Math 210B: Algebra, Homework 5

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February 9, 2014

Problem 1.
Show that as submodule of a cyclic module over a PID is also cyclic.

Solution.
Let $M$ be a cyclic module, so that $\varphi : R \to M$ is a surjection under $\varphi(r) = r \cdot m$ for some fixed $M$. Therefore $M$ is isomorphic to $R/\ker \varphi$, i.e. a quotient of $R$ by some ideal. Further, any submodule of $M$ is isomorphic to an ideal of $R/\ker \varphi$. Since $R/\ker \varphi$ is a PID also, any submodule of it is generated by one element. Hence any submodule $N \subset M$ is cyclic.

Problem 2.
Let $a$ and $b$ be nonzero elements of a PID $R$. Prove that $R/aR \oplus R/bR \cong R/cR \oplus R/dR$, where $c$ is the least common multiple and $d$ is the greatest common divisor of $a$ and $b$.

Solution.
Recall that we may decompose these cyclic modules in terms of their elementary divisors. Let $p_1, \ldots, p_n$ be all prime elements dividing $a$ or $b$. Then we may write

$$a = \prod_{i=1}^{n} p_i^{a_i}, \quad b = \prod_{i=1}^{n} p_i^{b_i}$$

where $a_i, b_i \geq 0$. Then

$$R/aR \oplus R/bR \cong \bigoplus_{i=1}^{n} \left( R/p_i^{a_i}R \oplus R/p_i^{b_i}R \right).$$

For each prime $p_i$, we may choose $c_i = \max\{a_i, b_i\}$ and $d_i = \min\{a_i, b_i\}$. Then let $c = \prod p_i^{c_i}$ and $d = \prod p_i^{d_i}$. By construction, $c$ is the least common multiple of $a, b$ and $d$ is the greatest common divisor. Finally, by rearrangement, we have

$$\bigoplus_{i=1}^{n} \left( R/p_i^{a_i}R \oplus R/p_i^{b_i}R \right) = \bigoplus_{i=1}^{n} \left( R/p_i^{c_i}R \oplus R/p_i^{d_i}R \right) \cong R/cR \oplus R/dR,$$

since we may recombine the appropriate coprime ideals as we separated them above. This completes the proof.
Problem 3.
Find the invariant factors of the factor group \( \mathbb{Z}^3/N \), where \( N \) is generated by \((-4, 4, 2), (16, -4, -8), \) and \((8, 4, 2)\).

Solution.
We can recover the invariant factors of this group from reducing the following matrix:
\[
\begin{pmatrix}
-4 & 16 & 8 \\
4 & -4 & 4 \\
2 & -8 & 2
\end{pmatrix}.
\]
Eliminating the first row and column, we obtain
\[
\begin{pmatrix}
12 & 0 & 0 \\
0 & 4 & -4 \\
0 & 2 & -8
\end{pmatrix}.
\]
Continuing, we obtain
\[
\begin{pmatrix}
12 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 4
\end{pmatrix}.
\]
Fiddling around to obtain the appropriate diagonal, we obtain
\[
\begin{pmatrix}
12 & 0 & 0 \\
0 & 12 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]
This yields the factorisation
\[
G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}.
\]

Problem 4.
Find the canonical form of the linear operator in \( \mathbb{R}^3 \) given by the matrix
\[
\begin{pmatrix}
-2 & 0 & 0 \\
-1 & -4 & -1 \\
2 & 4 & 0
\end{pmatrix}.
\]

Solution.
It is easy verified that the characteristic polynomial \( p_A \) of the given matrix is \((X + 2)^3\) via direct computation from the determinant. First, we need to calculate the minimal polynomial. We see that \( A + 2I \neq 0 \), but \((A + 2I)^2 = 0\). Hence the minimal polynomial of \( A \) is \((X + 2)^2\), so the invariant factors are \((X + 2), (X + 2)^2\). Hence the rational canonical form of \( A \) is
\[
\begin{pmatrix}
-2 & 0 & 0 \\
0 & 0 & -4 \\
0 & 1 & -4
\end{pmatrix}.
\]
Problem 5.
Find the Jordan canonical form of the linear operator in \( \mathbb{C}^2 \) given by the matrix
\[
\begin{pmatrix}
1 & -1 \\
1 & 3
\end{pmatrix}.
\]

Solution.
The characteristic polynomial here is \( p_A(X) = (X - 2)^2 \). Since \( A - 2 \neq 0 \), the minimal polynomial is \( (X - 2)^2 \) itself. Therefore the Jordan block of the eigenvalue 2 has dimension 2. Therefore the Jordan canonical form is
\[
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}.
\]

Problem 6.
Find the invariant factors of the zero operator in a vector space of dimension \( n \).

