1. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{4^{n+1}}$.

(A) $\frac{1}{12}$  
(B) $\frac{1}{9}$  
(C) $\frac{1}{6}$  
(D) $\frac{4}{3}$  
(E) $\frac{1}{3}$

2. A rectangle has dimensions $a$ units by $b$ units with $a > b$. A diagonal divides the rectangle into two triangles. A square, with sides parallel to those of the rectangle, is inscribed in each triangle. Find the distance between the vertices (of the squares) that lie in the interior of the rectangle.

(A) $\frac{(a - b) \sqrt{a^2 + b^2}}{a + b}$  
(B) $\sqrt{a^2 - b^2}$  
(C) $\frac{a^2 - b^2}{\sqrt{ab}}$  
(D) $\frac{a^2 - b^2}{\sqrt{a^2 + b^2}}$  
(E) $\frac{(a - b) \sqrt{a^2 - b^2}}{a + b}$
3. Find the index of the subgroup generated by the permutation
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 1 & 2 & 4 & 5
\end{pmatrix}
\]
in the alternating group \(A_5\).

(A) 20  \quad (D) 3

(B) 40  \quad (E) 5

(C) 24

4. In a pure radioactive decay model, the rate of change in the mass, \(M\), satisfies the differential equation, \(dM/dt = -M/10\). If the initial value of \(M\) is \(M_0\), find \(M\) in terms of \(M_0\) after 20 units of time, \(t\), have elapsed.

(A) \(\frac{1}{2} M_0\)

(B) \(\frac{1}{4} M_0\)

(C) \(\frac{M_0}{2e}\)

(D) \(\frac{M_0}{e}\)

(E) \(\frac{M_0}{e^2}\)

6. Let the set \(S\) be infinite and let the set \(T\) be countably infinite. Let \(\overline{S}\) denote the complement of \(S\). If \(S\) and \(T\) are both subsets of the real numbers, which of the following pairs of sets must be of the same cardinality?

(A) \(T\) and \(\overline{S} \cap T\)

(B) \(S\) and \(S \cup T\)

(C) \(T\) and \(\overline{S} \cup T\)

(D) Both (A) and (B)

(E) Both (A) and (C)

7. Find the maximum value of \(f(x) = 5 \sin 7x + 12 \cos 7x\).

(A) 12

(B) 5

(C) 7

(D) 17

(E) 13

8. In a computer program, separate loops with distinct indices produce \(M\) and \(N\) operations respectively. If these reside internally in a loop with an independent index producing \(P\)
operations, find the total number of operations represented by the three loops.

(A) \( p^{k+N} \)  
(B) \( (M + N)^p \)  
(C) \( P(M + N) \)

11. Define \( S_k = \{ \frac{k}{j} \mid j = k, k + 1, \ldots \} \) for \( k = 1, 2, \ldots \). Find the intersection \( \bigcap_{k=1}^\infty S_k \).

(A) \( S_1 \)  
(B) the empty set  
(C) the interval \([0, 1]\)

12. Evaluate \( \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{n(1 + \frac{k}{n})} \).

(A) 1  
(B) \( \ln 2 \)  
(C) 0

13. Evaluate the definite integral \( \int_0^a \frac{x^2 + b^2}{x^2 + a^2} \, dx \).

(A) \( \frac{(4 - \pi) a + \pi b^2}{4a} \)  
(B) \( 4a^2 + \pi (b^2 - a^2) \)  
(C) \( \frac{(2 - \pi) a + \pi b^2}{2a} \)
14. Let \( f(x, y) = x^3 - axy + y^2 - x \). Find the greatest lower bound for \( a \) so that \( f(x, y) \) has a relative minimum point.

(A) 0  (D) 6
(B) \( \sqrt{48} \)  (E) Does not exist
(C) 12

17. Define a function, \( f(x) \), to be differentiably redundant of order \( n \) if the \( n \)-th derivative \( f^{(n)}(x) = f(x) \) but \( f^{(k)}(x) \neq f(x) \) when \( k < n \). For easy examples, in this context, \( e^x \) is of order 1, \( e^{-x} \) is of order 2, and \( \cos x \) is of order 4. Which of the following functions is differentiably redundant of order 6?

