Problem 1.5.4.
Let $X$ and $Z$ be transversal submanifolds of $Y$. Prove that if $y \in X \cap Z$, then
$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

Solution.
Since $X \cap Z$ is itself a manifold and is contained in both $X$ and $Z$ we have $T_y(X \cap Z) \subset T_y(X)$ and $T_y(X \cap Z) \subset T_y(Z)$, hence we have the inclusion $T_y(X \cap Z) \subset T_y(X) \cap T_y(Z)$.

We claim that these spaces have the same dimension. Since $X \subseteq Z$, $\text{codim}(X \cap Z) = \text{codim}(X) + \text{codim}(Z)$. If $\dim Y = n$, then we may write $\dim(X \cap Z) = \dim X + \dim Z - n$. Therefore
$$\dim T_y(X \cap Z) = \dim T_y(X) + \dim T_y(Z) - n.$$  

From linear algebra, we know that
$$\dim(T_y(X) \cap T_y(Z)) = \dim T_y(X) + \dim T_y(Z) - \dim(T_y(X) + T_y(Z)).$$

Again by transversality, we must have $T_y(X) + T_y(Z) = T_y(Y)$, so that $\dim(T_y(X) + T_y(Z)) = n$. Combining these, we have
$$\dim(T_y(X) \cap T_y(Z)) = \dim T_y(X \cap Z).$$

Since these spaces have the same dimension and one includes the other, they must be equal. This completes the proof.

Problem 1.5.5.
Let $f : X \to Y$ be a map transversal to a submanifold $Z$ in $Y$. Then $W = f^{-1}(Z)$ is a submanifold of $X$. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $Df_x : T_x(X) \to T_{f(x)}(Y)$.

Solution.
Consider $f \circ i : W \to Z$, where $i : W \to X$ is the inclusion map. Then $f \circ i$ is a diffeomorphism, so $D(f \circ i)_x : T_x(W) \to T_{f(x)}(Z)$ is an isomorphism of tangent spaces. Note that $D(f \circ i)_x = Df_{i(x)} \cdot Di_x = Df_x \cdot Di_x$. Since $Di_x : T_x(W) \to T_x(X)$ is also the inclusion map, we see that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ as required.
Problem 1.5.9.
Let $V$ be a vector space, and let $\Delta$ be the diagonal of $V \times V$. For a linear map $A : V \to V$, consider the graph $W = \{(v, Av) : v \in V\}$. Show that $W \nparallel \Delta$ if and only if $+1$ is not an eigenvalue of $A$.

Solution.
Let $\dim V = n$. If $W \nparallel \Delta$, then $T_{(v,w)}(W) + T_{(v,w)}(\Delta) = T_{(v,w)}(V \times V)$ at every $(v, w) \in V \times V$.
Since we are working over vector spaces, this is equivalent to $W + \Delta = V \times V$. We see that $W \cap \Delta = \{v \in V : Av = v\}$, i.e. the eigenspace of $+1$. Since $W + \Delta = V \times V$, we have
\[
\dim V \times V = \dim W + \dim \Delta - \dim (W \cap \Delta).
\]
Since $\dim W = \dim \Delta = n$ and $\dim V \times V = 2n$, we must have $\dim (W \cap \Delta) = 0$, i.e. $A$ does not have an eigenspace of $+1$, so $+1$ is not an eigenvalue of $A$.

Conversely, suppose that $A$ does not have an eigenspace for $+1$. Then $\dim W + \dim \Delta = 2n$ by the above equation, hence $W + \Delta = V$. Again, since we are working over vector spaces, this implies $W \nparallel \Delta$.

Problem 1.5.10.
Let $f : X \to X$ be a map with fixed point $x$. If $+1$ is not an eigenvalue of $Df_x : T_x(X) \to T_x(X)$, then $x$ is called a Lefschetz fixed point of $f$. $f$ is called a Lefschetz map if all its fixed points are Lefschetz. Prove that if $X$ is compact and $f$ is Lefschetz, then $f$ has only finitely many fixed points.

Solution.
Using the results of the previous problem, we know that $x$ is a Lefschetz fixed point if and only if graph $f$ and $\Delta$ (the diagonal) are transversal in $X \times X$. Again, as we saw above, this implies that $\dim(\text{graph } f \cap \Delta) = 0$. The set we are interested in is precisely graph $f \cap \Delta$, since this implies that $(x, f(x)) = (x, x)$ i.e. $f(x) = x$. By earlier results, we have shown that a dimension zero compact manifold is exactly a finite number of isolated points, hence $f$ may only have finitely many fixed points.

