The goal is the following theorem.

**Theorem (Hopf).** Let $M$ be a compact $n$-manifold without boundary, and let $f, g : M \to S^n$ be two smooth maps.

(a) If $M$ is orientable, then $f \sim g$ if and only if $\deg f = \deg g$.

(b) If $M$ is not orientable, then $f \sim g$ if and only if $\deg_2 f = \deg_2 g$.

We proceed for case (a) by means of the exercises outlined in Guillemin and Pollack.

**Exercise 1** Let $f : U \to \mathbb{R}^k$ be any smooth map defined on a subset $U$ of $\mathbb{R}^k$, and let $x$ be a regular point with $f(x) = z$. Let $B$ be a sufficiently small closed ball centred at $x$, and define $\partial f : \partial B \to \mathbb{R}^k$ to be the restriction of $f$ to the boundary of $B$. Prove that $W(\partial f, z) = +1$ if $f$ preserves orientation at $x$ and $W(\partial f, z) = -1$ if $f$ reverses orientation at $x$.

**Proof.** By changing coordinates, we may let $x = 0 = z$. Let $A = Df_0$. Then since 0 is a regular value, $A$ is bijective. Therefore since there is a small ball $B \subset U$, there is some other small ball $C$ of radius $c > 0$ in $\mathbb{R}^k$ in the image of $A$, and $\partial B$ is mapped diffeomorphically onto $\partial C$. In particular, for $x \in U$ outside of $B$, say $x \in S^{k-1}$, we have $|Ax| > c$ because it is not mapped onto $C$. There by linearity, $|Ax| > c|x|$ for all $0 \neq x \in U$.

We may write $f(x) = Ax + \varepsilon(x)$, where $\varepsilon(x)/|x| \to 0$ as $|x| \to 0$. Additionally, let $f_t(x) = Ax + t\varepsilon(x)$ for $t \in [0, 1]$. Then $f_t$ is a homotopy of $f_0(x) = Ax$ and $f_1(x) = f(x)$. Choose the radius of $B$ small enough such that

$$\frac{|\varepsilon(x)|}{|x|} < \frac{c}{2}$$

for all $x \in \partial B$. Then on $\partial B$,

$$|f_t(x)| \geq |Ax| - t|\varepsilon(x)| > c|x| - \frac{c|x|}{2} = \frac{c|x|}{2} > 0$$

so $f_t(x) \neq 0$ anywhere. Therefore we have a homotopy $F_t(x) = \frac{f_t(x)}{|f_t(x)|}$ between $F_0(x) = \frac{Ax}{|Ax|}$ and $F_1(x) = \frac{\partial f(x)}{|\partial f(x)|}$. Hence the degrees of $F_0$ and $F_1$ are equal, and since these are the direction maps of $A$ and $\partial f$, $W(A, 0) = W(\partial f, 0)$.

We now need a lemma in §3.4.
Lemma (G&P pg. 110). Suppose that \( E \) is a linear isomorphism of \( \mathbb{R}^k \) that preserves orientation. Then \( E \) is homotopic to the identity map. If \( E \) reverses orientation, then \( D \) is homotopic to the reflection map \( (x_1, x_2, \ldots, x_k) \mapsto (-x_1, x_2, \ldots, x_k) \).

Therefore if \( A \) preserves orientation, \( W(A, 0) = +1 \) and if \( A \) reverses orientation, \( W(A, 0) = -1 \), so the same holds for \( W(\partial f, 0) \).

Exercise 2 Let \( f : B \to \mathbb{R}^k \) be a smooth map defined on some closed ball \( B \) in \( \mathbb{R}^k \). Suppose that \( z \) is a regular value of \( f \) that has no preimages on the boundary sphere \( \partial B \), and consider \( \partial f : \partial B \to \mathbb{R}^k \). Prove that the number of preimages of \( z \), counted with the usual orientation convention, equals the winding number \( W(\partial f, z) \).