(A) \( e^{-x} + e^{\frac{x}{2}} \cos \left( \frac{\sqrt{3}x}{2} \right) \)
(B) \( e^{-x} + \cos x \)
(C) \( e^{\frac{x}{2}} \sin \left( \frac{\sqrt{3}x}{2} \right) \)
(D) Both (A) and (C)
(E) (A), (B) and (C)

15. Given that \( S \) and \( T \) are subspaces of a vector space, which of the following is also a subspace?

(A) \( S \cap T \)  (D) Both (A) and (C)
(B) \( S \cup T \)  (E) Both (B) and (C)
(C) 2S

18. In a homogeneous system of 5 linear equations in 7 unknowns, the rank of the coefficient matrix is 4. The maximum number of independent solution vectors is

(A) 5  (D) 1
(B) 2  (E) 3
(C) 4

19. The number, up to isomorphism, of abelian groups of order 40 is

(A) 40  (B) 20
20. From the set of integers \( \{1, 2, \ldots, 9\} \), how many nonempty subsets sum to an even integer?
   
   (A) 512
   (B) 255
   (C) \( \frac{9!}{2!} \)
   (D) None of these
   
21. How many topologies are possible on a set of 2 points?
   
   (A) 5
   (B) 4
   (C) 3
   (D) 2
   (E) 1

22. In the \((\varepsilon, \delta)\) definition of the limit, \( \lim_{x \to c} f(x) = L \), let \( f(x) = x^3 + 3x^2 - x + 1 \) and let \( c = 2 \). Find the least upper bound on \( \delta \) so that \( f(x) \) is bounded within \( \varepsilon \) of \( L \) for all sufficiently small \( \varepsilon > 0 \).
   
   (A) \( \frac{\varepsilon}{8} \)
   (B) \( \frac{\varepsilon}{23} \)
   (C) \( \frac{(\varepsilon^2)}{4} \)
   (D) \( \frac{(\varepsilon^2)}{16} \)
   (E) \( \frac{\varepsilon}{19} \)

23. Find the remainder on dividing \( 3^{20} \) by 7.
   
   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) 5

24. In the partial fractions expansion of \( \frac{s^2 + 1}{(s^2 - 2)(s^2 + 3)} \), the numerator of the fraction with denominator \( s^2 + 3 \) is
   
   (A) \( 2s + 1 \)
   (B) \( \frac{3}{5} \)
   (C) \( \frac{2}{5} \)
   (D) \( \frac{1 - 2s}{5} \)
   (E) None of these

25. For how many distinct real coefficients, \( a \), will the system of equations \( y = ax^2 + 2 \) and \( x = ay^2 + 2 \) admit a solution with \( y = 1 \)?
   
   (A) 0
   (B) 1
   (C) 2
   (D) 3
   (E) 4
20. A subgroup \( H \) in group \( G \) is invariant if \( gH = Hg \) for every \( g \) in \( G \). If \( H \) and \( K \) are both invariant subgroups of \( G \), which of the following is also an invariant subgroup?

(A) \( H \cap K \)  (D) Both (A) and (B)
(B) \( HK \)  (E) Both (B) and (C)
(C) \( H \cup K \)

27. Let \( R \) be a ring such that \( x^2 = x \) for each \( x \in R \). Which of the following must be true?

(A) \( x = -x \) for all \( x \in R \)
(B) \( R \) is commutative
(C) \( xy + yx = 0 \) for all \( x, y \in R \)
(D) Both (A) and (C)
(E) (A), (B), and (C)

28. In the integral domain \( D = \{ r + s\sqrt{17} \mid r, s \text{ integers} \} \), which of the following is irreducible?

