Problem 3.2.19
Suppose that $Z$ is an oriented hypersurface in the oriented manifold $Y$, and let $\vec{n}$ be a smooth field of unit normal vectors along $Z$ in $Y$. Note that $\vec{n}$ defines an orientation in each normal space $N_x(Z;Y)$ – namely, the orientation in which the vector $\vec{n}(z)$ is a positively oriented basis. Show that if the direct sum orientation on $N_x(Z;Y) \oplus T_z(Z)$ and the orientation of $T_z(Y)$ agree at one point $z \in Z$, then they agree in the entire connected component of $Z$ containing $z$. Show that there is precisely one choice of $\vec{n}$ for which the orientations always agree; we shall call this $\vec{n}$ the outward unit normal field along $Z$. Check that for boundary orientations this is the usual outward normal.

Solution.
Let $z \in Z$, and suppose that $U$ is a chart for $z \in Y$. Then let $\varphi : U \to \mathbb{R}^{n+1}$ be the canonical map. Then the restriction $U \cap Z = \varphi^{-1}({\{0\} \times \mathbb{R}^n}) \cong \mathbb{R}^n$ since codim $Z = 1$. Therefore since we have reduced to the case of Euclidean space, the set of points $z' \in Z$ which agree with the orientation of $T_z(Z)$ is an open set. Analogously, the set of points $z' \in Z$ which disagree with the orientation of $T_z(Z)$ is an open set, and these two sets are disjoint with union $Z$. These two sets create a disconnection of $Z$, so the entire connected component of $z$ must share the orientation of $z$. Further, since the choice of $-\vec{n}$ reverses the orientation of $N_x(Z;Y) \oplus T_z(Z)$, there is only one choice of $\vec{n}$. In boundary orientations, the proof is identical, so we are done.

Problem 3.2.26
Prove that every simply connected manifold $X$ is orientable.

Solution.
Let $x \in X$ be any point and choose an orientation for $T_x(X)$. Let $x \neq y \in X$. Then construct an orientation for $T_y(X)$ in the following way: let $\gamma : I \to X$ be a path connecting $x$ and $y$. Choose a finite cover of the path $\gamma(I) \subset \bigcup_{i=1}^n U_i$ such that $U_i$ is a chart (i.e. diffeomorphic to an open ball in $\mathbb{R}^k$) and $U_i \cap U_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, n-1\}$. In particular, $x \in U_1$ and $y \in U_n$. Since orientation is a local property, for each $y_1 \in U_1$ we let $T_{y_1}(U_1)$ have the same orientation as $T_x(X)$, which makes sense by passing diffeomorphically to $\mathbb{R}^k$. Since $U_1 \cap U_2 \neq \emptyset$, we pick some point $y_1' \in$ the intersection and repeat the process for all $y_2 \in U_2$. In this way, we orient every point in each $U_i$, so eventually gives $T_y(X)$ the same orientation as $T_x(X)$.  

We claim that this process does not depend on the specific $\gamma$ chosen. Suppose that have two paths $\gamma, \gamma'$ from $x$ to $y$. Then since $X$ is simply connected, we may deform $\gamma'$ towards $\gamma$, so that their open covers intersect nontrivially. Again, since orientation is a local property, we must have the same orientation in each of the open covers along the paths, so $T_y(X)$ is consistent. Repeating this for every point $y \in X$ gives us an orientation for the space, so we are done.

Problem 3.3.1

Suppose that $f : X \to Y$ is a diffeomorphism of compact, connected manifolds. Check that $\deg f = +1$ if $f$ preserves orientation and $\deg f = -1$ if $f$ reverses orientation.

Solution.

Recall that $\deg f$ is the sum over all $x \in f^{-1}(y)$ of the relative orientations of $x$ and $y$ ($-1$ if reversing, $+1$ if preserving), and that we use $\deg$ when $f$ is connected since this is independent of the choice of $y$. Since $f$ is a diffeomorphism, it is bijective, so $|f^{-1}(y)| = 1$ everywhere. Say $\{x\} = f^{-1}(y)$. If $f$ preserves orientation, then $\deg f = +1$ since we add the single point $+1$. If $f$ reverses orientation, then $\deg f = -1$ since we add the single point $-1$. This completes the proof.

Problem 3.3.2

(a) Compute the degree of the antipodal map $S^k \to S^k$, $x \mapsto -x$.

(b) Prove that the antipodal map is homotopic to the identity if and only if $k$ is odd.

(c) Prove that there exists a nonvanishing vector field on $S^k$ if and only if $k$ is odd.

(d) Could mod 2 theory prove parts (b) and (c)?

Solution.

(a) The antipodal map is a diffeomorphism of compact, connected manifolds, so we may apply 3.3.1. By earlier considerations, this map is orientation preserving if and only if $k$ is odd. Therefore the degree of the antipodal map is $+1$ if $k$ is odd and $-1$ if $k$ is even.

(b) First, we know that homotopic maps have the same intersection number, and by extension the same degree in the connect case. Therefore if the antipodal map is homotopic to the identity, then its degree must be 1, hence $k$ is odd. Exercise 1.6.7 gives the converse, so we are done.

(c) By Exercise 1.6.8, if $S^k$ has a nonvanishing vector field, then its antipodal map is homotopic to the identity, which occurs if and only if $k$ is odd. Conversely, by Exercise 1.6.7, if $k$ is odd, then $S^k$ has a nonvanishing vector field, so we are done.

(d) Since $-1 = +1 \mod 2$, it could not.

Problem 3.3.7

Prove that the map $S^1 \to S^1$ defined by $z \mapsto \bar{z}^m$ has degree $-m$. 

