N. Therefore $\pi^*(\omega)$ is exact, since $\mathcal{F}^k(S^n) = 0$ for all $1 \le k < N$. Write $\pi^*(\omega) = d\theta$.

We now use the fact that we have a covering space. In particular, since S^N is compact, this covering space is finitely-sheeted. If G is the (finite) group of deck transformations of π, then $S^n/G \cong M$. In order for a form on S^N to descend to M, it needs to be G-equivariant. We can construct this:

$$\eta = \frac{1}{|G|}\sum_{g\in G} g^*(\theta).$$

This form is G-equivariant. Therefore we can view it as a form on M. We claim that $d\eta = \omega$. We have

$$d\eta = \frac{1}{|G|}\sum_{g\in G} d(g^*(\theta)) = \frac{1}{|G|}\sum_{g\in G} g^*(d\theta) = \frac{1}{|G|}\sum_{g\in G} g^*(\pi^*(\omega)) = \omega,$$

because adding up all the the g^* conjugates of $\pi^*(\omega)$ will get us back the form ω that we started with. Therefore ω is exact. □

Problem 6.
Calculate the singular homology of $\mathbb{R}^n$, $n > 1$, with k points removed, $k \ge 1$.

Solution.
First, assume that $n = 2$. Then if we remove one point, we have a deformation retract to S^1. Therefore if we remove k points from $\mathbb{R}^2$, we have a deformation retract onto the wedge of k copies of S^1. Since we know how to calculate the singular homology of S^1 and know that wedge products become direct sums under homology (by the Mayer-Vietoris sequence), we are done. Indeed, we see that the general case is analogous. Removing k points from $\mathbb{R}^n$

gives a deformation retract onto the wedge of k copies of S^{n-1}. Therefore we can summarise all the homology thus:

$$H_i(\mathbb{R}^n \setminus \{k \text{ points}\}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^k & i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

□

Problem 7.

(a) Explain what is meant by adding a handle to a 2-sphere for a 2-dimensional orientable surface in general.

(b) Show that a 2-sphere with a positive number of handles cannot be simply connected.

Solution.

(a) Let M be such a surface. Then adding a handle happens in the following way: we recall that the connected sum # of two 2-spaces X, Y is accomplished thus: remove a disc homeomorphic to $\mathbb{R}^2$ from each space and glue X to Y along the boundary of this disc. If these spaces are oriented, then ∂X and ∂Y (which is homeomorphic to S^1) are naturally endowed with an orientation, and so we glue the space together as to preserve orientation.

We add a handle to an orientable space M by taking first taking the sum $M \# Z$, where Z is a hollow cylinder and remove one end of the cylinder to accomplish the sum. Then we connect the other end of Z to $M \setminus Z$ in the same way. This adds a tube between two discs on M, which looks much like a handle on a coffee mug if one visualises it properly.

(b) We claim that S^2 with g handles attached is homeomorphic to the torus of genus g, which we denote M_g. Indeed, it is clear that S^2 with one handle is homeomorphic to T^2 by taking the two open sets cut out of S^2 to be the open northern and southern hemispheres. Repeating this process increases the genus by 1 each time.

Now, we have $\pi_1(T^2) = \mathbb{Z}^2$, so the space is not simply connected. It is easy to see that the torus of genus g retracts onto the torus of genus $g-1$, which induces an injection $\pi_1(M_{g-1}) \hookrightarrow \pi_1(M_g)$. Therefore adding more handles can only enlarge the fundamental group, so a a sphere with a positive number of handles cannot be simply connected.

□

Problem 8.

(a) Define the degree $\deg f$ of a smooth map $f : S^2 \to S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choice you need to make in your definition.

(b) Prove in detail that for each integer k, there is a smooth map $f : S^2 \to S^2$ of degree k.

Solution.

(a) We know that $f : S^2 \to S^2$ induces a map on homology $f_* : H_2(S^2) \to H_2(S^2)$. Since $H_2(S^2) \cong \mathbb{Z}$, we know that $f_* \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. Therefore we associate to f_* an integer which we call its degree, where $f_*(x) = \deg f \cdot x$ for any $x \in H_2(S^2)$ (the domain).

(b) We will use the other definition of degree for this part: for any closed orientable manifolds of the same dimension M, N $f : M \to N$, and $p \in N$, $f^{-1}(p) = \{x_1, \dots, x_k\}$ is a finite set of discrete points. Further, at each of these points, f is a local diffeomorphism. As such, f is either orientation preserving or orientation reversing. Let

$$\mathrm{sgn}_x(f) = \begin{cases} 1 & f \text{ is orientation preserving near } x \\ -1 & f \text{ is orientation reversing near } x \end{cases}$$

Then we define $\deg_p f = \sum_{x \in f^{-1}(p)} \mathrm{sgn}_x(f)$. One can verify that since M is path connected by assumption that the choice of p is irrelevant, and so we can write $\deg f$ alone.

In our case, define $f : S^2 \to S^2$ in the following way: in the domain sphere, take k disjoint open discs. Then f will map those discs while preserving orientation onto the codomain $S^2 \setminus \{p\}$, where p is the north pole, and f maps the complement of the open discs to p. This map by construction is continuous and has $\deg f = k$ by taking any point $q \neq p$ and doing the above calculation. We can make this map smooth without changing the degree by creating a smooth bump at the border of each of the k open discs. To get maps with negative degree, simply glue the open discs onto $S^2 \setminus \{p\}$ while reversing orientation. This completes the proof.

□

Problem 9.
Explain how Stokes' theorem for manifolds with boundary gives, as a special case, the classical divergence theorem.

Solution.
See Spring 2008, #3. □

Problem 10.

(a) Show that every map $F : S^n \to T^k$ is nulhomotopic.

(b) Show that there is a map $F : T^n \to S^n$ such that F is not nulhomotopic.

(c) Show that every (smooth) map $F : S^n \to S^{n_1} \times \cdots \times S^{n_k}$, $n_1 + \cdots + n_k = n$, $n_i > 0$, $k \geq 2$ has degree 0.

Solution.

(a) We assume $n > 1$, otherwise this is patently false. The universal cover of T^k is $\mathbb{R}^k$, so any map $F : S^n \to T^k$ has a unique lift to $F' : S^n \to \mathbb{R}^k$ because $\pi_1(S^n)$ is trivial. We claim that any such map is nulhomotopic, which is obvious since $\mathbb{R}^k$ is contractible. Therefore, if we let $p : \mathbb{R}^k \to T^k$ be the usual covering map, since $F = p \circ F'$, a homotopy of F' to the constant map can descend to one on F.

(b) Let $U \subset T^n$ be some open set diffeomorphic to $\mathbb{R}^n$. Then define a map F as follows: let F map U diffeomorphically (orientation preserving) onto $S^n \setminus \{n\}$, where n is the north pole, and map $T^n \setminus U$ to n. This map is continuous by construction, and can be made smooth if necessary by the density of smooth functions in continuous functions. Then using the definition of degree by orientated intersection theory, we take a regular value of F, which is any point $x \neq n$. As such, $F^{-1}(x)$ has only one point of matching orientation. Hence $\deg F = 1$. Since degree is a homotopy invariant, if F were nulhomotopic we would need $\deg F = 0$, so F is not nulhomotopic.

(c) We now use the definition of degree via differential forms: let ω_i be a nowhere vanishing n_i-form on S^{n_i}, which exists because S^{n_i} is orientable. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is a nowhere vanishing n-form on the product of spheres. If we normalise this, we get a nowhere vanishing form ω on the n-form that integrates to 1. Then

$$\begin{aligned}\deg F = \int_{S^n} F^*\omega &= \int_{S^n} F^*(\omega_1 \wedge \cdots \wedge \omega_k) \\ &= \int_{S^n} F^*\omega_1 \wedge \cdots \wedge F^*\omega_k.\end{aligned}$$

First, since $F^*\omega_i$ is an closed n_i-form on S^n (as pullback sends closed forms to closed forms), it must be exact since $0 < n_i < n$. Then we let $F^*\omega_i = d\eta_i$ for each i. Then

$$\int_{S^n} \mathbb{F}^*\omega = \int_{S^n} d\eta_1 \wedge \cdots \wedge d\eta_k = \int_{S^n} d(\eta_1 \wedge d\eta_2 \wedge \cdots \wedge d\eta_k)$$

Applying Stokes' theorem,

$$\int_{S^n} F^*\omega = \int_{\partial S^n} \eta_1 \wedge d\eta_2 \wedge \cdots \wedge d\eta_k = 0$$

since $\partial S^n = \varnothing$. Therefore $\deg F = 0$ for any F. By the Hopf degree theorem, any degree zero map is nulhomotopic.