Solution.
The matrix of this operator in any basis of \( V \) is the zero matrix. Therefore the characteristic polynomial of the zero operator is \( X^n \). The minimal polynomial of the zero operator is just \( X \), since \( A = 0 \). Therefore since all invariant factors divide the minimal polynomial, we have the invariant factors of the zero operator are just \( n \) copies of \( X \). 

Problem 7.
Prove that an \( n \times n \) matrix \( A \) is similar to a diagonal matrix if and only if the elementary divisors of \( A \) are all linear.

Solution.
First, suppose all the elementary divisors of \( A \) are linear. Then the minimal polynomial \( p_A \) factors as
\[
p_A(X) = \prod_{i=1}^{n} (X - \lambda_i)
\]
The elementary divisor decomposition of \( A \) is therefore the diagonal matrix \( \text{diag}(\lambda_1, \ldots, \lambda_n) \).

Now suppose that \( A \) is has a nonlinear elementary divisor. It suffices to show this case for \( A \) has only one nonlinear elementary divisor, and we may further assume it is of degree 2. Then let that elementary divisor be \( X^2 - aX - b \). Suppose \( A \) were similar to a diagonal matrix \( \text{diag}(\lambda, \mu) \). Then we would need
\[
(X - \lambda)(X - \mu) = X^2 - aX - b.
\]
If \( \lambda \neq \mu \), then the diagonal matrix evidently has two elementary divisors, which is a contradiction. Therefore the diagonal matrix is \( \text{diag}(\lambda, \lambda) = \lambda I \). But the minimal polynomial of \( \lambda I \) is \( X - \lambda I \), which is linear, and the minimal polynomial of \( A \) is quadratic. Therefore \( A \) cannot be diagonalisable. Through generalisation, this shows that any nonlinear elementary divisor makes \( A \) undiagonalisable, which completes the proof.
Problem 8.
Show that the minimal polynomial of an \( n \times n \) matrix \( A \) has the same irreducible divisors as the characteristic polynomial of \( A \).

Solution.
Let \( f_1, \ldots, f_s \) be the invariant factors of \( A \), so \( f_s \) is the minimal polynomial. Then
\[
p_A = \prod_{i=1}^{s} f_i.
\]
Suppose that \( g \) is an irreducible polynomial such that \( g \mid p_A \). Then \( g \mid f_j \) for some \( j \in \{1, \ldots, s\} \) since irreducible polynomials are the prime elements of \( F[X] \). Since \( f_j \mid f_s \) for every choice of \( j \), we also have \( g \mid f_s \). This completes the proof. \( \square \)

Problem 9.
Let \( A \) be a nilpotent \( n \times n \) matrix. Show that the invariant factors of \( A \) are the powers of \( X \). Prove that \( A^n = 0 \).

Solution.
If \( A \) is a nilpotent matrix, then \( A^N = 0 \) for some \( N \in \mathbb{N} \). Assume that \( m \) is the minimal such \( N \), i.e. \( A^{m-1} \neq 0 \) but \( A^m = 0 \). Therefore the minimal polynomial of \( A \) is \( X^m \). Since all invariant factors divide the minimal polynomial, they are all powers of \( X \). Finally, since the minimal polynomial has degree at most \( n \), \( n - m \geq 0 \), so we have
\[
A^n = A^m \cdot A^{n-m} = 0 \cdot A^{n-m} = 0.
\]
\( \square \)

Problem 10.
Prove that an \( n \times n \) matrix is similar to its transpose \( A^t \).

Solution.
First, recall that matrices are similar if and only if they have the same elementary divisors. Therefore, we need only prove this theorem in the case where \( A \) has only one elementary divisor, since two block diagonal matrices are similar if and only if each block is similar up to some permutation.

Therefore let \( p_A = f^m \) for some irreducible polynomial \( f \), where \( p_A \) is also the minimal polynomial of \( A \). Then \( p_{A^t} = p_A \) since
\[
p_A = \det(X \cdot I - A) = \det(X \cdot I^t - A^t) = p_{A^t}.
\]
Further, let the minimal polynomial of \( A^t \) be \( g \mid p_{A^t} = p_A \). Then
\[
0 = g(A') = \sum_{i=1}^{m} a_i(A')^i = \sum_{i=1}^{m} a_i(A^t)^{i \text{ transpose}} \rightarrow 0 = \sum_{i=1}^{m} a_i A^i = g(A).
\]
Therefore \( g = p_A \) since \( p_A \) was the minimal polynomial of \( A \). Hence, in our case, \( A \) and \( A^t \) have only one invariant factor, namely \( p_A \) itself, so they have the same rational canonical form, so they are similar. Applying this to finitely many elementary divisors of any matrix, we conclude that a matrix is similar to its transpose. \( \square \)