Proof. Let \( f^{-1}(z) = \{b_1, \ldots, b_n\} \) be the preimages of \( z \). Then let \( b_i \subset U_i \) be a small neighbourhood around each point such that \( U_i \cap \partial B = \emptyset \) for each \( i \). Let \( U = \bigcup U_i \) and \( B' = B \setminus U \). Then we have a new map \( \tilde{f} : B' \to \mathbb{R}^k \setminus \{z\} \cong S^{k-1} \). Then let \( u : \partial B' \to S^{k-1} \) be the directional map on \( B' \), namely

\[
u(x) = \frac{\tilde{f}(x) - z}{|\tilde{f}(x) - z|}.
\]

Then this map extends to all of \( B' \) since \( f(x) \neq z \) for any \( z \in B' \) by construction. Therefore \( W(f|_{\partial B'}, z) = \deg u \), and \( \deg u = 0 \) by the proposition of §3.3.

Lemma (G&P pg. 128). Suppose that \( f : X \to Y \) is a smooth map of compact oriented manifolds having the same dimension and that \( X = \partial W \). If \( f \) can be extended to all of \( W \), then \( \deg f = 0 \).

Now, we have \( \partial B' = \partial B \setminus \bigcup_{i=1}^n \partial U_i \), and each \( \partial U_i \) inherits an orientation opposite to that of \( \partial B \). Hence

\[0 = W(f|_{\partial B'}, z) = W(\partial f, z) - \sum_{i=1}^n W(f_i, z)\]

so \( W(\partial f, z) = \sum_{i=1}^n W(f_i, z) \) as required.

Exercise 3 Let \( B \) be a closed ball in \( \mathbb{R}^k \), and let \( f : \mathbb{R}^k \setminus \text{Int} B \to Y \) be any smooth map defined outside the open ball \( \text{Int} B \). Show that if the restriction \( \partial f : \partial B \to Y \) is homotopic to a constant, then \( f \) extends to a smooth map defined on all of \( \mathbb{R}^k \) into \( Y \).

Proof. Assume that \( B = B(0, r) \) is centred at the origin, and let \( g_t : \partial B \to Y \) be a homotopy such that \( g_1 = \partial f \) and \( g_0 = \text{constant} \). Then consider the maps \( f(tx) = f_t(x) \) where \( x \in \partial B \) and \( t \in [0, 1] \). Then \( f_t(x) = \partial f(x) \) and as \( t \to 0 \), \( f_t \) passes through all of \( B \) since each \( 0 \neq y \in B \) may be written uniquely in the form \( y = tx \) for \( x \in \partial B \). Since we have a homotopy from \( f_1 = \partial f \) to \( f_0 = \text{constant} \), consider

\[F(x) = \begin{cases} f(x) & x \in \mathbb{R}^k \setminus \text{Int} B \\ f_t(y) & x \in B \text{ such that } x = ty, y \in \partial B, t \in [0, 1] \end{cases}\]

On \( \partial B \), we have \( f = \partial f = f_1 \), hence \( F \) is well defined on \( \mathbb{R}^k \setminus \text{Int} B \). On \( B \setminus \{0\} \), by the unique representation of each \( x \in B \) as above, \( F \) is still well defined. Finally, at \( 0 \), \( F \) is well
defined since $g_0$ is a constant function so does not depend on the choice of representative with $t = 0$. By gluing all of these pieces together, $F$ is a continuous function.

Further, $F$ can be made smooth by creating a buffer around $\partial B$ using the standard trick of Exercise 1.6.1. Modifying the homotopy $f_t$ slightly (as in that exercise), we can assure that the transition at $\partial B$ is smooth, so we have extended $F$ smoothly to all of $\mathbb{R}^k$. \qed

We now have the following special case the Hopf theorem.

**Theorem (Special Case).** Any smooth map $f : S^l \to S^l$ having degree zero is homotopic to a constant map.

**Exercise 4** Check that the special case implies the following corollary.

**Corollary.** Any smooth map $f : S^l \to \mathbb{R}^{l+1} \setminus \{0\}$ having winding number zero with respect to the origin is homotopic to a constant.