□

10 Spring 2009

Problem 1.

(a) Show that a closed 1-form θ on $S^1 \times (-1, 1)$ is dF for some function $F : S^1 \times (-1, 1) \to \mathbb{R}$ if and only if $\int_{S^1} i^*\theta = 0$, where $i : S^1 \to S^1 \times (-1, 1)$ is defined by $i(p) = (p, 0)$.

(b) Show that a 2-form ω on S^2 is $d\theta$ for some 1-form on S^1 if and only if $\int_{S^2} \omega = 0$.

Solution.

(a) We know that $S^1 \times (-1, 1)$ is homotopic to S^1 by contracting each line $\{x\} \times (-1, 1)$. Therefore $H^1(S^1) \cong H^1(S^1 \times (-1, 1))$, and $i^* : H^1(S^1 \times (-1, 1) \to H^1(S^1)$ is an isomorphism. In particular every cohomology class in $H^1(S^1 \times (-1, 1))$ is uniquely determined by a cohomology class of S^1. Hence if we have closed form ω on $S^1 \times (-1,1)$, it is exact if and only if its corresponding form $i^*\omega$ is exact on S^1.

That $i^*\omega$ is exact if and only if $\int_{S^1} i^*\omega = 0$ follows from Stokes theorem. If $i^*\omega = df$, then

$$\int_{S^1} i^*\omega = \int_{\partial S^1} f = 0,$$

since $\partial S^1 = \varnothing$. Conversely, we can construct the function f if $i^*\omega$ integrates to 0. Let $p \in S^1$ be some basepoint. Then for any $q \in S^1$, there is an arc $\gamma : [0,1] \to S^1$ so that $\gamma(0) = p$ and $\gamma(1) = q$. Define

$$f(q) := \int_\gamma \omega.$$

By the fundamental theorem of calculus, we have $df = \omega$. We need only show that f is well defined. Suppose that γ' is another path from p to q. Then let $\bar{\gamma}'$ be the path from q to p around the other side of S^1. We have

$$\int_\gamma \omega - \int_\gamma' \omega \int_{\bar{\gamma}' \circ \gamma} \omega = \int_{S^1} \omega = 0.$$

Therefore f is well-defined, so we are done.

(b) This works similarly. Note that the θ in the question might as well be viewed as form on S^2. If we have a form on S^2, we obtain a form on S^1 induced by the inclusion of the equator. Now, if ω is exact, then let $\omega = d\theta$. We have

$$\int_{S^2} \omega = \int_{\partial S^2} \theta = 0.$$

Conversely, suppose $\int_{S^2} \omega = 0$. We will construct θ piecewise. Let $L = S^2 \setminus \{n\}$ and $U = S^2 \setminus \{s\}$, where n is the north pole and s is the south pole. Then $U, L \simeq D^2$, so they are contractible. Now $\omega|_U$ and $\omega|_L$ are (degree at least 1) closed forms on a contractible domain, so they are exact by the Poincaré lemma. Therefore let θ_U and θ_L be the corresponding 1-forms. We can restrict these forms to $U \cap L$, and we claim that θ_U and θ_L differ by an exact form on $U \cap L$. Indeed, if i^* denotes the map induced by an inclusion map,

$$d(i^*_{U\cap L}(\theta_U - \theta_L)) = i^*_{U\cap L}(d\theta_U - d\theta_L) = \omega|_{U\cap L} - \omega|_{U\cap L} = 0.$$

Therefore $\theta_U - \theta_L$ is a closed 1-form on $U \cap L \simeq S^1$. Further, we know that $\int_{S^1} \theta_U = \int_{S^1} \theta_L$, so in fact $\theta_U - \theta_L$ is exact, say it equals df for $f : U \cap L \to \mathbb{R}$.

We can now glue together θ_U and θ_L on S^2. We have $\theta_L = \theta_U + df$. Therefore on S^2 less a neighbourhood of the south pole, define $\theta = \theta_U$, and on that neighbourhood of the south pole we can smoothly interpolate $\theta_U + t \cdot df$ so that the value is correct at the south pole. Then all the cohomology classes work out properly, and $d\theta = \omega$ globally.

$\square$

Problem 2.
Suppose that M and N are connected smooth manifolds of the same dimension $n \geq 1$ and $F : M \to N$ is a smooth map such that $dF : T_pM \to T_{F(p)}N$ is surjective for each $p \in M$.

(a) Prove that if M is compact, then F is onto and F is a covering map.

(b) Find an example of such an everywhere nonsingular equidimensional map where N is compact, F is onto, $F^{-1}(p)$ is finite for each $p \in N$, but F is not a covering map.

Solution.

(a) F is a submersion, so it is an open map. Since M is compact, F is proper and $F(M)$ is a compact subset of N, and in particular is closed. Therefore $F(M)$ is both open and closed in N, so $F(M) = N$ because N is connected. We have a proper surjective continuous map, but we need to show that it is a local diffeomorphism. But since $\dim T_pM = \dim T_{F(p)}N = n$, dF_p is actually a linear isomorphism everywhere, so we are done.

(b) Take $F : \mathbb{R} \to S^1$ so that $[0, 1]$ maps diffeomorphically onto $S^1 \setminus U$ for some connected open set U and maps $(-\infty, 0)$ and $(1, \infty)$ onto U diffeomorphically. Then we have a local diffeomorphism, so $d\pi$ is an isomorphism everywhere. Further, $F^{-1}(p)$ has cardinality 1 or 2 at any point. But this map is not even-sheeted, so it cannot be a covering map.

□

Problem 3.

(a) Suppose that M is a smooth connected manifold. Prove that, given an open subset U of M and a finite set of points $p_1, \ldots, p_k$, there is a diffeomorphism $F : M \to M$ such that $f(\{p_1, \ldots, p_k\}) \subset U$.

(b) Use part (a) to show that if M is compact and $\chi(M) = 0$, then there is a vector field on M which vanishes nowhere. You may assume that if a vector field has isolated zeroes, then the sum of the indices at the zero points equals $\chi(M) = 0$.

Solution.

(a) We proceed by induction. Let $p \in M$ be any point, and let $x \in U$. Then since M is connected, it is path connected, so let $\gamma : I \to M$ be a path with $\gamma(0) = p$ and $\gamma(1) = x$. We may take a tubular neighbourhood V of γ in M to obtain a tube diffeomorphic to $\mathbb{R}^n$. Then we may construct a diffeomorphism of M that is the identity outside of V and inside V moves p along the path γ near $x \in U$.

For k points, we compose diffeomorphisms $F_1, \ldots, F_k$ which repeat the above process.

(b) Let V be any vector field on M with (finitely many) isolated zeroes at $p_1, \ldots, p_k$. Then we know that the sum of the indices of these zeroes equals zero. By (a), we may move all these zeroes into an arbitrarily small neighbourhood U diffeomorphic to $\mathbb{R}^n$. Then we know the index of V around U is zero. Therefore we may 'delete' the vector field as given inside U and replace it with a nonvanishing vector field, since locally constructing such a thing is trivial. The new vector field V we obtain is nowhere vanishing.

□

Problem 4.
A smooth vector field V on $\mathbb{R}^3$ is said to be 'gradient-like' if, for each $p \in \mathbb{R}^3$, there is a neighbourhood U_p of p and a function $\lambda_p : U_p \to \mathbb{R} \setminus \{0\}$ such that $\lambda_p V$ on U_p is the gradient of some smooth function on U_p. Suppose V is nowhere zero on $\mathbb{R}^3$. Then show that V is gradient-like if and only if $\operatorname{curl} V$ is perpendicular to V at each point of $\mathbb{R}^3$.

Solution.
See Spring 2012, #4. □

Problem 5.
Suppose that M is a compact smooth manifold of dimension n.

(a) Show that there is a positive integer k such that there is an immersion $F : M \to \mathbb{R}^k$.

(b) Show that if $k > 2n$, there is a $k-1$-dimensional subspace H of $\mathbb{R}^k$ such that $P \circ F$ is an immersion, where $P : \mathbb{R}^k \to H$ is orthogonal projection.

Solution.

(a) See Fall 2011, #1.