Problem 1.6.1.
Suppose that $f_0, f_1 : X \to Y$ are homotopic. Show that there exists a homotopy $\tilde{F} : X \times I \to Y$ such that $\tilde{F}(x, t) = f_0(x)$ for all $t \in [0, \frac{1}{4}]$, and $\tilde{F}(x, t) = f_1(x)$ for all $t \in [\frac{3}{4}, 1]$.

Solution.
We are guaranteed a smooth homotopy $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Taking the hint, we will construct a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ manipulating $t$. We recall that we have created a bump function $\rho$ (1.1.18) such that, for any $a < b$,
\[
\rho(x) = \begin{cases} 
0 & x \leq a \\
1 & x \geq b
\end{cases}
\]
with smooth interpolation in between. Let $a = \frac{1}{4}$ and $b = \frac{3}{4}$. Since the composition of smooth functions are still smooth we define $\tilde{F} = F(x, \rho(t))$, which is smooth. It satisfies exactly what we want, so we are done.
Problem 1.6.2.
Prove that homotopy is an equivalence relation (i.e. show transitivity).

Solution.
Suppose that \( f \sim g \) and \( g \sim h \) on \( X \to Y \). Let \( H_0 : X \times I \to Y \) and \( H_1 : X \times I \to Y \) be these homotopies, respectively. We use the preceding problem here. Since the choice of \( \frac{1}{4} \) and \( \frac{3}{4} \) was arbitrary in 1.6.1, we know there exists \( \tilde{H}_0 \) such that \( \tilde{H}_0(x, t) = f(x) \) for all \( t \in [0, \frac{1}{2}] \) and \( \tilde{H}_0(x, t) = g(x) \) for all \( t \in [\frac{3}{4}, 1] \). Further, there exists \( \tilde{H}_1 \) such that \( \tilde{H}_1(x, t) = g(x) \) for all \( t \in [0, \frac{3}{4}] \) and \( \tilde{H}_1(x, t) = h(x) \) for all \( t \in [\frac{1}{2}, 1] \).

We therefore can combine these into one homotopy. Define \( H_2 : X \times I \to Y \) by

\[
H_2(x, t) = \begin{cases} 
\tilde{H}_0(x, t) & t \in [0, \frac{1}{2}] \\
\tilde{H}_1(x, t) & t \in [\frac{1}{2}, 1].
\end{cases}
\]

This is still a homotopy because \( \tilde{H}_0 \) and \( \tilde{H}_1 \) agree near \( t = 1/2 \), and the entire function is smooth and takes the appropriate values at \( t = 0 \) and \( t = 1 \). This completes the proof.

Problem 1.6.6.
Check that all contractible spaces are simply connected, but convince yourself that the converse is false.

Solution.
By 1.6.4, if \( X \) is contractible, then all maps from an arbitrary manifold \( Y \to X \) are homotopic. Therefore every map \( S^1 \to X \) is homotopic to the constant map from \( S^1 \to X \), so \( X \) is simply connected.

We showed in 1.7.6 (on the last homework) that \( S^k \) is simply connected for \( k > 1 \). However, they are not contractible.

Problem 1.6.7.
Show that the antipodal map \( x \mapsto -x \) of \( S^k \to S^k \) is homotopic to the identity if \( k \) is odd.

Solution.
We will construct this antipodal map explicitly. Let \( k = 2n - 1 \), where \( n \geq 1 \). Consider \( S^k \subset \mathbb{C}^n \cong \mathbb{R}^{2n} \) as the set of points \( \{ z = (z_1, \ldots, z_n) : |z| = 1 \} \), where \( |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2} \), and \( |z_i| \) is the usual complex norm. Consider the map \( F : S^k \times I \to S^k \) given by

\[
F((z_1, \ldots, z_n), t) = (e^{i\pi t} z_1, \ldots, e^{i\pi t} z_n).
\]

This satisfies \( F(z, 0) = z \) and \( F(z, 1) = -z \). Further, it is smooth since \( e^{i\pi t} \) is smooth and at any point \( t \in (0, 1) \),

\[
|F(z, t)| = |e^{\pi it}| \cdot |z| = 1,
\]

so \( F \) stays on \( S^k \). Thus it is a homology between the identity and the antipodal map, so we are done.
Problem 2.1.8.
Show that there are precisely two unit vectors in $T_x(X)$ that are perpendicular to $T_x(\partial X)$ and that one lies inside $H_x(X)$, the other outside. Denote the outward unit normal by $n(x)$. Note that if $X$ sits in $\mathbb{R}^k$, $n(x)$ may be considered to be a map of $\partial X$ into $\mathbb{R}^k$. Prove that $n$ is smooth.