Solution.
As hinted, the map $z \mapsto \bar{z}$ is an orientation-reversing diffeomorphism. That it is a diffeo-
morphism is clear, and the orientation is reversed because under conjugation any tangent
vector points the opposite direction around the circle. By Exercise 3.3.1, $\deg(z \mapsto \bar{z}) = -1$.
In general, $z \mapsto \bar{z}^m$ is the $m$-wrapped circle with orientation reversed everywhere. Therefore
the preimage of any point contains $m$ points with opposite orientation, so $\deg(z \mapsto \bar{z}^m) = -m$. \qed

Problem 3.3.8
According to Exercise 2.4.8, for any map $f : S^1 \to S^1$, there exists a map $g : \mathbb{R} \to \mathbb{R}$ such that
$$f(\cos t, \sin t) = (\cos g(t), \sin g(t)).$$
Moreover, $g$ satisfies $g(t + 2\pi) = g(t) + 2\pi q$ for some integer $q$. Show that $\deg f = q$.

Solution.
Let $g' : \mathbb{R} \to \mathbb{R}$ be defined by $g'(t) = tq$. We claim that $g \sim g'$. This homotopy can be
constructed explicitly in the form of Exercise 3.3.9 below, and for the same reason induces
a homotopy $f \sim f'$ in $S^1$. It is easy to verify that $f'(z) = z^q$ is the corresponding map, the
$q$-fold wrapping of the circle. This is an orientation-preserving variant of Exercise 3.3.7, so
it is clear that $\deg f = \deg f' = q$. \qed

Problem 3.3.9
Prove that two maps of the circle $S^1$ into itself are homotopic if and only if they have the
same degree.

Solution.
It is true in any situation that homotopic maps have the same degree, since homotopic maps
have the same intersection number. To prove the converse, suppose that $f_0, f_1 : S^1 \to S^1$
have the same degree, say $q$. Then the corresponding maps $g_0, g_1 : \mathbb{R} \to \mathbb{R}$ are of the form
$g_i(t + 2\pi) = g_i(t) + 2\pi q$. Then consider $g : I \times \mathbb{R} \to \mathbb{R}$ where $g(s, -) = g_s = sg_1 + (1 - s)g_0$.
Then
$$g_s(t + 2\pi) = s(g_1(t) + 2\pi q) + (1 - s)(g_0(t) + 2\pi q) = sg_1(t) + (1 - s)g_0(t) + 2\pi q$$
so each $g_s$ induces a map $f_s : S^1 \to S^1$ and further a map $f : I \times S^1 \to S^1$ such that
$f(0, -) = f_0$ and $f(1, -) = f_1$. Therefore $f_0$ and $f_1$ are homotopic, so we are done. \qed

Problem 3.3.10
Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are given. Prove that $\deg g \circ f = \deg f \cdot \deg g$.

Solution.
We proceed by counting points in the preimage. Let $z \in Z$ be a regular value of $g \circ f$. Then
$z$ is also a regular value of $g$ and $g^{-1}(z)$ is a regular value of $f$, so preimages make sense.
Let $g^{-1}(z) = \{y_1, \ldots, y_k\}$ and let $\varepsilon_i = \pm 1$ for each $i \in \{1, \ldots, k\}$ depend on whether the
orientations are consistent on $g$, and similarly define $f^{-1}(y_i) = \{x_{i1}, \ldots, x_{i\ell}\}$ and $\delta_{ij}$ for the map $f$. Let $\varphi_1, \ldots, \varphi_{k\ell}$ be $\pm 1$ depending on if $x_{ij}$ is preserved under $g \circ f$. Then

$$\deg g \circ f = \sum_{n=1}^{k\ell} \varphi_n$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{\ell} \varepsilon_i \cdot \delta_{ij}$$

$$= \sum_{j=1}^{\ell} \delta_{ij} \cdot \sum_{i=1}^{k} \varepsilon_i$$

$$= \sum_{j=1}^{\ell} \delta_{ij} \cdot \deg g = \deg f \cdot \deg g.$$  

This completes the proof. \qed

Problem 3.3.11

Prove that a map $f : S^1 \to S^1$ extends to the whole ball $B = \{|z| \leq 1\}$ if and only if $\deg f = 0$.

Solution.

First, suppose that $f$ can be extended to $B$. Then by the boundary theorem, since $S^1 = \partial B$, we have $\deg f = 0$.

Conversely, suppose that $\deg f = 0$. Then $f \sim 0$, the constant map by Exercise 3.3.9. Let $H : I \times S^1 \to S^1$ be a strict homotopy such that $H(0, -) = H_0 = f$ and $H(1, -) = H_1 = 0$. Then we may view $H$ as a function $g$ on the annulus $A = \{\frac{1}{2} \leq |z| \leq 1\}$ by

$$g(z) = H \left( \frac{1 - |z|}{2}, \frac{z}{|z|} \right), \text{ i.e. if } z = re^{i\theta}, \ g(re^{i\theta}) = H \left( r, e^{i\theta} \right).$$

Moreover for any small $\varepsilon > 0$, just as we would deform a homotopy we can retract $g$ such that $g(z) = 0$ for all $|z| < 1/2 + \varepsilon$. Therefore we may extend $g$ to the entire $B$ by letting $g(z) = 0$ for all $|z| \leq 1/2$. Since there is a neighbourhood of the circle $|z| = 1/2$ on which $g$ is constant, this ‘insertion’ of the inner circle is smooth. Therefore we have extended $f$ to $B$ (since $f(z) = g(z)$ for $|z| = 1$), so we are done. \qed