Problem 1.
Let $H$ be a $p$-subgroup of a finite group $G$. Show that if $H$ is not a Sylow $p$-subgroup, then $N_G(H) \neq H$.

Solution.
Since $H$ is a $p$-group, it is contained in some Sylow $p$-subgroup $P \subset G$. Since $P$ is a solvable group, then by in class lemma proper subgroups of $P$ satisfy $H \subset N_P(H)$. Since $N_P(H) \subset N_G(H)$, we have $H \subset N_G(H)$ as required.

Problem 2.
Let $G$ be a $p$-group and let $k$ be a divisor of $|G|$. Prove that $G$ contains a normal subgroup of order $k$.

Solution.
We proceed by induction. Let $|G| = p^n$. Since every divisor of $p^n$ is $p^k$ for some $0 \leq k \leq n$, so we replace $k$ by $p^k$. By earlier arguments, a $p$-group has a nontrivial centre. Since $Z(G)$ is itself a $p$-group, it contains an element of order $p$. Therefore let $P_1$ be a group of order $p$ contained in $Z(G)$, so that $P_1 \vartriangleleft G$.

Now assume that $G$ contains a normal subgroup of order $p^k$, $k < n$, say $P_k$. Then $G/P_k$ is itself a $p$-group, so it contains a normal subgroup of order $p$, say $P_{k+1}$. By earlier considerations, since $\pi : G \to G/P_k$ is a surjection and $P_{k+1} \triangleleft G/P_k$, $\pi^{-1}(P_{k+1}) \triangleleft G$. Further, $|\pi^{-1}(P_{k+1})| = p^{k+1}$. This completes the proof.

Problem 3.
Prove that if $G$ contains a subgroup $H$ of finite index, then $G$ contains a normal subgroup $N$ of finite index such that $N \subset H$.

Solution.
Let $|G : H| = n < \infty$. Taking the hint, examine the action of $G$ onto the finite set of $n$ cosets $aH$ by left translation. This induces a homomorphism $\varphi : G \to S_n$. By earlier considerations, $\ker \varphi = N$ is a normal subgroup of $G$. We claim that it has finite index. By the first isomorphism theorem, $G/N \cong K$ for some subgroup $K \subset S_n$. Thus $|G/N| < \infty$ since $S_n$ is a finite group. But $|G : N| = |G/N| < \infty$, thus $N \triangleleft G$ is a normal subgroup of finite index. Finally, if $n \in N = \ker \varphi$, then in particular $n \cdot H = H$, where $H$ is a coset. Therefore $n \in H$. Hence $N \subset H$, so we are done.
Problem 4.
Let $G$ be a $p$-group and $H$ a normal subgroup in $G$ of order $p$. Show that $H \subset Z(G)$.

Solution.
If $H$ is normal in $G$, then we may write $H = \bigcup C_i$, where $C_i$ is a conjugacy class in $G$. By the class equation, the nontrivial conjugacy classes of $G$ have orders divisible by $p$. Since $\{e\} \subset H$ is always a conjugacy class, there cannot exist a conjugacy class of order $\geq p$ in $H$. Hence all conjugacy classes contained in $H$ are trivial, i.e. $gxg^{-1} = x$ for all $x \in H$, $g \in G$. Therefore $H \subset Z(G)$.

Problem 5.
(a) A subgroup $H \subset G$ is called characteristic if $f(H) = H$ for every automorphism $f$ of $G$. Show that a characteristic subgroup $H$ is normal in $G$.

(b) Prove that if $K$ is a characteristic subgroup of $H$ and $H$ is a characteristic subgroup of $G$, then $K$ is characteristic in $G$.

Solution.
(a) By earlier considerations, we know that conjugation by any element is an automorphism of $G$. Since $H$ is characteristic, $gHg^{-1} = H$ for all $g \in G$. Hence $H$ is normal in $G$.

(b) Let $\varphi$ be an automorphism of $G$. Then since $\varphi(H) = H$, $\varphi|_H$ as an automorphism of $H$. Therefore since $\varphi|_H(K) = K$, we have $\varphi(K) = K$ as well.

Problem 6.
For a group $G$, let $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geq 0$. Show that $G^{(i)}$ is a characteristic subgroup of $G$.

Solution.
We prove this by induction. First, it is clear that $G$ is a characteristic subgroup of $G$. Now assume that $G^{(i)}$ is characteristic. Let $\varphi$ be an automorphism of $G$ and let $ghg^{-1}h^{-1} \in G^{(i+1)}$, where $g, h \in G^{(i)}$. Then

$$\varphi(ghg^{-1}h^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1} \in G^{(i+1)}$$

because $\varphi$ preserves $G^{(i)}$. Therefore $\varphi(G^{(i+1)}) \subset G^{(i+1)}$, and since $\varphi$ is a bijection, we are done.

Problem 7.
(a) Show that if $K$ and $H$ are normal in $G$, then so is $[K, H]$.