Solution.
If $x \in \partial X$, then there is a neighbourhood of $x$ which is diffeomorphic to $H^k$. The tangent space of $T_x(\partial X)$ is isomorphic to the hyperplane $\mathbb{R}^{k-1} \subset H^k$. At any point, there are exactly two unit vectors orthogonal to the plane, since its codimension is 1. One of these vectors has $x_k < 0$, which corresponds to the outward vector, and the other has $x_k > 0$ which is the inward vector (taking G&P’s definition of $H^k$).

To see this is smooth, let $\gamma : I \to H_x(X)$ be a smooth arc lying inside a sufficiently small neighbourhood $U$ of $x$. Then let $\varphi(t) = D\gamma_t/|D\gamma_t|$. By the chain rule, $\varphi(t)$ is smooth and $-D\varphi(t)(0)$ is a preimage of $n(x)$. Since the inclusion of $\partial X$ into $\mathbb{R}^k$ is also smooth, $n(x) : \partial X \to U \to \mathbb{R}^k$ will also be a smooth map.

Problem 2.1.9.
(a) Show that $\partial X$ is a closed subset of $X$.
(b) Find some examples in which $\partial X$ is compact but $X$ is not.

Solution.
(a) We use the result of 2.1.11. Given that $\partial x$ may be given by $f^{-1}(0)$ by a smooth function $f$, in particular $f$ is continuous. Then $\partial X = f^{-1}(0)$ is closed since $\{0\} \subset \mathbb{R}$ is closed.

We may also see this by noting that the interior of $X$ is open (since any point is covered by a chart). Since $\partial X = X \setminus \text{int} X = \text{int} X^c$, $\partial X$ is closed in $X$.

(b) Let $X = H^1$. Then $\partial X = \{0\}$, which is compact, but $X$ itself is not.

Problem 2.1.10.
Let $x \in \partial X$ be a boundary point. Show that there exists a smooth nonnegative function $f$ on some open neighbourhood $U$ of $x$, such that $f(z) = 0$ if and only if $z \in \partial U$, and if $z \in \partial U$, then $Df_x(n(z)) < 0$ (where $n(z)$ is the unit outward normal vector).

Solution.
Let $\pi : H^k \to \mathbb{R}$ be given by $\pi(x_1, \ldots, x_k) = x_k$. We adopt G&P’s convention and define $H^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$. Then $\pi \geq 0$ everywhere. Further, $\pi(z) = 0$ only when $z \in \partial H^k$.

Let $\varphi : U \to V \subset H^k$ be a chart around $z \in X$. Then let $f : U \to \mathbb{R}$ by $f = \pi \circ \varphi$. Then $f \geq 0$ and $f(z) = 0$ if and only if $\varphi(z) \in \partial H^k$ if and only if $z \in \partial X$. Since the derivative is linear, we know $Df_z(n(x)) < 0$ if and only if $Df_z(-n(z)) > 0$. Now, $Df_z = D(\pi \circ \varphi)_z = D\pi_{\varphi(z)} \cdot D\varphi_z$. 

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We claim that \( D\varphi_z(-n(z)) \) points inward into \( H^k \). To see this, we have for any curve satisfying \( \gamma(0) = z \) and \( \gamma'(0) = -n(z) \),

\[
D\varphi_z(-n(z)) = \lim_{t \to 0} \frac{\varphi(\gamma(t)) - \varphi(\gamma(0))}{t} = \lim_{t \to 0} \frac{\varphi(\gamma(t)) - \varphi(z)}{t}.
\]

Since \( \varphi(\gamma(t)) \in H^k \setminus \partial H^k \) for all \( t \) and \( \varphi(z) \in \partial H^k \). Since \( \pi \) is linear, \( D\pi\varphi(z) = \pi \), so \( \pi(D\varphi_z(-n(z))) > 0 \) since \( D\varphi_z(-n(z)) \in H^k \). This completes the proof. \( \square \)

**Problem 2.1.11.**
Show that if \( X \) is any manifold with boundary, then there exist a smooth nonnegative function \( f \) on \( X \), with a regular value at 0, such that \( \partial X = f^{-1}(0) \).

**Solution.**
Let \( U_i \) be a locally finite cover of \( X \), which is guaranteed to exist. Then by 2.1.9, we have functions \( f_i : U_1 \to \mathbb{R} \) such that \( f_i \geq 0 \) and \( f_i(x) \) if and only if \( x \in \partial U_i \). By locally finite, we may define \( f : X \to \mathbb{R} \) by \( f = \sum f_i \). Then by construction, \( f = 0 \) if and only if \( x \in \bigcup \partial U_i = \partial X \), and \( f \geq 0 \) in general. This completes the proof. \( \square \)