(A) \( 8 + 2\sqrt{17} \)  (D) \( 7 + \sqrt{17} \)
(B) \( 3 - \sqrt{17} \)  (E) \( 13 + \sqrt{17} \)
(C) \( 9 - 2\sqrt{17} \)

29. For which value of \( k \) is \( x^4 \) a solution for the differential equation \( x^2y'' - 3xy' - 4y = 0 \)?

(A) 4  (D) 1
(B) 3  (E) None of these
(C) 2

30. Which of the following is a factor of \( x^4 + 1 \)?

(A) \( x + 1 \)  (D) \( x^2 + x - 1 \)
(B) \( x^2 + 1 \)  (E) None of these
(C) \( x^2 - \sqrt{2}x + 1 \)

31. The surface given by \( z = x^2 - y^2 \) is cut by the plane given by \( y = 3x \), producing a curve in the plane. Find the slope of this curve at the point \( (1, 3, -8) \).

(A) 3  (D) 0
(B) \(-16\)  (E) \( \frac{18}{\sqrt{10}} \)
(C) \(-8\sqrt{\frac{2}{5}}\)
32. If $A$ is an $n \times n$ matrix with diagonal entries, $a$, and other entries, $b$, then one eigenvalue of $A$ is $a - b$. Find another eigenvalue of $A$.

(A) $b - a$  (D) 0
(B) $nb + a - b$  (E) None of these
(C) $nb - a + b$

33. In ancient Egypt, the formula $A = \left(\frac{8d}{9}\right)^2$ was used for the area of a circle of diameter $d$. Using the correct multiple, relating the volume of a sphere to the area of a circle, what should the Egyptian formula be for the volume of the sphere of diameter $d$?

(A) $\frac{\pi d^3}{3}$  (D) $\frac{2^8 \pi d^3}{3^5}$
(B) $\left(\frac{8d}{9}\right)^3$  (E) $\frac{2^7 d^3}{3^5}$
(C) $4\left(\frac{8}{9}d\right)^3$

34. Find $k$ so that the matrix

$$A = \begin{pmatrix} k & 1 & 2 \\ 1 & 2 & k \\ 1 & 2 & 3 \end{pmatrix}$$

has eigenvalue $\lambda = 1$.

(A) $\frac{1}{2}$  (B) $-\frac{1}{2}$

35. Express $z^{14}$ in the form $a + bi$ if $z = \frac{1+i}{\sqrt{2}}$.

(A) $-i$  (D) $i$
(B) $-1$  (E) $\frac{1+i}{128}$
(C) 1

36. Which number is nearest a solution of $e^{\frac{x}{100}} - e^{-x} = e^{-1}$?

(A) $1 - \frac{1}{2} \ln 4$  (D) $-1$
(B) $1 - \frac{1}{2} \ln 3$  (E) 0
(C) $1 - \frac{1}{2} \ln 2$

37. In the Laurent series for $f(z) = \frac{1}{(z-4)}$ centered at $z = 1$, the coefficient of $(z-1)^{-2}$ is

(A) 9  (D) 3
(B) $-9$  (E) $-1$
(C) $-3$
38. If \( f(x) = \int_1^3 \frac{dt}{1 + t^3} \) then \( f'(2) \) is

(A) \( \frac{4}{65} \)  
(B) \( \frac{1}{9} \)  
(C) \( \ln \left( \frac{65}{2} \right) \)

39. The series \( \sum_{n=2}^{\infty} \frac{1}{n \cdot 3^n} \) must

(A) converge to a value greater than 1/4  
(B) converge to a value greater than 1/9  
(C) converge to a value less than 1/18  
(D) converge to a value less than 1/12  
(E) diverge

41. In the finite field, \( \mathbb{Z}_p \), the multiplicative inverse of 10 is

(A) 13  
(B) 12  
(C) 11  
(D) 9  
(E) 7

42. The system of equations \( x^2 - y = a \) and \( y - x = b \) has exactly one value of \( x \) in its solution(s). This value of \( x \) must be

(A) 0  
(B) 1  
(C) \( \frac{3}{2} \)  
(D) \( -\frac{1}{2} \)  
(E) \( \frac{1}{2} \)

43. \( \log_4 64 \) is identical to

(A) \( \log_4 343 \)  
(B) \( \frac{\log_{10} 64}{\log_{10} 4} \)  
(C) \( \log_2 256 \)  
(D) Both (A) and (B)  
(E) Both (A) and (C)
44. Let \( S = \{ \tan(k) \mid k = 1, 2, \ldots \} \). Find the set of limit points of \( S \) on the real line.