(b) See Guillemin and Pollack, p.51. Define the orthogonal projection P in the following way: define $g : TM \to \mathbb{R}^k$ by $g(x, v) = dF_x(v)$. Since $\dim TM = 2n < k$, there is a point $a \in \mathbb{R}^k$ that is not in the image of g. Then let H be the orthogonal complement of a, and P the associated orthogonal projection. We claim that $P \circ F$ is still an immersion.

Suppose that $d(P \circ F)_x(v) = 0$ for nonzero $(x, v) \in TM$. Then since P is linear, we have
$$d(P \circ F)_x(v) = P(dF_x(v)) = 0.$$
Therefore $dF_x(v) \in \mathbb{R}a$. Since our original F was an immersion, this implies that $dF_x(v) = t \cdot a$ has $t \neq 0$. But then $dF_x(v/t) = g(x, v/t) = a$, a contradiction.

□

Problem 6.
Let $\mathrm{GL}^+(n, \mathbb{R})$ be the set of $n \times n$ matrices with determinant > 0. Note that $\mathrm{GL}^+(n, \mathbb{R})$ can be considered to be a subset of $\mathbb{R}^{n^2}$ and this subset is open.

(a) Prove that $\mathrm{SL}^+(n, \mathbb{R}) = \{A \in \mathrm{GL}^+(n, \mathbb{R}) : \det A = 1\}$ is a submanifold.

(b) Identify the tangent space of $\mathrm{SL}^+(n, \mathbb{R})$ at the identity matrix I_n.

(c) Prove that, for every $n \times n$ matrix B, the series
$$\exp(B) = 1 + B + \frac{B}{2!} + \cdots$$
converges to some $n \times n$ matrix.

(d) Prove that if $\exp(tB) \in \mathrm{SL}^+(n, \mathbb{R})$ for all $t \in \mathbb{R}$, then $\operatorname{tr} B = 0$.

(e) Prove that if $\operatorname{tr} B = 0$, then $\exp(B) \in \mathrm{SL}^+(n, \mathbb{R})$.

Solution.

(a) We have the continuous determinant map $\det : \mathrm{GL}^+(n, \mathbb{R}) \to \mathbb{R}^+$, and thus we can define $\mathrm{SL}^+(n, \mathbb{R}) = \det^{-1}(1)$. If we prove that 1 is a regular value of det, then we are done by the preimage theorem. Since the rank of $T_1\mathbb{R}$ is 1, we need to only show that the derivative of det is not identically zero. But this is clear, since perturbing a matrix in almost any direction changes the value of the determinant. Therefore det is a submersion at the value 1, so it is a regular value.

(b) We claim this tangent space is $\mathfrak{sl}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \operatorname{tr} A = 0\}$. To see this, we use that the tangent space is the space of derivations of paths through the identity. We take a path $I_n + tA$ through the identity. We need to see when we have $\det(I_n + tA) = 1$ for small t. We can illustrate what happens when $n = 2$, with the general case working in the same way. We have

$$\begin{aligned} \det\left(I_2 + t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \det\begin{pmatrix} ta + 1 & tb \\ tc + 1 & td \end{pmatrix} \\ &= (ta + 1)(td + 1) - t^2bc = t^2(ad - bc) + t(a + d) + 1 \\ &= t^2 \cdot \det A + t \cdot \operatorname{tr} A + 1 = t^2 + t \cdot \operatorname{tr} A + 1. \end{aligned}$$

Hence for this to be 1 as $t \to 0$, we need to have $\operatorname{tr} A = 0$. Since the derivative of the path $I_n + tA$ is just A, this means that $A \in T_{I_n} \mathrm{SL}^+(n, \mathbb{R})$, as claimed.

(c) We have proved this below in Spring 2008 #5.

(d) We did not prove this directly, but from what we did prove, we know that $\operatorname{tr} tB = 0$ for all $t \in \mathbb{R}$. This means that $\operatorname{tr} B = 0$ since trace is linear.

(e) See (c).

□

Problem 7.

(a) Define complex projective space $\mathbb{CP}^n$.

(b) Calculate the homology of $\mathbb{CP}^n$.

Solution.

(a) The easiest description of $\mathbb{CP}^n$ is as follows: define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \iff \exists \lambda \in \mathbb{C}^\times \text{ such that } \lambda x_i = y_i \text{ for all } i.$$

Then $\mathbb{CP}^n := \mathbb{C}^{n+1}/\sim$. We denote elements by homogenous coordinates

$$[x_0 : \cdots : x_n] \text{ with } x_i \in \mathbb{C}$$

(b) It is best to use cellular homology, and so we must define a cell structure on $\mathbb{CP}^n$. We do recursively. First, $\mathbb{CP}^0$ is just a point. Now for $\mathbb{CP}^n$ in general, from the above description we know that we could take a quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ by $x \sim y$ if and only if $x = \lambda y$ for $|\lambda| = 1$. We can given an even more useful description, though.

Consider the closed unit ball D^{2n}. We want to construct $\mathbb{CP}^n$ as a quotient of this space along its boundary. Suppose $x \in S^{2n+1}$ has last coordinate real and nonnegative. Then it is of the form $(z, \sqrt{1-|z|})$ for $z \in \mathbb{C}^n$ and $|z| \leq 1$. This is the graph of the function $\mathbb{C}^n \to \mathbb{R}$ given by $z \mapsto \sqrt{1-|z|}$, which is a disk D with boundary $\partial D = \{(z, 0) : |z| = 1\}$.

Now, we know that any vector $x \in S^{2n+1}$ is equivalent to a vector of the above form by

$$(x_0, \dots, x_n) \sim \left(\frac{x_0}{x_n}, \dots, 1\right)$$

assuming that $x_n \neq 0$. If $x_n = 0$, then we may view $x \in S^{2n-1} \subset S^{2n+1}$. Further, these associated vectors are unique in D. Therefore $\mathbb{CP}^n$ is a quotient of D by its boundary $\partial D \cong S^{2n-1}$, which is identified as in $\mathbb{CP}^{n-1}$. Hence we can construct $\mathbb{CP}^n$ by gluing disks $D^0, D^2, \dots, D^{2n}$ successively in the way described above. This shows that the cellular structure of $\mathbb{CP}^n$ is supposed only in even dimensions, viz.

$$C_i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}.$$

Our chain complex is therefore $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots$. Hence the homology is, for $0 \leq i \leq 2n$,

$$H_i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}.$$

□

Problem 8.
Let $p : E \to B$ be a covering space and $f : X \to B$ a map. Define $E^* = \{(x, e) \in X \times B : f(x) = p(e)\}$. Prove that $q : E^* \to X$ by $q(x, e) = x$ is a covering space.

Solution.
We have the following diagram:

$$\begin{array}{ccc} E^* & \longrightarrow & E \\ {\scriptstyle q}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X & \xrightarrow{\ f\ } & B \end{array}$$

where the top map is given by projection in the other coordinate, i.e. $g(x, e) = e$. We will chase this diagram and show q is a covering space. First, take any point $x \in X$ and consider $f(x) \in B$. Then there is a neighbourhood V of $f(x)$ so that $p^{-1}(V)$ is a disjoint union of open sets homeomorphic to V. Therefore let $U = f^{-1}(V)$. Let $e \in E$ such that $p(e) = f(x)$. Let W be a neighbourhood of e homeomorphic to V. Let $e' \in W$. Then $p(e') \in V$, so there exists $x' \in U$ so that $f(x') = p(e')$. Through this method, we can construct a preimage of W in E^*, which must be open. If we repeat this method for each $e_i \in p^{-1}(f(x))$, then we obtain neighbourhoods U_i over U. These neighbourhoods are disjoint since the neighbourhoods W_i are disjoint in E. Further, they are homeomorphic to U since they are the graph of the function $g : U \to E$ by the method above. This completes the proof. $\square$

Problem 9.

(a) Explain carefully and concretely what it means for two smooth maps of S^1 into $\mathbb{R}^2$ to be transversal.

(b) Do the same for maps of S^1 into $\mathbb{R}^3$.

(c) Explain what it means for transversal maps to be 'generic' and prove that they are indeed generic in the cases above.

Solution.

(a) Let $f, g : S^1 \to \mathbb{R}^2$. In general, we say that two submanifolds $X, Z \subset Y$ are transversal if they satisfy
$$T_y X + T_y Z = T_y Y$$
for every $y \in X \cap Z$. For two maps, we instead look at the pushforwards of the tangent spaces. Concretely, if $x, y \in S^1$ so that $f(x) = g(y) = z \in \mathbb{R}^2$, then we require
$$df(T_x S^1) + dg(T_y S^1) = T_z \mathbb{R}^2.$$
In our case, since $\dim T_x S^1$ is constantly 1 everywhere (as a loop is an embedding), we need $df(T_x S^1)$ to be linearly independent from $dg(T_y S^1)$ for these two maps to be transversal. This means that the two loops do not intersect tangentially.