**Proof.** $f : S^l \to \mathbb{R}^l \setminus \{0\}$ has winding number zero if and only if the map direction map $u : S^l \to S^l$ has degree zero. By the special case, if a map $S^l \to S^l$ has degree zero, then it is homotopic to the constant map. In this case, we have $u(x) = \frac{f(x)}{|f(x)|}$. Then we have a homotopy

$$f_t(x) = tf(x) + (1 - t) \frac{f(x)}{|f(x)|}$$

between $f(x)$ and $u(x)$. Since homotopies are transitive, we have $f \sim \text{constant}$. \qed

We now prove the special case. By Exercise 3.3.9, two smooth maps $f, g : S^1 \to S^1$ are homotopic if and only if they have the same degree. Therefore if $f : S^1 \to S^1$ is a smooth map of degree zero, it is homotopic to a constant map. This proves the base case $l = 1$ of the special case. Now we assume the special case for $l = k - 1$. We will need the following exercise in the proof.

**Exercise 5** Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map with 0 was a regular value. Suppose that $f^{-1}(0)$ is finite and that the number of preimage points in $f^{-1}(0)$ is zero when counted with the usual orientation convention. Assuming the special case in dimension $k - 1$, prove that there exists a mapping $g : \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = f$ outside of a compact set.

**Proof.** Let $B$ be a ball centred around the origin sufficiently large to contain all $f^{-1}(0)$ in its interior. Let $\partial f : \partial B \to \mathbb{R}^k \setminus \{0\}$. By Exercise 2, we have $W(\partial f, 0) = 0$. Since $\partial B \cong S^{k-1}$, by the inductive step, $\partial f \sim \text{constant}$.

Now we may replace our original map $f$ by the restriction $f|_{\mathbb{R}^k \setminus \text{Int } B} : \mathbb{R}^k \setminus \text{Int } B \to \mathbb{R}^k \setminus \{0\}$. Therefore Exercise 3 applies, so there exists a smooth extension $g : \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $f = g$ outside of $B$. Since $B$ was compact, this completes the proof. \qed

Note that since $f$ and $g$ are equal outside of a compact set, the homotopy $tf + (1 - t)g$ is compactly supported.

**Exercise 6** Establish the special case in dimension $k$. 

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Proof. By Sard’s theorem, let \( a, b \in S^k \) be two regular values of \( f : S^k \to S^k \). Let \( f^{-1}(a) = \{a_1, \ldots, a_n\} \) and \( f^{-1}(b) = \{b_1, \ldots, b_m\} \) be the finite preimages of these values. Since these sets are finite, we may apply the isotopy lemma and its corollary to \( S^k \setminus f^{-1}(b) \) (which is connected for \( k > 1 \)) to find an open neighbourhood \( U \subset S^k \setminus f^{-1}(b) \) of \( f^{-1}(a) \) diffeomorphic to \( \mathbb{R}^k \). That is, there is a neighbourhood \( U \subset S^k \) of \( f^{-1}(a) \) avoiding all \( b_i \).

Let \( \alpha : \mathbb{R}^k \to U \) be a map induced by the chart and let \( \beta : S^k \setminus \{b\} \to \mathbb{R}^k \) such that \( \beta(a) = 0 \). Then examine the map \( \beta \circ f \circ \alpha : \mathbb{R}^k \to \mathbb{R}^k \) which satisfies the premises of Exercise 5. Therefore there exists \( g \sim \beta \circ f \circ \alpha \) such that \( g = \beta \circ f \circ \alpha \) outside a compact set. Then on the neighbourhood \( U \) of \( a, f \sim \beta^{-1} \circ g \circ \alpha^{-1} : U \to S^k \setminus \{a, b\} \). Further, \( f = \beta^{-1} \circ g \circ \alpha^{-1} \) outside of \( \alpha^{-1}(B) \). Thus we have a smooth map

\[
h : S^k \to S^k \setminus \{a\}
\]

which is homotopic to \( f \), where we let \( h = \beta^{-1} \circ g \circ \alpha^{-1} \) on \( \alpha^{-1}(B) \). Since \( S^k \setminus \{a\} \cong \mathbb{R}^k \) is contractible, every map into it is homotopic to a constant map. Hence \( f \sim \text{constant} \). \( \square \)

The Hopf theorem is a particular instance of the following theorem.