(b) Prove that $[G, H]$ is normal in $G$ for every subgroup $H \subset G$.

Solution.
(a) We know \([K, H]\) is the subgroup generated by \(\{khk^{-1}h^{-1} : k \in K, h \in H\}\). We will use 9 to prove this. Let \(g \in G\). Then

\[
gkhk^{-1}h^{-1}g^{-1} = (gk^{-1})(ghg^{-1})(gk^{-1}g^{-1})(gh^{-1}g^{-1}) = k' h' k'^{-1} h'^{-1},
\]

where \(k' = kgh^{-1} \in K\) and \(h' = ghg^{-1} \in H\) and \((gk^{-1})^{-1} = gk^{-1}g^{-1}\) as claimed. Therefore since \(gSg^{-1} \subset S\) for all \(g \in G\), the subgroup generated by it is normal.

(b) We use 9 here as well. Then if \(ghg^{-1}h^{-1}\) is a generator of \([G, H]\) and \(k \in G\),

\[
khg^{-1}h^{-1}k^{-1} = kghg^{-1}(k^{-1}h^{-1}hk)h^{-1}k^{-1} = kgh(kg)^{-1}h^{-1} \cdot hhk^{-1}k^{-1}.
\]

The first term is in \([G, H]\) and the second term is in the subgroup generated by \([G, H]\), since it is the inverse of \(khk^{-1}h^{-1} \in [G, H]\). By the reasoning of 9, the conjugate of a product is the product of the conjugates. Since each conjugate of a generator is still contained in the subgroup generated by \([G, H]\), the conjugate of each product remains in the subgroup generated by \([G, H]\). Therefore this subgroup is normal in \(G\).

\(\square\)

**Problem 8.**
Let \(G\) be a group and \(Z(G)\) the centre of \(G\). Show that if \(G/Z(G)\) is nilpotent, then so is \(G\).

**Solution.**
Let \(G/Z(G) = G_0 \subset \ldots \subset G_n = \{eZ(G)\}\) be central series, so that \(G_{i+1} \triangleleft G_i\) and \(G_i/G_{i+1} \subset Z(G_0/G_{i+1})\), i.e. \([G_0, G_i] \subset G_{i+1}\). By the correspondence theorem (proved in the last homework), normal subgroups of \(G/Z(G)\) are in bijective correspondence to those of \(G\) containing \(Z(G)\). Let \(G'_1\) be the normal subgroup corresponding to \(G_1\). Then \(G = G'_0 > G'_1\).

\(\square\)

**Problem 9.**
Assume that a subset \(S \subset G\) of a group satisfies \(gSg^{-1} \subset S\) for all \(g \in G\). Prove that the subgroup generated by \(S\) is normal in \(G\).

**Solution.**
Let \(h = s_1 \cdots s_n\) be an element of \(\langle S \rangle\), where \(s_i\) or \(s_i^{-1} \in S\). Then

\[
ghg^{-1} = gs_1 \cdots s_ng^{-1} = (gs_1g^{-1})(gs_2g^{-1}) \cdots (gs_ng^{-1}).
\]

Since \(gs_ig^{-1} \in S\) for each \(s_i \in S\) and \((gs_i^{-1})^{-1} = gs_i^{-1}g^{-1} \in S\) for each \(s_i^{-1} \in S\), \(ghg^{-1} \in \langle S \rangle\). Hence \(g\langle S \rangle g^{-1} \subset \langle S \rangle\), so \(\langle S \rangle\) is normal.

\(\square\)

**Problem 10.**
Let \(N\) be an abelian normal subgroup in a finite group \(G\). Assume that the orders \(|G/N|\) and \(|\text{Aut}(N)|\) are relatively prime. Prove that \(N \subset Z(G)\).
Solution.
Consider the group action from $G/N$ on $N$ by conjugation. This induces a group homomorphism $\varphi : G/N \to \text{Aut} N$ given by $gN \mapsto c_g$, the automorphism given by conjugation by $g$. This is well defined because, if $gN = g'N$, then suppose $g = g'n$. Then for any $h \in N$,
\[
ghg^{-1} = g'n h (g'n)^{-1} = g'n h n'^{-1} g' = g'h g'^{-1}
\]
since $N$ is abelian.

Since $\text{im} \varphi$ is a subgroup of $\text{Aut} N$, $|\text{im} \varphi|$ divides $|\text{Aut}(N)|$. Additionally, since $G$ is finite, $|G/N| = |\text{ker} \varphi| \cdot |\text{im} \varphi|$. Therefore $|\text{im} \varphi|$ divides $|G/N|$ as well, hence $|\text{im} \varphi|$ divides the gcd of $|G/N|$ and $|\text{Aut}(N)|$, which is 1. Therefore $\text{im} \varphi = \{e\}$, so $c_g$ is the trivial automorphism for every $gN \in G/N$, i.e. $N$ commutes with $g$ for all $gN$. Therefore since $N$ commutes with itself and with $G/N$, $N \subset Z(G)$. This completes the proof. \qed