(b) For this case, since $\dim(df(T_x S^1) + dg(T_y S^1)) \leq 2$ and $\dim T_z \mathbb{R}^3 = 3$, we can never have the above equality. Therefore we must have no points x, y so that $f(x) = g(y)$, i.e. the loops are disjoint.

(c) 'Generic' means that, given any two maps, they may be perturbed by an arbitrarily small amount so that they are transversal. For (a), suppose first that f, g intersect tangentially on a discrete set. Then we may take small neighbourhoods around each point which are disjoint. Then in these neighbourhoods, may take one curve and move it past the other, creating two non-tangential intersections which are transversal. If they intersect tangentially on a non-discrete set, then these problem areas are discrete open sets where the two loops overlap. A translation will pull these areas apart, so that the maps are trivially transversal at those points. For (b), we do essentially the same thing, except we can pull apart all areas of intersection by using the third dimension in this space. If f and g intersect non-tangentially at $f(x) = g(y) = z \in \mathbb{R}^3$, then $df(T_x S^1) + dg(T_y S^1)$ is a plane in $T_z \mathbb{R}^3$, which we identify with a neighbourhood of z. Then we may pull the loops apart in the direction perpendicular to this plane. This completes the proof

$\square$

Problem 10.
Let M be the 3-manifold with boundary obtained as the union of a two-holed torus in 3-space and the bounded component of its complement. Let X be the space obtained from M by deleting k points from the interior of M.

(a) Calculate the fundamental group of X.

(b) Calculate the homology of X.

Solution.

(a) We will see what this will deformation retract onto. If we take first M, we may 'squish' it to obtain two overlapping annuli. These will deformation retract onto the wedge of two circles. Now, say we remove a point from M to obtain M'. Then we may deform M' so that we have a closed ball with a point removed glued onto M. Thus we can deform M as before, and deform the closed ball without a point into S^2. This is true for the removal of any number of points, so we have

$$X \simeq S^1 \vee S^1 \vee \bigvee^k S^2 \implies \pi_1(X) \cong \pi_1(S^1 \vee S^1 \vee \bigvee^k S^2) \cong \pi_1(S^1)^2 \times \pi_1(S^2)^k$$
$$\cong \mathbb{Z} * \mathbb{Z}$$

(b) This space is connected, so $H_0(X) = \mathbb{Z}$. We have calculated $\pi_1(X)$ above, so $H_1(X) = \mathbb{Z}^2$, the abelianisation of $\mathbb{Z} * \mathbb{Z}$. Finally, $H_2(X)$ depends only on the 2- and 3-skeleton of X, we see that

$$H_2(X) \cong H_2\left(\bigvee^k S^2\right) \cong H_2(S^2)^k = \mathbb{Z}^k.$$

□

Problem 11.
Let P be a finite polyhedron.

(a) Define the Euler characteristic $\chi(P)$.

(b) Prove that if $P_1, P_2 \subset P$ are subpolyhedra such that $P_1 \cap P_2$ is a point and $P_1 \cup P_2 = P$, then $\chi(P) = \chi(P_1) + \chi(P_2) - 1$.

(c) Suppose that $p : E \to P$ is an n-sheeted covering space. Prove that $\chi(E) = n \cdot \chi(P)$.

Solution.

(a) Let us assume that a finite polyhedron is one obtained by gluing a finite number of simplices together. The Euler characteristic is the alternating sum of the number of simplices in each dimension. In general, we can define the Euler characteristic as the alternating sum of the Betti numbers, i.e.

$$\sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(P).$$

These notions give the same result on simplicial complexes.

(b) We see that we are counting every simplex once except for the point $P_1 \cap P_2$. Since we are double-counting this point in the sum, we must subtract 1 from the final tally.

(c) Take an open cover of P by open sets whose preimage is an n-sheeted homeomorphic copy. Since P is compact, we can take a finite subcover $U_1, \dots, U_n$ of this cover. Therefore these sets have a minimum diameter, say $\delta > 0$. Then through barycentric subdivision, we can create a finer simplicial structure on P such that every simplex is contained in some U. Therefore the preimage of every simplex is n copies of that simplex. This shows that $\chi(E) = n \cdot \chi(P)$ as required.

□

Problem 12.
Let $f : T \to T = S^1 \times S^1$ be the map of the torus inducing $f_\pi : \pi_1(T) \to \pi_1(T) = \mathbb{Z}^2$ and let F be a matrix representing f_π. Prove that the determinant of F equals the degree of f.

Solution.
Define a new map $\phi : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ induced by the action of F on $\mathbb{R}^2$. Then $\phi_\pi = f_\pi$, so we will use this more convenient map.

We want to show $\deg \phi = \det F$ first. We use the cohomological definition: if ω is a volume form on T, then

$$\deg \phi = \int_T \phi^* \omega.$$

By multivariable calculus, $\phi^*\omega = J(\phi)\omega$, where J is the Jacobian of ϕ. Since we clearly have $\deg \phi = J(\phi) = \det F$, we are done here.

We now need to show that $\phi \sim f$, so that their degrees are equal. Consider the map $f - \phi$, which is well defined on the torus since it is a topological group. Then $(f - \phi)_\pi = 0$, so the map $f - \phi : T \to T$ lifts to a map $\widetilde{f - \phi} : T \to \mathbb{R}^2$, since $\mathbb{R}^2$ is the universal cover. But in $\mathbb{R}^2$, all maps are nulhomotopic. Such a nulhomotopy projects down to $f - \phi : T \to T$, whence $f - \phi \sim 0$, i.e. $f \sim \phi$.

□

11 Fall 2008

Problem 1.
Let $G(k, n)$ be the collection of all k-dimensional linear subspaces of $\mathbb{R}^n$.

(a) Define a natural topological and smooth structure on $G(k, n)$, and show that with respect to the structures you defined, $G(k, n)$ is a smooth manifold.

(b) Show that $G(k, n)$ is diffeomorphic to $G(n - k, n)$.

Solution.

(a) We use a theorem out of Lie group theory. Note that the Lie group $O(n)$ acts transitively on $G(k, n)$. We have the following proposition: if G is a Lie group, and G acts on a set X transitively such that the stabiliser of any point $p \in X$ is a closed Lie subgroup of G, then X has a unique smooth manifold structure such that the given action is smooth.

In our case, we need to show that the stabiliser of a k-subspace of $\mathbb{R}^n$ is a closed subgroup of $O(n)$. For simplicity, consider $p = \langle e_1, \dots, e_k \rangle \in G(k, n)$. Then the stabiliser of p is the set of matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A \in O(k), B \in O(n-k).$$

This is evidently a closed Lie subgroup of $O(n)$. This gives us the smooth structure on $G(k, n)$.

(b) The diffeomorphism is given by sending a space to its orthogonal complement. This map is bijective, and smoothness of the map should follow easily.

□

Problem 2.
Let M and N be two smooth manifolds, and $f : M \to N$ a smooth map. Assume that $df_x : T_xM \to T_{f(x)}N$ is surjective for all $x \in M$ and that the inverse image $f^{-1}(y)$ is compact for all $y \in N$.

(a) Show that for any $y \in N$ there is an open neighbourhood of V of y such that $f^{-1}(V)$ is diffeomorphic to $V \times f^{-1}(y)$.

(b) Assuming further that N is connected, can you take V to be N in (a)?

Solution.

(a) First, note that f is an open map, so is a surjection onto some collections of connected components of N. If $y \in N$ is not in this collection, then there is a neighbourhood V of y contained in its connected component such that $f^{-1}(V) = \varnothing$, so the claim holds trivially. For the remainder of the question, this problem becomes Ehresmann's fibration theorem.

Let $y \in \operatorname{im} f$. Then $f^{-1}(y) = \{x_1, \dots, x_n\}$ is a finite (disjoint) collection of points. As such, we can take open neighbourhoods U_i around each x_i. Then consider $\bigcap f(U_i)$. Since f is an open map, this is a finite intersection of open sets in N, so is a neighbourhood V of y. Then we can consider the disjoint collection V_i of open sets around each x_i obtained by $U_i \cap f^{-1}(V)$. Then all the V_i are diffeomorphic to each other and disjoint, and $V_i \cong f^{-1}(V)$. Therefore we have $f^{-1}(V) \cong V \times f^{-1}(y)$.

