Problem Spring 2011, 1.
Show that if $X$ is a smooth vector field on a (smooth) manifold of dimension $n$ and if $X_p$ is nonzero for some point of $p$, then there is a coordinate system defined in a neighbourhood of $p$, say $(x^1, \ldots, x^n)$, such that on a neighbourhood of $p$, $X = (x^i)$.

Solution.
If $X_p$ is nonzero at $p$, then it is also nonzero in a neighbourhood of $p$. Let $U$ be such a neighbourhood, and let $\psi : U \to \mathbb{R}^n$ be a diffeomorphism. It suffices to prove this claim in $\mathbb{R}^n$ at the point $p = 0$. Notice that at 0, the vector $X_0 \in T_0 \mathbb{R}^n$ can be extended to a basis of $T_0 \mathbb{R}^n$. This yields a system of coordinates $(t^i)$ so that $\frac{\partial}{\partial t^1}|_0 = X_0$. Let $\phi_t$ be the flow of $X$. We will use this to extend $t$ to a system of coordinates in a neighbourhood $V$ of 0.

Define a map $\psi$ on $V$ by

$$\psi(x^1, \ldots, x^n) = \phi_{x^1}(0, x^2, \ldots, x^n).$$

Then at a point $x = (x^i) \in V$, we have

$$\psi_* \left( \frac{\partial}{\partial t^1} \right)_x (f) = \left. \frac{\partial}{\partial t} \right|_x (f \circ \psi)$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ f(\psi(x^1 + h, x^2, \ldots, x^n)) - f(\psi(x)) \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ f(\phi_{x^1+h}(0, x^2, \ldots, x^n)) - f(\psi(x)) \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ f(\phi_h(\psi(x))) - f(\psi(x)) \right]$$

$$= (L_X f)(\psi(x)).$$

For $i > 1$, we have

$$\psi_* \left( \frac{\partial}{\partial t^i} \right)_x (f) = \left. \frac{\partial}{\partial t^i} \right|_0 (f \circ \psi)$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ f(\psi(0, \ldots, h, \ldots, 0)) - f(0) \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ f(0, \ldots, h, \ldots, 0) - f(0) \right]$$

$$= \left. \frac{\partial f}{\partial t^i} \right|_0.$$
Therefore the coordinate system \( y = \psi^{-1} \) is a coordinate system in a neighbourhood of 0, which satisfies \( X = \frac{\partial}{\partial y} \) since \( \psi^*(\frac{\partial}{\partial y}) = X \circ \psi \).

**Problem Spring 2011, 2.**
(a) Demonstrate the formula \( L_X = d i_X + i_X d \), where \( L \) is the Lie derivative and \( i \) is the interior product.

(b) Use this formula to show that a vector field \( X \) on \( \mathbb{R}^3 \) has a flow (locally) that preserves volume if and only if the divergence of \( X = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z} \) is everywhere 0, where the divergence of \( X \) is \( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \).

**Solution.**
(a) We can assume that \( X = \frac{\partial}{\partial x} \) use this on a basis form \( \omega = f \, dy \) and extend by linearity and through the wedge product. We know that
\[
L_X \omega = L_X f \, dy + f L_X dy = \frac{\partial f}{\partial x} \, dy + f \frac{\partial}{\partial x} dy = \frac{\partial f}{\partial x} \, dy + f \delta_i(j).
\]
This is exactly the formula required. Since the Lie derivate follows the same Leibniz rule (i.e. with the introduced \((-1)^k\)) as the inner product, it extends through the wedge product. Therefore this is the proper formula.

(b) Let \( \omega = dx \wedge dy \wedge dz \) be the standard volume form on \( \mathbb{R}^3 \). A vector field \( X \) preserves volume if and only if \( L_X \omega = 0 \). We use Cartan’s magic formula above
\[
L_X \omega = d(i_X \omega) + i_X (d\omega) = d(i_X \omega),
\]
where the last term vanishes since \( \omega \) is a top form. Let \( X \) have the form as above. Then
\[
i_X \omega = f \, dy \wedge dz + g \, dz \wedge dx + h \, dx \wedge dy.
\]
Hence
\[
d(i_X \omega) = \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dx \wedge dy \wedge dz.
\]
Hence \( X \) preserves volume if and only if \( \text{div} \, X = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0 \) everywhere, as required.