**Theorem (Extension).** Let \( W \) be a compact, connected, oriented \( k + 1 \)-dimensional manifold with boundary, and let \( f : \partial W \to S^k \) be a smooth map. Then \( f \) extends to a globally defined map \( F : W \to S^k \), with \( \partial F = f \), if and only if the degree of \( f \) is zero.

To prove it, we need the following lemma.

**Exercise 7** Let \( W \) be any compact manifold with boundary, and let \( f : \partial W \to \mathbb{R}^{k+1} \) be any smooth map whatsoever. Prove that \( f \) may be extended to all of \( W \).

**Proof.** By Whitney’s embedding theorem, \( W \subset \mathbb{R}^N \) for some \( N \) as a closed subset. Applying the \( \varepsilon \)-neighbourhood theorem to \( \partial W \) (which is a closed set), we may extend \( f \) to a map \( F \) defined on an open neighbourhood \( U \) of \( \partial W \) in \( \mathbb{R}^N \).

Let \( \rho \) be a smooth function supported on \( \partial W \) and zero outside a compact subset of \( U \), with smooth interpolation in between. Then we can extend \( F \) to \( F' : W \to \mathbb{R}^{k+1} \) by letting

\[
F'(x) = \begin{cases} 
\rho(x) \cdot F(x) & x \in U \\
0 & \text{otherwise}
\end{cases}
\]

This extends \( f \) to all of \( W \). \( \square \)

**Exercise 8** Prove the Extension Theorem.

**Proof.** By the above lemma in the proof of Exercise 1, we have the forward direction.

For the converse, by Exercise 7, we can extend \( f \) to \( F : W \to \mathbb{R}^{k+1} \). By the transversality extension theorem, we may assume 0 is a regular value of \( F \). Again applying the isotopy lemma, we may place all the points in the (finite) preimage \( F^{-1}(0) \) in some \( U \subset \text{Int} W \) diffeomorphic to \( \mathbb{R}^{k+1} \). Let \( B \) be a closed ball in \( U \) containing all \( F^{-1}(0) \) in its interior.

Consider \( \partial F = F|_{\partial B} : \partial B \to \mathbb{R}^{k+1} \setminus 0 \). As in Exercise 2, we can extend \( F/|F| \) to \( W \setminus \text{Int} B \), since the preimage of 0 is contained in \( \text{Int} B \). Hence \( W(F|_{\partial (W \setminus \text{Int} B)}, 0) = \deg(F/|F|) = 0 \).
We also have \( W(F|_{\partial W}, 0) = W(f, 0) = \deg f = 0 \), by our assumption. Since we have \( \partial(W \setminus \text{Int } B) = \partial W \cup (-\partial B) \), we have

\[
W(F|_{\partial(W \setminus \text{Int } B)}, 0) = W(F|_{\partial W}, 0) - W(\partial F, 0),
\]

which implies that \( W(\partial F, 0) = 0 \). Therefore by the corollary to the special case, \( \partial F \sim \text{constant} \). By Exercise 3, \( F|_W \) extends to a smooth map \( F' : W \to \mathbb{R}^{k+1} \setminus \{0\} \). Therefore \( F'/|F'| : W \to S^k \) is the required extension of \( f \), and we are done. \( \square \)

**Exercise 9** Conclude:

**Theorem (Hopf Degree).** Two maps of a compact, connected, oriented \( k \)-manifold \( X \) into \( S^k \) are homotopic if and only if they have the same degree.

**Proof.** Let \( f_0, f_1 : X \to S^k \), and let \( W = X \times I \) be a compact, oriented \( k+1 \)-dimensional manifold with boundary. Define \( f : \partial W \to S^k \) by \( f = f_0 \) on \( X \times \{0\} \) and \( f = f_1 \) on \( X \times \{1\} \). Then by the extension theorem, \( f \) extends to map on all of \( W \) if and only if \( \deg f = 0 \). Such an extension is equivalent to a homotopy of \( f_0 \) and \( f_1 \). Further, since \( \partial W = (X \times \{1\}) \cup -(X \times \{0\}) \), we have \( \deg f = \deg f_1 - \deg f_0 \), so \( \deg f = 0 \) if and only if \( \deg f_0 = \deg f_1 \). This completes the proof. \( \square \)

This proves the orientable case. For the non-orientable case, we will prove a non-orientable extension theorem.