(b) No. If we could take V to be N, then f would be a fibration. However, we do not know that $f : M \to N$ is even-sheeted. For example, let $N = S^1$, and let $M = \mathbb{R}$. For manifolds of equal dimension, a fibration is a covering map. We can construct a cover of S^1 by $\mathbb{R}$ that is 1-sheeted on an open connected subset of S^1 by 2-sheeted on its complement, viz. Fall 2013, #1(b).

□

Problem 3.
Let M be a connected smooth manifold. Show that for any two points $x, y \in M$ there is a diffeomorphism f of M such that $f(x) = y$.

Solution.
Fix x. Let $N \subset M$ be the set

$$N = \{y \in M : \text{there exists a diffeomorphism } f : M \to M \text{ such that } f(x) = y\}.$$

The problem is equivalent to $N = M$. First, take any coordinate chart $\phi : U \to \mathbb{R}^n$ containing x. Then we know that there is an isomorphism in $\mathbb{R}^n$ that sends $\phi(x)$ to $\phi(y)$, namely the shift by $\phi(y) - \phi(x)$. Pulling this back gives us a diffeomorphism of M which acts trivially on every other chart not intersecting U. Therefore N is an open set. Further, by the same argument, $M \setminus N$ is also open. If $y \in M \setminus N$ with a coordinate chart V around y and $z \in V \cap N$, then we could construct a diffeomorphism sending z to y. Composition of these diffeomorphisms being another diffeomorphism, we have a contradiction. Therefore N and $M \setminus N$ is a partition of M by open sets. Since M is connected, we must have $N = \varnothing$ or $N = M$. Since $x \in N$, we are done. □

Problem 4.
Let $\theta = \sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$ be a 1-form defined on $\mathbb{R}^{2n}$, where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are the coordinates of $\mathbb{R}^{2n}$. Consider the $2n-1$-dimensional distribution $D = \ker \theta$. Is D integrable?

Solution.
We know from Fall 2013, #5 that $\ker \theta$ is integrable if and only if $\theta \wedge d\theta = 0$. As such, ths is what we will try to show. Since we differentiate linearly, we have

$$d\theta = \sum_{i=1}^{n} dx_i \wedge dy_i - dy_i \wedge dx_i = 2\sum_{i=1}^{n} dx_i \wedge dy_i.$$

Since the wedge product is bilinear,

$$\frac{1}{2}(\theta \wedge d\theta) = \sum_{i,j=1}^{n} x_i dy_i \wedge dx_j \wedge dy_j - y_i dx_i \wedge dx_j \wedge dy_j$$

The terms with $i = j$ are identically zero, so removing them yields

$$= \sum_{i \neq j} x_i dy_i \wedge dx_j \wedge dy_j - y_i dx_i \wedge dx_j \wedge dy_j$$

Rearranging these terms in order to use the standard basis for 3-forms,

$$\begin{aligned} &= \sum_{i<j} (-x_i dx_j \wedge dy_i \wedge dy_j - y_i dx_i \wedge dx_j \wedge dy_j) \\ &\qquad + \sum_{i>j} (x_i dx_j \wedge dy_j \wedge dy_i + y_i dx_i \wedge dx_i \wedge dy_j) \\ &= 0. \end{aligned}$$

Therefore D is integrable. □

Problem 5.
Let D be a bounded domain in $\mathbb{R}^n$ with a smooth boundary S, $j : S \to \mathbb{R}^n$ be the inclusion map and X be a smooth vector field defined on $\mathbb{R}^n$.

(a) Denote the standard volume form $dx_1 \wedge \cdots \wedge dx_n$ by ω. Show that $j^*(i_X\omega) = \langle X, N\rangle \, dS$, where N is the outward unit normal vector field along S and $\langle -, -\rangle$ the Euclidean inner product. Here $i_X\omega$ is the contraction of ω along X, dS is the 'area' form on S. Explain carefully the definition and the geometrical meaning of the term dS.

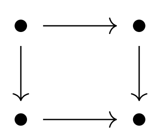
(b) Use (a) and Stokes' Theorem to show that

$$\int_D \mathcal{L}_X\omega = \int_S \langle X, N\rangle \, dS.$$

Solution.
This is essentially the same as Spring 2008, #3 below. □

Problem 6.
Find $\pi_1(T^2 \setminus \{k \text{ points}\})$, where T^2 is the 2-dimensional torus.

Solution.
First, picture T^2 as the unit square with edges identified appropriately. If we puncture this square, then we can deformation retract it to the following hollow square:



Gluing this together yields $T^2 \setminus \{p\} \simeq S^1 \vee S^1$. If we were to remove k points, we may assume they are located at $(1/2k + (i-1)/k, 1/2)$ for $i = 1, \dots, k$. Then we can deformation retract this square so that we have the above picture but with edges at i/k for $i = 0, \dots, k$. Thus when we deformation retract, we obtain a wedge of $k+1$ circles (one for each edge). Therefore

$$\pi_1(T^2 \setminus \{k \text{ points}\}) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k+1 \text{ times}}.$$

□

Problem 7.
Find the homology groups $H_i(\Delta_n^{(k)})$, $i = 0, \dots, k$. Here $\Delta_n^{(k)}$ is the k-skeleton of the n-simplex Δ_n with $k \le n$.

Solution.
Fix n and let $\Delta_n = X$ and $\Delta_n^{(k)} = X^k$ for ease of notation. Since only X^k is disconnected for only $k = 0$, we have $H_0(X^k) = \mathbb{Z}$ for $k \neq 0$ and $H_0(X^0) = \mathbb{Z}^{n+1}$.

Now, we examine $H_k(X^k)$ for $k > 0$. If we take Z_k and B_k to be cycles and boundaries respectively, we know $B_k = 0$ as there are no $k+1$-simplices in X^k. However, we know that the boundary of every 'expected' $k+1$-simplex is a cycle in Z_k. We know that there are $\binom{n+1}{k+1}$ simplices in X^{k+1}, so this is the rank of Z_k. Therefore the rank of $H_k(X^k)$ is $\binom{n+1}{k+1}$.

Since X^k is a CW-complex, we know that $H_i(X^k)$ depends only on X^i and X^{i+1}. Therefore we see that $H_i(X^k) = H_i(X)$ for $i < k$. This shows that $H_i(X^k) = 0$ for $0 < i < k$. We summarise this solution below:

$$H_i(\Delta_n^{(k)}) = \begin{cases} \mathbb{Z} & i = 0, k > 0 \\ 0 & k > i > 0 \\ \mathbb{Z}^{\binom{n+1}{k+1}} & i = k \end{cases}.$$

□

Problem 8.
Let G be a topological group with identity element e. For any two continuous loops $\gamma_1, \gamma_2 : S^1 \to G$ sending $1 \in S^1$ to $e \in G$, define $\gamma_1 * \gamma_2 : S^1 \to G$ by $\gamma_1 * \gamma_2(t) = \gamma_1(t) \cdot \gamma_2(t)$.

(a) Show that the product $*$ induces a product structure on $\pi_1(G, e)$ and this new product on $\pi_1(G, e)$ is the same as the usual one.

(b) Is $\pi_1(G, e)$ commutative?

Solution.
The argument for this problem is known as the Eckmann-Hilton argument, but we do not have to use its full strength. We need to show that $\gamma_1\gamma_2 = \gamma_1 * \gamma_2$, where the first product is the usual concatenation of loops. First, let us view the loops as maps $\gamma_i : [0,1] \to S^1$ with $\gamma_i(0) = \gamma_i(1) = 1$. Second, reparametrise as follows:

$$\bar{\gamma}_1(t) = \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ e & t \in [1/2, 1] \end{cases}, \qquad \bar{\gamma}_2(t) = \begin{cases} e & t \in [0, 1/2] \\ \gamma_2(2t-1) & t \in [1/2, 1] \end{cases}.$$

We have $[\gamma_i] = [\bar{\gamma}_i]$. In this case, it is easy to see that $\bar{\gamma}_1\bar{\gamma}_2 = \bar{\gamma}_1 * \bar{\gamma}_2$ for all t. Therefore we have shown

$$[\gamma_1\gamma_2] = [\bar{\gamma}_1\bar{\gamma}_2] = [\bar{\gamma}_1 * \bar{\gamma}_2] = [\gamma_1 * \gamma_2].$$

It is also clear that the identity loop is the same in both cases, so these multiplication is the same.