**Problem Spring 2011, 3.**
(a) Explain some systematic reason why there is a closed 2-form on \( \mathbb{R}^3 \setminus \{0\} \) that is not exact. You may do this by exhibiting such a form explicitly and checking that it is closed but not exact or you may argue using theorems that such a form must exist.

(b) With \( \phi \) such a form, discuss why, for any smooth map \( F : S^2 \to S^2 \),
\[
\deg F = \frac{\int_{S^2} F^* \phi}{\int_{S^2} \phi}.
\]
This includes explaining why the denominator cannot be 0.
Solution.

(a) First, note that the cohomology on $\mathbb{R}^3 \setminus \{0\}$ is that of $S^2$, since there is a homotopy retract from the former manifold to the latter. Therefore we need only define a closed 2-form on $S^2$ that is not exact. If we have an exact 2-form, then by Stokes’ theorem,

$$ \int_{S^2} d\omega = \int_{D^2} d^2\omega = 0, $$

where $D^2$ is the solid unit ball. Therefore demonstrating a 2-form with positive integral is sufficient in our case. Let $p$ be a point of $S^2$ in a local chart $U$. Then we can define a closed form on $U$ such that $\int_U \omega \neq 0$, and moreover we may let $\omega$ be compactly supported in $U$. Then we can extend $\omega$ to all of $S^2$ via a partition of unity, which we may make subordinate to $U$ and furthermore finite since $S^2$ is compact. Hence $\int_U \omega = \int_{S^2} \omega' \neq 0$. Further, since the partition of unity is finite, it commutes with the derivative, so that $d\omega' = 0$ as well, so it is closed.

(b) For this proof, we will need $F$ to be a proper map. But this is always satisfied on a closed manifold, so it works in our case. We will use the stack of records theorem:

**Lemma.** First, we know the denominator is not zero since nonexact forms have nonzero integral, as proved above. Hence we aim to show

$$ \deg F \int_{S^2} \omega = \int_{S^2} F^* \omega. $$

Now, Let $F : M \to N$ be a smooth map between manifolds of the same dimension, $M$ compact. Let $y \in N$ be a regular value. Then $F^{-1}(y)$ is finite, and there is a neighbourhood $V$ of $y$ such that $F^{-1}(V)$ is a disjoint union of open sets $U_i$ in $M$ such that $F|_{U_i} : U_i \to V$ is a diffeomorphism.

Our $F$ satisfies these conditions. Then for any point $q$ and chart $V$, its preimage is a finite set $p_i$ in charts $U_{p_i}$ where $U_{p_i} \cong V$. Also, in this case, $\deg F|_{U_{p_i}} = \pm 1$ since it is a diffeomorphism. Let $\omega \in \Omega^2(V)$, and view $\omega$ as a form on $\Omega^2(S^2)$ in the natural way (since it is compactly supported). We have

$$ F^* \omega = \sum_{p \in F^{-1}(q)} F^*|_{U_p} \omega. $$

Therefore we have

$$ \int_{S^2} F^* \omega = \sum_{p \in F^{-1}(q)} \int_{U_p} F^*|_{U_p} \omega = \sum_{p \in F^{-1}(q)} \varepsilon(p) \int_V \omega = \deg F \int_{S^2} \omega, $$

where $\varepsilon(p) = \pm 1$ depending on the orientation of the local diffeomorphism at $p$. Since every form on $S^2$ may be written as a finite sum of forms on local charts, extending by linearity this holds for all forms.
Problem Spring 2011, 4.
Show without using De Rham’s Theorem (but you may use the Poincaré Lemma without proof) that a 2-form $\phi$ on $S^2$ that has integral 0 is exact.