(A) \((-\infty, \infty)\)  (D) \((-\infty, 0]\)

(B) \([-\frac{\pi}{2}, \frac{\pi}{2}]\)  (E) The empty set

(C) \([0, \infty)\)

47. Which of the following is a solution to the differential equation \( y'' = 2y' + 8y \)?

(A) \(e^{2x} - e^{-2x}\)  (D) \(e^{2x} - e^{-2x}\)

(B) \(xe^{4x}\)  (E) \(xe^{-2x}\)

(C) \(e^{2x}\)

45. Given the sequence \( a_n = \sin n, n \geq 1 \), find \(\lim_{n \to \infty} \inf\{a_n\} - \lim_{n \to \infty} \sup\{a_n\}\).

(A) 2  (D) -1

(B) 1  (E) -2

(C) 0

48. Let \( f(x) \) be defined and strictly increasing on \((a, b)\). Find the maximum value of \( g(x) = -f(x) \) on \((a, b)\).

(A) \(g(b)\)  (D) \(f(a)\)

(B) \(f(b)\)  (E) Does not exist

(C) \(g(a)\)

46. Find the length of a diagonal of a regular pentagon of side length 1.

(A) \(2 \cos \frac{\pi}{5}\)  (D) \(\sqrt{2}\)

(B) \(\sqrt{2} \left(1 + \cos \left(\frac{\pi}{5}\right)\right)\)  (E) \(\sqrt{2} + \sqrt{2\cos \left(\frac{2\pi}{5}\right)}\)

(C) \(\sqrt{2} \left(1 + \cos \left(\frac{2\pi}{5}\right)\right)\)

49. Find the number of distinguishable permutations of six colored blocks if one is red, two are yellow, and three are blue.

(A) 360  (D) 120

(B) 60  (E) 240

(C) 720
50. Two vertices of an isosceles triangle are (1,2) and (4,6). The inradius of the triangle is 3/2. Find the maximum possible area for the triangle.

(A) $\frac{45}{4}$  
(B) $\frac{9\sqrt{19}}{4}$  
(C) $\frac{45}{2}$  
(D) $\frac{9\sqrt{19}}{2}$  
(E) None of these

53. A family of curves is represented by the differential equation $ydx - xdy = 0$. Which of the following best describes the family of orthogonal trajectories to this given family?

(A) all parabolas with vertex at (0, 0)  
(B) all hyperbolas with centers at (0, 0)  
(C) all lines through (0, 0)  
(D) all circles with centers at (0, 0)  
(E) all lines parallel to the y-axis

51. The volume, $V$, of the region in space bounded above by the surface $x^2 + y^2 + z^2 = 4$ and below by $z = -\sqrt{x^2 + y^2}$ is represented by a triple integral in spherical coordinates as

$$\iiint_V \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$  
\text{find the upper limit of integration for } \phi .

(A) $\pi$  
(B) $\frac{3\pi}{4}$  
(C) $\frac{\pi}{2}$  
(D) $\frac{\pi}{4}$  
(E) $\frac{\pi}{6}$

54. Which of the following sequences is a solution for the difference equation $x_{n+1} + 36 = x_{n+1} + 6x_n$?

(A) $3^n + 6n$  
(B) $2^n + 6n + 1$  
(C) $6^n + 6n$  
(D) $3^n + 6 + n$  
(E) $(-1)^n 2^{n+1} + 6n + 1$

52. The reliability of component $C$ is $R$. A system is designed as a series of $n$ subcomponents each of which is doubly redundant using two components, $C$. Find the reliability of the system.