To show commutativity, notice that in the above argument that at every $t \in [0,1]$, one of $\bar{\gamma}_i$ is the identity, i.e. that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ commute. Therefore $[\gamma_1 * \gamma_2] = [\gamma_2 * \gamma_1]$, so the fundamental group must be commutative. □

Problem 9.

(a) Show that any continuous map $f : S^2 \to T^2$ is nulhomotopic.

(b) Show that there exists a continuous map $f : T^2 \to S^2$ which is not nulhomotopic.

Solution.

(a) Recall that the universal cover of the torus is $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Therefore such an f has a unique lift to $\tilde{f} : S^2 \to \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, any continuous map into it is nulhomotopic. Therefore $f = p \circ \tilde{f}$ is also nulhomotopic.

(b) Identify $T^2 \cong S^1 \times S^1$ and $S^2 = \{\alpha \in \mathbb{R}^3 : |\alpha| = 1\}$. Let $A = \{(x, y, z) \in S^2 : z = 0$ and $B = \{(x, y, z) \in S^2 : y = 0\}$. A and B are both homeomorphic to S^1 and create pseudo-coordinate axes on S^2. Define $f : T^2 \to S^2$ by (θ_1, θ_2) maps to the same points with respect to A and B. This map is surjective but not injective, e.g. $f(\pi, 0) = f(0, \pi)$. Moreover, it is a degree 2 map, so it cannot be nulhomotopic.

□

Problem 10.
Let A and B be two chain complexes with boundary operators ∂_A and ∂_b and let $f : A \to B$ be a chain map. Define a new chain complex C whose ith chain group is $C_i = A_i \oplus B_{i+1}$ and boundary operator defined by $\partial_C(a, b) = (\partial_A(a), \partial_B(b) + (-1)^{\deg a} f(a))$.

(a) Show that C is indeed a chain complex and there is a short exact sequence of chain complex

$$0 \to B \to C \to A \to 0$$

such that $B_{i+1} \mapsto C_i$ and $C_i \mapsto A_i$.

(b) Write down the long exact sequence of the homology groups associated to the above short exact sequence. What is the connecting homomorphism in the long exact sequence?

(c) Let $(f_*)_i : H_i(A) \to H_i(B)$ be the induced map of f on the ith homology group. Show that it is an isomorphism for all i if and only if $H_i(C) = 0$ for all i.

Solution.
(a) We will show that $\partial_C^2 = 0$. We have

$$\begin{aligned}\partial_C^2(a, b) &= \partial_C(\partial_A(a), \partial_B(b) + (-1)^{\deg a} f(a)) \\ &= (\partial_A^2(a), \partial_B(\partial_B(b) + (-1)^{\deg a} f(a)) + (-1)^{\deg a - 1} f(\partial_A(a)))\end{aligned}$$

The first coordinate is zero, as is the $\partial_B^2(b)$ in the second coordinate. We are left with

$$\partial_B\left[(-1)^{\deg a} f(a)\right] + (-1)^{\deg a - 1} f(\partial_A(a)).$$

Because f is a chain map, it satisfies the following commutative diagram:

$$\begin{array}{ccc} A_i & \xrightarrow{\partial_A} & A_{i-1} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ B_i & \xrightarrow{\partial_B} & B_{i-1} \end{array}$$

This shows that $\partial_B(f(a)) = f(\partial_A(a))$. Since $(-1)^{\deg a}$ and $(-1)^{\deg a - 1}$ have opposite signs, the above expression is zero. This shows that C is a chain complex.

Now, we need to check that the above sequence is short exact. Call the maps $j : B \to C$ and $p : C \to A$. It is clear that $\operatorname{im} j \subset \ker p$. Further, if $c \in \ker p$, then it must be supported only in the B_{i+1} terms, for which we can always find a preimage under j

(b) The induced sequence is

$$\cdots \to H_{i+1}(B) \to H_i(C) \to H_i(A) \xrightarrow{\delta_i} H_i(B) \to \cdots$$

We need to construct the appropriate map from the snake lemma $\delta_i : \ker \partial_A \to \operatorname{coker} \partial_B$, where both of these boundary maps are in degree i.

$$\begin{array}{ccccccccc}
 & & & & & & \ker \partial_A & & \\
 & & & & & & \big\downarrow & & \\
0 & \longrightarrow & B_{i+1} & \xrightarrow{j_i} & C_i & \xrightarrow{p_i} & A_i & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \partial_B} & & \big\downarrow{\scriptstyle \partial_C} & & \big\downarrow{\scriptstyle \partial_A} & & \\
0 & \longrightarrow & B_i & \xrightarrow{j_{i-1}} & C_{i-1} & \xrightarrow{p_{i-1}} & A_{i-1} & \longrightarrow & 0 \\
 & & \big\downarrow & & & & & & \\
 & & \operatorname{coker} \partial_B & & & & & &
\end{array}$$

Let $a \in \ker \partial_A$. Then since p_i is surjective, we may choose a preimage $(\alpha, b) \in C_i$. By construction, we know that $\alpha = a$, so we will write it as such now. Then $\partial_A(p_i(a, b)) = p_{i-1}(\partial_C(a, b)) = 0$ Therefore $\partial_C(a, b) \in \ker p_{i-1} = \operatorname{im} j_{i-1}$, so there exists $\beta \in B_i$ so that $j_{i-1}(b) = \partial_C(a, b)$. In our case, we know that $\partial_C(a, b) = (\partial_A(a), \partial_B(b) + (-1)^i f(a))$. By construction, we know that $\beta = \partial_B(b) + (-1)^i f(a)$. Now, we may project b into the appropriate member of $\operatorname{coker} \partial_B$ to complete the map, and which gives $(-1)^i f(a)$. Therefore since we have defined this map on the level of $\ker \partial_A$, it descends to a map on homology. Thus $\delta_i(a) = (-1)^i f_*(a)$.

(c) If $H_i(C) = 0$ for all i, then our long exact sequence becomes

$$0 \to H_i(A) \xrightarrow{\delta_i} H_i(B) \to 0$$

at each degree. Therefore δ_i is an isomorphism. If i is even, then we have nothing else to show; if i is odd, then we have an inverse $(f_*)_i^{-1} = -\delta_i^{-1}$. The converse is clear given this reasoning.

□

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Problem 1.
Let M and N be smooth manifolds, and let $f : M \to N$ be a smooth map.

(a) Define the map f^* of p-forms on N to p-forms on M.

(b) Prove that if ω is a p-form on N, then $f^*(d_N \omega) = d_M(f^* \omega)$.

Solution.

(a) It suffices to give a definition in local coordinates, which we do. In a coordinate chart, we can use the coordinates x_i to construct a basis for the space of p-forms, namely $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ with $i_1 < \cdots < i_p$. A general form therefore looks like

$$\omega = \sum g_I dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where g_I is a smooth function and I is the multiindex corresponding to $(i_1, \ldots, i_p)$. Then we define

$$f^*\omega = \sum (g_I \circ f) df_{i_1} \wedge \cdots \wedge df_{i_p},$$

where $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

(b) It suffices to prove this on a form of the type

$$\omega = g dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

by linearity. Then we see that

$$\begin{aligned} f^*(d\omega) &= f^* \left(dg \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}\right) \\ &= f^* \left(\sum \frac{\partial g}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right) \\ &= \sum f^* \left(\frac{\partial g}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right) \\ &= \sum \left(\frac{\partial g}{\partial x_i} \circ f \right) df_i \wedge df_{i_1} \wedge \cdots \wedge df_{i_p} \\ &= d(g \circ f) df_{i_1} \wedge \cdots \wedge df_{i_p} = d(f^*(\omega)). \end{aligned}$$

□

Problem 2.
Let M be a smooth manifold and X a smooth vector field on M.

(a) Suppose $X_p \neq 0$ for some $p \in M$. Show, using the flow of X, that there is a neighbourhood U of p and a coordinate system $(x_1, \ldots, x_n)$ on U so that $X = \frac{\partial}{\partial x_1}$ on U.

(b) Use the above to prove that if Y is another smooth vector field on M with $[X, Y] = 0$ everywhere, then $\phi_s(\psi_t(p)) = \psi_t(\phi_s(p))$ for all s, t sufficiently small, where ϕ and ψ are the flows of X and Y. (Hint: Write Y near p in terms of the coordinate system of part (a).)

Solution.