Solution.
Let $I$ be the integration function from $\Omega^{k+1}([0, 1] \times M) \to \Omega^k(M)$, and let $J_t : M \to [0, 1] \times M$ be the inclusion function. Then the Poincaré Lemma tells us that closed forms are exact on contractible manifolds. First, we can separate $S^2$ into two contractible manifolds $U$ and $D$ by moving the south pole and north pole respectively. Then we have $\phi|_U = d\theta_U$ and $\phi|_D = d\theta_D$. Let $T$ denote the top hemisphere of the circle and $B$ the bottom, each with boundary $S^1$ but with opposite orientations. Using Stokes’ theorem,

$$0 = \int_{S^2} \omega = \int_T d\theta_U + \int_B d\theta_D = \int_{S^1} \theta_U - \theta_D.$$

Hence $\theta_U - \theta_D$ is exact on $U \cap D$, so let it be $df$ for some function $f$. Consider $\rho$ a partition of unity subordinate to $U$ and $D$ so that $\rho = \rho_U + \rho_D$, each supported on the appropriate subscript. Define functions

$$f_U = \rho_D \cdot f, \quad f_D = \rho_U \cdot f.$$

Then define

$$\theta = \begin{cases} \theta_U - df_U & \text{on } U \\ \theta_D + df_D & \text{on } D. \end{cases}$$

Then by construction, $d\theta = \omega$. And indeed, we have

$$\theta_U = \theta_D + d(f_U + f_D) = \theta_D + df \text{ on } U \cap D.$$

Hence we have shown that $\omega$ is an exact form.

Problem Fall 2012, 5.
Assume that $\Delta = \{X_1, \ldots, X_k\}$ is a $k$-dimensional distribution spanned by vector fields on an open set $\Omega \subset M^n$ in an $n$-dimensional manifold. For each open subset $V \subset \Omega$ define

$$Z_V = \{u \in C^\infty(V) : X_i u = 0 \text{ for all } i \in \{1, \ldots, k\}\}.$$

Show that the following two statements are equivalent:

(a) The distribution $\Delta$ is integrable.

(b) For each $x \in \Omega$ there exists an open neighbourhood $x \in V \subset \Omega$ and $n - k$ functions $u_1, \ldots, u_{n-k} \in Z_V$ such that the differentials $du_1, \ldots, du_{n-k}$ are linearly independent at each point in $V$.  

\[ \square \]
Solution.
If $\Delta$ is integrable, then for each $x \in \Omega$ there is a $k$-dimensional submanifold $N \subset \Omega$ so that $T_x N = \Delta$ locally. Since $T_x N \subset T_x \Omega$, this means that there are $n - k$ coordinate vectors in $(T_x N)^\perp$ which span this complement. Their derivatives are also linearly independent, so (b) is satisfied.

For the converse, we use the Frobenius theorem. $\Delta$ is an integrable distribution if and only if the ideal of forms (with respect to the wedge product) annihilating $\Delta$ is closed under exterior differentiation. Let $\ell(\Delta)$ be this ideal. Indeed, we see that $Z_V$ generates $\ell(\Delta)$.

Assume that (b) holds. We claim that $du_1, \ldots, du_{n-k}$ generate $\ell(\Delta)$, i.e. $u_i$ span $Z_V$. Let $g$ be a form such that $dg$ is not in the span of $du_i$. Then in $\Delta$, we must have $\sum g_j dx_j \neq 0$, where the $x_j$ are coordinate functions with respect to the $X_j$ by dimensionality. Therefore there is an element $X_j$ of $\Delta$ that is not annihilated by $dg$, which corresponds to the term $g_j dx_j \neq 0$ above. Hence $\ell(\Delta)$ is generated by $du_i$. Therefore it is closed under exterior derivative, so applying the Frobenius theorem we are done.

Problem Fall 2012, 6.
On $\mathbb{R}^n \setminus \{0\}$, define the $(n-1)$-forms
$$
\sigma = \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \quad \omega = \frac{1}{|x|^n} \sigma.
$$

(a) Show that $\omega = r^* \circ i^*(\sigma)$, where $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is the natural inclusion of the unit sphere and $r(x) = \frac{x}{|x|} : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the natural retraction.

(b) Show that $\sigma$ is not a closed form.