(A) $R^n(1 - R)^n$  
(B) $R^n(2 - R)^n$  
(C) $R^n(1 - R)^n$

55. $R^{2^n}$  
(D) $\frac{R^n}{2^n}$  
(E) $1 - (1 - R)^n$
55. Let $f$ be a mapping from a topological space $\overline{X}$ onto itself. Which of the following is true for continuous $f$?

(A) Every open set in $\overline{X}$ is the image of an open set in $\overline{X}$

(B) $f^{-1}(B)$ is bounded for each bounded set $B$ in $\overline{X}$

(C) $f$ is one-to-one

(D) Both (A) and (B)

(E) Both (A) and (C)

56. At the point $(2, -1, 2)$ on the surface $z = xy^2$, find a direction vector for the greatest rate of decrease of $z$.

(A) $\hat{i} - 2\hat{j}$

(D) $-\hat{i} + 4\hat{j}$

(B) $\hat{i} - 4\hat{j}$

(E) $\hat{i} + \hat{j}$

(C) $\frac{(\hat{i} - 4\hat{j})}{\sqrt{17}}$

57. Let $G$ be a polyhedron with 27 vertices and 40 edges. Find the number of faces on $G$.

(A) 12

(D) 15

(B) 13

(E) Cannot be decided based on the given information

(C) 14

58. Find the coefficient of $x^2y^3$ in the binomial expansion $(x - 2y)^5$.

(A) $-160$

(D) 8

(B) 80

(E) $-8$

(C) $-80$

59. In the evaluation of the integral $\int \frac{dx}{x(2 + 3x)^4}$, the coefficient of $\ln(2 + 3x)$ is

(A) $-\frac{3}{4}$

(D) $\frac{1}{4}$

(B) $-\frac{1}{4}$

(E) $-\frac{1}{36}$

(C) $\frac{3}{4}$

60. A biased coin is tossed repeatedly until the first "tail" occurs. The expected number of tosses required to produce the first tail is estimated as $T$. Assuming this is true, find the probability of at least two tails in $3T$ tosses.

(A) $\frac{T^{3T} - (T - 1)^{3T-1}(4T)}{T^{3T}}$

(B) $\frac{T^{3T} - (T - 1)^{3T-1}(3T)}{T^{3T}}$

(C) $\frac{T^{3T} - (T - 1)^{3T-1}(3T - 1)}{T^{3T}}$
61. The vertices of a quadrilateral are (0, 0), (1, 4), (3, 2), and (5, 5). Find the first coordinate of the centroid of the region.

(A) \( \frac{9}{4} \)

(B) \( \frac{13}{6} \)

(C) \( \frac{7}{3} \)

(D) \( \frac{17}{6} \)

(E) None of these

63. If \( f(x) = \ln x \), find

\[
\lim_{m \to 0} \left[ \lim_{n \to 0} \frac{f(2 + m + n) - f(2 + m) - f(2 + n) + f(2)}{mn} \right]
\]

(A) \( \frac{1}{4} \)

(B) 1

(C) \( -1 \)

(D) \( \frac{1}{2} \)

(E) \( -\frac{1}{4} \)

64. Let \( S = \{ x_1, x_2, \ldots, x_n, \ldots \} \) be a topological space where the open sets are \( U_n = \{ x_1, \ldots, x_n \}, n = 1, 2, \ldots \). Let \( E = \{ x_2, x_3, \ldots, x_n, \ldots \} \). Find the set of cluster points of \( E \).

(A) \( S \setminus \{ x_1 \} \)

(B) \( \{ x_1 \} \)

(C) \( \{ x_2 \} \)

(D) \( E \setminus \{ x_2 \} \)

(E) \( S \setminus E \)

65. A stiff beam on two supports that are 20 ft. apart is loaded by two uniform blocks with dimensions and weights as shown below.

![Diagram of a beam with two supports and blocks loaded on it.]

How much of the total weight is supported at the left support?
66. Given that $\sum_{n=1}^{\infty} a_n$ converges to $L$, which conclusion is valid for $\sum_{n=1}^{\infty} a_n^2$?

(A) It may diverge

(B) It converges absolutely

(C) It converges to $M < L$

(D) It converges to $M > L$

(E) It converges to $L^3$