(a) This can be found in Spivak's *Differential Geometry*, Volume 1, p.148. We may assume that we are working in $\mathbb{R}^n$ and that $p = 0$. Let $(t_1, \ldots, t_n)$ be the standard coordinates. We can further assume that $X(0) = \frac{\partial}{\partial t_1}\Big|_0$. The idea of the proof hinges on the integral curves running through points near 0.

Let ϕ_t be the flow of X. Consider $\chi : U \to \mathbb{R}^n$ for a neighbourhood U of the origin given by

$$\chi(a_1, \dots, a_n) = \phi_{a_1}(0, a_2, \dots, a_n).$$

Then for $a = (a_1, \dots, a_n)$,

$$\begin{aligned}
\chi_* \left(\frac{\partial}{\partial t_1}\Big|_a \right)(f) &= \frac{\partial}{\partial t_1}\Big|_a (f \circ \chi) \\
&= \lim_{h\to 0} \frac{1}{h} \left[\chi(a_1 + h, a_2, \dots, a_n) - f(\chi(a)) \right] \\
&= \lim_{h\to 0} \frac{1}{h} \left[f(\phi_{a_1+h}(0, a_2, \dots, a_n)) - f(\chi(a)) \right] \\
&= \lim_{h\to 0} \frac{1}{h} \left[f(\phi_h(\chi(a))) - f(\chi(a)) \right] \\
&= (\mathcal{L}_X f)(\chi(a)).
\end{aligned}$$

Further, for $i > 1$, we have

$$\begin{aligned}
\chi_* \left(\frac{\partial}{\partial t_i}\Big|_0 \right)(f) &= \frac{\partial}{\partial t_i}\Big|_0 (f \circ \chi) \\
&= \lim_{h\to 0} \frac{1}{h} \left[f(\chi(0, \dots, h, \dots, 0) - f(0) \right] \\
&= \lim_{h\to 0} \frac{1}{h} \left[f(0, \dots, h, \dots, 0) - f(0) \right] \\
&= \frac{\partial f}{\partial t_i}\Big|_0.
\end{aligned}$$

Since $X(0) = \frac{\partial}{\partial t_i}|_0$, $\chi_*(0) = I$ is nonsingular. Therefore χ^{-1} is a coordinate system that we require.

(b) This is found on p.157 of the same reference. We write

$$0 = [X, Y] = \mathcal{L}_X Y = \lim_{h\to 0} \frac{1}{h} \left[Y_q - (\phi_{h*} Y)_q \right] \text{ for all } q \in M.$$

For a point $p \in M$, let c be a curve in M_p given by $c(t) = (\phi_{t*} Y)_p$. We see that

$$\begin{aligned}
c'(t) &= \lim_{h\to 0} \frac{1}{h} \left[c(t+h) - c(t) \right] \\
&= \lim_{h\to 0} \frac{1}{h} \left[(\phi_{(t+h)*} Y)_p - (\phi_{t*} Y)_p \right] \\
&= \lim_{h\to 0} \frac{1}{h} \left[\phi_{t*} (\phi_{h*} Y)_{\phi_{-t*}(p)} - \phi_{t*} Y_{\phi_{-t*}(p)} \right] \\
&= \phi_{t*} \left[\lim_{h\to 0} \frac{1}{h} \left[(\phi_{h*} Y)_{\phi_{-t*}(p)} - Y_{\phi_{-t*}(p)} \right] \right] \\
&= \phi_{t*}(0) = 0,
\end{aligned}$$

where the last line uses the point $q = \phi_{-t}(p)$. Therefore $c(t) = c(0)$ everywhere so $\phi_{t*} Y = Y$. We know that the flow of $\phi_{t*} Y$ is $\phi_t \circ \psi_s \circ \phi_{-t}$, hence

$$\phi_t \circ \psi_s \circ \phi_{-t} = \psi_s \implies \psi_t \circ \psi_s = \psi_s \circ \phi_t.$$

□

Problem 3.
Gauss' Divergence Theorem asserts that if U is a bounded domain in $\mathbb{R}^3$ with smooth boundary and if X is a smooth vector field defined in a neighbourhood of U, then

$$\int_U \operatorname{div} X d(\text{vol}) = \int_{\partial U} \langle X, N \rangle d(\text{area}),$$

where N is the unit outward normal vector for ∂U. Show how the divergence theorem follows from Stokes' theorem.

Solution.
Stokes' theorem tells us that, for a domain U and $(n-1)$-form ω,

$$\int_U d\omega = \int_{\partial U} \omega.$$

Therefore we would like show that $d\left(\langle X, N\rangle d(\text{area})\right) = \operatorname{div} X d(\text{vol})$. If we are in local coordinates (which we may assume), then we may let $d(\text{vol}) = dx \wedge dy \wedge dz$ and

$$X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z} \implies \operatorname{div} X = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

for functions $f, g, h : \mathbb{R}^3 \to \mathbb{R}$. Direct calculation shows that

$$d(f\, dy \wedge dz - g\, dx \wedge dz + h\, dx \wedge dy) = \operatorname{div} X dx \wedge dy \wedge dz.$$

Therefore we need to prove that the above form, which we call η, is equal to $\langle X, N\rangle d(\text{area})$.

Now, take local coordinates for ∂U, say some α, β. If ϕ is the chart map, then we have

$$N = \phi_* \frac{\partial}{\partial \alpha} \times \phi_* \frac{\partial}{\partial \beta},$$

where $\times$ is the cross product in $\mathbb{R}^3$. Further,

$$d(\text{area}) = \phi_*(d\alpha \wedge d\beta) = d\phi_\alpha \wedge d\phi_\beta.$$

Then using the change of variables formula, we transform η by ϕ into

$$f \circ \phi\, d(\phi^* y) \wedge d(\phi^* z) - g \circ \phi\, d(\phi^* x) \wedge d(\phi^* z) + h \circ \phi\, d(\phi^* x) \wedge d(\phi^* y).$$

Since we can write $\phi = (\phi_1, \phi_2, \phi_3)$, we have

$$d(\phi^* x) = d\phi_1 = \frac{\partial \phi_1}{\partial \alpha} d\alpha + \frac{\partial \phi_1}{\partial \beta} d\beta$$

and similar. But these coefficients come from $\phi_* \frac{\partial}{\partial \alpha}$ and $\phi_* \frac{\partial}{\partial \beta}$. Direct computation shows the mess we get above is precisely equal to $\langle X, N\rangle$ in these coordinates. □

Problem 4.

(a) Let θ be a 1-form on S^2 with $d\theta = 0$. Construction a function f on S^2 with $df = \theta$.

(b) Let θ be a 1-form on $S^1 \times (0,1)$ with $d\theta = 0$. Show that there is a function $f : S^1 \times (0,1) \to \mathbb{R}$ with $df = \theta$ if and only if

$$\int_{S^1 \times \{1/2\}} \theta = 0.$$

(c) Use the above to show that if ω is a 2-form on S^2 with $\int_{S^2} \omega = 0$ then there is a 1-form θ on S^2 with $d\theta = \omega$. (Hint: You may use the Poincaré Lemma so that $\omega = d\theta_1$ on $S^2 \setminus \{\text{NP}\}$ and $\omega = d\theta_2$ on $S^2 \setminus \{\text{SP}\}$. Use Stokes' theorem to show $\theta_1 - \theta_2$ satisfies the integral condition of (b).)

Solution.
This problem mirrors Spring 2009, #3, and in particular the construction in (a) mirrors (b) above. □

Problem 5.
Let $\text{SO}(3) = \{A \in \text{SL}(3) : A^{-1} = A^{\perp}\}$. Also, let

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!} = 1 + A + \frac{A^2}{2!} + \cdots .$$

(a) Prove that $\exp(A)$ always converges, so that $\exp : M_3(\mathbb{R}) \to M_3(\mathbb{R})$. You may assume henceforth that exp is smooth and can be differentiated termwise.

(b) Show that exp is injective on some neighbourhood of the zero matrix in $M_3(\mathbb{R})$. (Hint: inverse function theorem.)

(c) Show that $\exp(B) \in \text{SO}(3)$ if $B^{\perp} = -B$.

(d) Show that exp restricted to the space of antisymmetric matrices is a surjective map from some neighbourhood of the zero matrix to a neighbourhood of $I_3 \in \text{SO}(3)$. (Hint: Note that every element of SO(3) is a rotation around some axis.)

(e) Discuss how to combine the above parts to give coordinate charts on SO(3) and thus make it a differentiable manifold.

Solution.