(c) Show that $\omega$ is a closed form that is not exact.

Solution.
(a) On some vector fields $X_1, \ldots, X_{n-1}$, we have
$$
r^* \circ i^* \sigma(X_1, \ldots, X_{n-1}) = \sigma(di \circ dr(X_1), \ldots, di \circ dr(X_{n-1})) = \sigma(X_1/|x|, \ldots, X_{n-1}/|x|).
$$
But then $\omega = 1/|x|^n \sigma$ is clear since we have
$$
\sigma(X_1/|x|, \ldots, X_{n-1}/|x|)
= \sum_{i=1}^n (-1)^{i-1} x^i (x/|x|) d(x^1/|x|) \wedge \cdots \wedge d(\widehat{x^i}/|x|) \wedge \cdots \wedge d(x^n/|x|)
= \frac{1}{|x|^n} \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.
$$

(b) $d\sigma = \sum_{i=1}^n 1 \cdot dx^1 \wedge \cdots \wedge dx^n = n \cdot dx^1 \wedge \cdots \wedge dx^n \neq 0.$
(c) As above, $d\sigma$ is the scalar of a volume form. However, $d(i^*\sigma)$ is evidently 0, so $r^*d(i^*\sigma) = d\omega = 0$.

To show $\omega$ is not exact, we use Stokes’ theorem as in 3(a) above. If $\omega$ were exact, then its integral would be 0. But we know that

$$\int_{S^{n-1}} i^*\omega = \int_{S^{n-1}} i^*\sigma.$$ 

But $d\sigma$ is a scalar multiple of a volume form, so we have

$$\int_{S^{n-1}} i^*\sigma = \int_{D^{n-1}} d\sigma \neq 0.$$

Therefore $i^*\omega$ is not exact, so $\omega$ cannot be exact either.

\[\square\]

Problem Fall 2012, 7.
Let $n \geq 0$ be an integer. Let $M$ be a compact, orientable, smooth manifold of dimension $4n + 2$. Show that $\dim H^{2n+1}(M; \mathbb{R})$ is even.

Solution.
Consider the map $F : H^{2n+1}(M) \times H^{2n+1}(M) \rightarrow H^{4n+2}(M)$ given by $(\omega, \eta) \mapsto \omega \wedge \eta$. Let $\dim k = H^{2n+1}$, and since $M$ is compact and orientable, we have $\dim H^{4n+2}(M) = \mathbb{R}$. This this map is $\mathbb{R}$-bilinear, so induces a map

$$G : H^{2n+1}(M) \rightarrow H^{2n+1}(M)^\vee = \text{Hom}_\mathbb{R}(H^{2n+1}(M), \mathbb{R}).$$

where $\omega \mapsto -\wedge \omega$. By Poincaré duality (or linear algebra), $\dim H^{2n+1}(M)^\vee = k$ as well, and in particular this map is an isomorphism. $G$ is representable by a matrix $\alpha \in M_k(\mathbb{R})$. Since the wedge product is skew symmetric, $\alpha$ is a skew symmetric matrix as well. Hence we have

$$\det \alpha^\perp = \det \alpha = \det -\alpha^\perp = (-1)^k \det \alpha^\perp.$$

If $\det \alpha \neq 0$, then we must have $k$ is even so that $(-1)^k = 1$. But we know that this is always the case, since the map $G$ is an isomorphism. This completes the proof. \[\square\]

Problem Fall 2012, 9.
Consider the coordinate axes in $\mathbb{R}^n$:

$$L_i = \{(x_1, \ldots, x_n) : x_j = 0 \text{ for all } j \neq i\}.$$ 

Calculate the homology groups of the complement $\mathbb{R}^n \setminus (L_1 \cup \cdots \cup L_n)$.

Solution.
Let $M^n$ be this manifold. Note that since we have removed the origin, we may smoothly retract this manifold to $S^{n-1}$. In this case, removing the coordinate axes is homotopy equivalent to removing $2n$ distinct points.
First, in the case $n = 1$, the manifold obtained is the empty manifold, so there is nothing to compute. In the case $n = 2$, we have $S^1$ with four points removed is homotopy equivalent to four disconnected unit intervals. Hence $H^0(M^2) = \mathbb{R}^4$ and $H^1(M^1) = 0$ as each interval is contractible.