**Theorem (Non-orientable Extension).** Let \( W \) be a compact, connected, non-orientable \( k+1 \)-dimensional manifold with boundary, and let \( f : \partial W \to S^k \) be a smooth map. Then \( f \) extends to a globally defined map \( F : W \to S^k \), with \( \partial F = f \), if and only if the degree modulo 2 of \( f \) is zero.

**Proof.** The forward case is clear just as in the orientable version of the theorem.

Conversely, suppose \( \deg_2 f = 0 \). We can extend \( f \) to \( F : W \to \mathbb{R}^{k+1} \) by Exercise 7. By the transversality extension theorem, we may assume 0 is a regular value of \( F \). Again applying the isotopy lemma, we may place all the points in the (finite) preimage \( F^{-1}(0) \) in some \( U \subset \text{Int } W \) diffeomorphic to \( \mathbb{R}^{k+1} \). Let \( B \) be a closed ball in \( U \) containing all \( F^{-1}(0) \) in its interior.

As in the orientable case, we extend \( F/|F| \) to \( F' : W \setminus \text{Int } B \). By the boundary theorem, \( \deg_2 F'|_{\partial W} = \deg_2 F'|_{\partial B} = 0 \). Therefore \( \deg F'|_{\partial B} \) is an even number, counted with respect to the orientation of \( B \). We need to prove that \( \deg F'|_{\partial B} = 0 \).

We need the following lemma.

**Lemma.** Given a point \( y \in Y \) in a connected non-orientable manifold \( Y \), choose an orientation of \( T_y Y \). Then there exists a diffeomorphism \( h : Y \to Y \) isotopic to the identity such that \( h(y) = y \) and the degree of \( h \) at \( y \) with respect to the chosen orientation is \(-1\). Moreover, this isotopy may be taken to be compactly supported.

**Proof.** Choose a loop from \( y \) to \( y \) which reverses the orientation of \( T_y Y \). We may cover this loop with finitely many open balls giving rise to the opposite orientation. Then we may apply the isotopy lemma to every ball. Gluing these diffeomorphisms together, we are left with a map isotopic to the identity, compactly supported, reversing the orientation at \( y \). \( \square \)
Now let \( y \in F^{-1}(0) \). Then given an orientation-reversing \( h \) as above, the degree of \( F' \circ h \) in a small neighbourhood of \( y \) is changed by \(-1\). Choosing the appropriate number of \( h \), we have \( \deg(F' \circ h_1 \circ \ldots \circ h_m)|_{\partial B} = 0 \). Since \( F' \sim F' \circ h_1 \circ \ldots \circ h_m \), we have \( \deg F'|_{\partial B} = 0 \). Therefore we can extend \( F' \) to all of \( W \) just as in the orientable case.

We now apply this theorem to prove what we want.

**Theorem (Hopf, Non-orientable).** Two maps of a compact, connected, non-orientable \( k \)-manifold \( X \) into \( S^k \) are homotopic if and only if they have the same degree modulo 2.

*Proof.* Let \( f_0, f_1 : X \to S^k \), and let \( W = X \times I \) be a compact, non-orientable \( k + 1 \) dimensional manifold with boundary. Define \( f : \partial W \to S^k \) by \( f = f_0 \) on \( X \times \{0\} \) and \( f = f_1 \) on \( X \times \{1\} \). Then by the non-orientable extension theorem, \( f \) extends to map on all of \( W \) if and only if \( \deg_2 f = 0 \). Such an extension is equivalent to a homotopy of \( f_0 \) and \( f_1 \). Further, since \( \partial W = (X \times \{1\}) \cup -(X \times \{0\}) \), we have \( \deg_2 f = \deg_2 f_1 - \deg_2 f_0 \), so \( \deg_2 f = 0 \) if and only if \( \deg_2 f_0 = \deg_2 f_1 \). This completes the proof. \( \square \)