(a) Let $\|\cdot\|$ be the operator norm for matrices, i.e. $\|A\| = \sup\{|Ax| : |x| = 1\}$. This norm is submultiplicative, so we have $\|A^n\| \leq \|A\|^n$. Therefore we have

$$0 \leq \left\| \frac{A^n}{n!} \right\| \leq \frac{\|A\|^n}{n!} \implies 0 \leq \|\exp(A)\| \leq \exp(\|A\|).$$

where we also use subadditivity. Since $\|A\|$ is just a finite number, $\exp(\|A\|) < \infty$, so our sequence converges to a matrix with finite operator norm. This is sufficient to prove that the sequence converges in $M_3(\mathbb{R})$ as this space is complete.

(b) As expected, we can see that $d\exp = \exp$, i.e. this function is its own derivative. At $0 \in M_3(\mathbb{R})$, we have $d\exp|_0 = \exp(0) = I_3$, so by the inverse function theorem exp is invertible in some neighbourhood of the zero matrix. In particular, it must be injective.

(c) Suppose that $B^\perp = -B$. We need to prove that $\det(\exp B) = 1$ and $B^{-1} = B^\perp$. First, B being antisymmetric implies that $\operatorname{tr} B = 0$. We see this because each diagonal entry must be its own negative. We claim that

$$\det(\exp(A)) = \exp(\operatorname{tr}(A)).$$

This obviously holds true for diagonal matrices, and we can see it holds true for (complex) diagonalisable matrices as well: suppose A is diagonalisable and $A = PDP^{-1}$ for a diagonal D. Note that $(PDP^{-1})^n = PD^nP^{-1}$ for all n by internal cancellation. Then

$$\det(\exp(A)) = \det(\exp(PDP^{-1})) = \det(P\exp(D)P^{-1}) = \det(\exp(D)) = \exp(\operatorname{tr}(D)).$$

Since the trace is not affected by conjugation, noting $\operatorname{tr}(D) = \operatorname{tr}(A)$ proves this statement. Now we claim that the set of diagonalisable matrices is dense in $M_3(\mathbb{R})$. Indeed, the only matrices that are not diagonalisable are $\{B : \det(B) = 0\}$. Any matrix with zero determinant may be perturbed by an arbitrarily small amount so that it has nonzero determinant since the determinant function is continuous. Therefore since $\det, \exp, \operatorname{tr}$ are all continuous, what holds on a dense subset actually holds on the entire space.

This proves that $\det(\exp(B)) = \exp(\operatorname{tr}(B)) = \exp(0) = 1$, as required. To show that these matrices are orthogonal, note that (as expected again), $\exp(A + A') = \exp(A)\exp(A')$. Additionally, $\exp(A)^\perp = \exp(A^\perp)$ since $\perp$ distributes through sums. Therefore

$$\exp(B)\exp(B)^\perp = \exp(B)\exp(B^\perp) = \exp(B + B^\perp) = \exp(B - B) = I_3.$$

This completes the proof.

(d) Since the tangent space to $I_3 \in \mathrm{SO}(3)$ is isomorphic to $\mathrm{SO}(3)$ itself, we need to check that for every $A \in \mathrm{SO}(3)$, there exists an antisymmetric $B \in M_3(\mathbb{R})$ so that $\exp(A) = B$. We take the hint that every $A \in \mathrm{SO}(3)$ is a rotation of 3-space around an axis, then up to a change of orthonormal basis we may assume that

$$A = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

Direct calculation shows that

$$B = \left(\begin{array}{cc|c} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

is a logarithm of A. B is antisymmetric, and so is the conjugate of B by orthonormal matrices, so this completes the proof.

(e) We know that antisymmetric matrices have a natural smooth structure inherited from $M_3(\mathbb{R})$. Further, exp is a smooth submersion of antisymmetric matrices onto SO(3). Therefore we can define charts on SO(3) by taking the images of charts on antisymmetric matrices.

□

Problem 6.
Let M and N be two compact, oriented manifolds of the same dimension. Let ω be a nowhere vanishing n-form on N with $\int_N \omega = 1$. Let $F : M \to N$ be a smooth map.

(a) Set $\deg_\omega F = \int_M F^*\omega$. Show that $\deg_\omega F$ is independent of the choice of ω. (You may assume de Rham's theorem).

(b) Show that there is a smooth map from $S^2 \times S^2 \to S^4$ of degree 1.

(c) Show that no map from $S^4 \to S^2 \times S^2$ has degree 1.

Solution.

(a) Let ω_1, ω_2 be two forms satisfying the above. Then we claim that $\omega_1 - \omega_2$ is exact. Indeed, de Rham's theorem tells us that the integration map $I : H^i_{dR}(M) \to H^i(M; \mathbb{R})$ given by

$$I(\omega) = c \mapsto \int_c \omega \in \operatorname{Hom}(H_i(M; \mathbb{R}), \mathbb{R}).$$

Therefore since we are looking at top-dimensional forms on N on compact, oriented manifolds, we know that the element $\omega_1 - \omega_2$ maps to the zero homomorphism, since $\int_N \omega_1 = \int_N \omega_2$. Therefore $\omega_1 - \omega_2$ must be represented by the zero element of $H^i_{dR}(M)$, i.e. it is an exact form. Therefore we have $\omega_1 - \omega_2 = d\eta$ for some $(n-1)$-form η. Then

$$\deg_{\omega_1} F - \deg_{\omega_2} F = \int_M F^*(\omega_1 - \omega_2) = \int_M F^*(d\eta) = \int_M dF^*(\eta).$$

By Stokes' theorem, we have

$$\int_M dF^*(\eta) = \int_{\partial M} F^*(\eta) = 0$$

since $\partial M = \varnothing$ by assumption.

(b) See Spring 2010, #10 for a relatively similar treatment of this problem. Use the fact that $S^2 \times S^2$ has a 4-cell in its decomposition.

(c) Ibid.

□

Problem 7.
Describe carefully the basic algebraic construction of algebraic topology, namely, how to go from a short exact sequence of chain complexes to a long exact sequence in homology. Give explicitly, in particular, the construction of the 'connecting homomorphism' and prove exactness at its image. You need not prove exactness of the long exact sequence elsewhere.

Solution.
See Spring 2011 #6(b) and Spring 2012 #5(b) for the two bits of this question. □

Problem 8.

(a) Prove that S^n is simply connected if $n > 1$.

(b) Prove that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$, $n > 1$.

(c) Prove that $\mathbb{RP}^n$ is orientable if n is odd ($n > 1$).

Solution.

(a) We claim that any map $f : S^1 \to S^n$ for $n > 1$ cannot be surjective. This follows from Sard's theorem. Since $\dim T_x S^1 < \dim T_{f(x)} S^n$, every point in the image of f is a critical value. Therefore the image of f has measure zero, so there must be a point $y \in S^n$ that is not in the image of f. As such, f factors through to a map $f : S^1 \to \mathbb{R}^n$ via the diffeomorphism $S^n \setminus \{y\} \cong \mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, f is nulhomotopic. Therefore every loop in S^1 is nulhomotopic, so S^n is simply connected.

(b) We know that only the 1- and 2-skeleton of $\mathbb{RP}^n$ determines π_1. Further, since we have embeddings $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$, we know that $\pi_1(\mathbb{RP}^n) \cong \pi_1(\mathbb{RP}^2)$. Since $\mathbb{RP}^2$ is formed (as a CW-complex) by a copy of S^1 with a 2-cell attached to it by a double cover, we have

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \pi_1(\mathbb{RP}^2) \to 0$$

is exact. Therefore $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$, as needed.

(c) We have shown this in Spring 2012, #6 and elsewhere.

□

Problem 9.
Find by any method the homology groups of $\mathbb{RP}^n$ with integer coefficients.

Solution.
See Spring 2012, #6. □

Problem 10.

(a) Define complex projective space $\mathbb{CP}^n$.

(b) Show that $\mathbb{CP}^n$ is compact.

(c) Show that $\mathbb{CP}^1 \cong S^2$. (Homeomorphic is enough.)

(d) Show that $\mathbb{CP}^n$ is simply connected.

(e) Find the homology of $\mathbb{CP}^n$.

Solution.
(a), (b), and (e) have been done in Spring 2012, #7. To show (c), we note that $\mathbb{CP}^1$ is formed by gluing the boundary of a 2-cell to a 0-cell, which is clearly homeomorphic to S^2. To show (d), we know by its cell decomposition that $\mathbb{CP}^n$ has no 1-skeleton, so it must have a trivial fundamental group. □