In higher dimensions, removing points from $S^{n-1}$ does not disconnect it, so we proceed with a general lemma.

**Lemma.** If $M$ is a connected manifold of dimension $n \geq 2$, then for any point $p \in M$,

$$H^k(M \setminus \{p\}) \cong H^k(M)$$

for $k < n - 1$, and $H^n(M \setminus \{p\}) = 0$. Additionally, if $M$ is compact and orientable, then $H^{n-1}(M \setminus \{p\}) \cong H^{n-1}(M)$. Otherwise, $\dim H^{n-1}(M \setminus \{p\}) = 1 + \dim H^{n-1}(M)$.

**Proof.** This follows Spivak 11.2(a). Since $n \geq 2$, $M \setminus \{p\}$ is connected. Let $U$ be a chart around the point $\{p\}$, so that $U \cup M \setminus \{p\} = M$. Then $U \cong \mathbb{R}^n$, and $U \cap M \setminus \{p\} \cong S^{n-1}$. Therefore we have a sequence

$$H^{k-1}(S^{n-1}) \rightarrow H^k(M) \rightarrow H^k(\mathbb{R}^n) \oplus H^k(M \setminus \{p\}) \rightarrow H^k(S^{n-1}) \rightarrow H^{k+1}(M),$$

where $k > 0$ since we know what $H^0(M \setminus \{p\})$ is since we assumed it was connected. For $k < n - 1$, we have

$$0 \rightarrow H^k(M) \rightarrow H^k(M \setminus \{p\}) \rightarrow 0$$

so $H^k(M) \cong H^k(M \setminus \{p\})$. The last part of this sequence, starting at $k = n - 1$, is

$$0 \rightarrow H^{n-1}(M) \rightarrow H^{n-1}(M \setminus \{p\}) \rightarrow \mathbb{R} \rightarrow H^n(M) \rightarrow H^n(M \setminus \{p\}) \rightarrow 0$$

If $M$ is nonorientable or noncompact, then $H^n(M) = 0$, so we obtain

$$0 \rightarrow H^{n-1}(M) \rightarrow H^{n-1}(M \setminus \{p\}) \rightarrow \mathbb{R} \rightarrow 0 \rightarrow H^n(M \setminus \{p\}) \rightarrow 0,$$

so $H^n(M \setminus \{p\}) = 0$ as well and $\dim H^{n-1}(M \setminus \{p\}) = 1 + \dim H^{n-1}(M)$. If $M$ is compact and orientable, then $H^n(M) = \mathbb{R}$. However $M \setminus \{p\}$ is not compact anymore, so $H^n(M \setminus \{p\}) = 0$. Hence we obtain

$$0 \rightarrow H^{n-1}(M) \rightarrow H^{n-1}(M \setminus \{p\}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0.$$  

Since $\mathbb{R} \rightarrow \mathbb{R} \rightarrow 0$ is surjective, $H^{n-1}(M \setminus \{p\}) \rightarrow \mathbb{R}$ must be the zero map. Hence we again have

$$0 \rightarrow H^{n-1}(M) \rightarrow H^{n-1}(M \setminus \{p\}) \rightarrow 0$$

so $H^{n-1}(M) \cong H^{n-1}(M \setminus \{p\})$. \hfill \square

For simplicity we compute the homology of $M^{n+1}$, for $n \geq 2$. Then this reduces to $S^n$ minus $2(n + 1)$ points. Removing the first point, by the lemma, yields the same homology as $S^n$ except for the top homology is now 0. By induction now on noncompact manifolds, removing $2n + 1$ points gives

$$\dim H^k(S^n \setminus \{2n + 2 \text{ points}\}) = \begin{cases} 1 & k = 0 \\ 0 & k = 1, \ldots, n-2, n. \\ 2n+1 & k = n-1 \end{cases}$$

This completes the classification. \hfill \square