SOLUTIONS TO MATH GRE FORM GR0568

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The questions for this solution guide can be found here.

Solution 1. (B) The equation for parametrised arc length is

$$s = \int_a^b \sqrt{f_x^2 + f_y^2}\, dt.$$

Here, $f_x = -\sin t$ and $f_y = \cos t$, so that $f_x^2 + f_y^2 = 1$. This makes the integral

$$\int_0^\pi 1\, dt = \pi.$$

Solution 2. (E) Very straightforward. $y' = 1 + e^x$, so $y'(0) = 2$. $y(0) = 1$ so all in all we get $y - 1 = 2(x - 0)$. Rearranging gives the answer.

Solution 3. (D) We know that $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$. We know that $\dim(U + V)$ is 2, 3, or 4. That makes $\dim(U \cap V)$ either 0, 1, or 2.

Solution 4. (B) $e^x + x$ is an increasing function, so we will have $e^x + x = 2$ exactly once. Letting $f(x) = e^x + x$, we have $f(0) = 1$ and $f(1) = e + 1 > 2$. By the Intermediate Value Theorem, we get our single solution somewhere in $[0, 1]$.

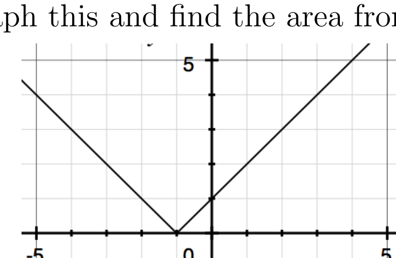
Solution 5. (B) It looks like we have a root at $x = 2$. This means that

$$0 = 3 \cdot 4 + 2b + 12 \implies b = -12.$$

Put another way, $f(x) = 3(x - 2)^2$. Then $f(5) = 3 \cdot 9 = 27$.

Solution 6. (C) We can try to do this algebraically, but non-algebraically is simpler. Graphing $y = x^2 - 4$ shows that the graph crosses the x-axis at ± 2. Therefore a circle of radius 1 or $\sqrt{2}$ will not intersect the parabola at all. A circle of radius 3 will intersect four times – twice above and twice below the x-axis. A circle of radius 4 will only intersect at one point below the x-axis (and twice above), and a circle of radius 5 will only intersect at the two points above.

Solution 7. (C) Better to graph this and find the area from the triangles.



On the left, we have a triangle of length and height 2. On the right, we have a triangle of length and height 4. That gives us a total area of $2 + 8 = 10$.

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Solution 8. (A) There would be two ways to approach this problem: either you know what kind of triangle is going to maximise the area ahead of time or you need to do some optimisation. Let us assume we don't know what we want. Let α denote the angle measure of the triangle at the vertex of the centre of the circle. We know that the base of the triangle is 1, the radius of the circle, and its height is determined by α. Specifically, its height is $\sin\alpha$. Therefore the area of the triangle is $(\sin\alpha)/2$. Since $\sin\alpha$ takes its maximum when $\alpha = \pi/2$ (i.e. the triangle is a right triangle), that gives our answer.

Solution 9. (A) We know that, for $x \in [0, 1]$,

$$\sqrt{1-x^4} \leq \sqrt{1-x^8} \leq 1 \leq \sqrt{1+x^4}.$$

Taking integrals will preserve these inequalities, giving us $J < L < 1 < K$.

Solution 10. (B) If the above is the graph of g', then we can order values for $g(x)$ by doing some graphical integration. Starting from $x = 0$, the most positive area is attained when we reach $x = 2$, so $g(2)$ should be the right answer.

Solution 11. (E) 1.5 is nearly 1.44, so $\sqrt{1.44} = 1.2$. 266 is approximately 256, so $\sqrt{256} = 16$, and $16^3 = 4096$. We know we are looking for a number larger than this, which leaves only 5300.

Alternatively, square the whole thing to get $1.5 \cdot (266)^3$. We can remove one of the factors of 266 and multiply it by the 1.5 to obtain $399 \cdot (266)^2$. Square rooting this expression gives approximately $20 \cdot 266$, which is again close to 5300.

Solution 12. (C) This condition makes the matrix of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

There is no reason that $a = 0$ or $b = 0$, so there is no reason $(1, 0)$ or $(0, 1)$ should be eigenvectors. But it is easy to verify that $(1, 1)$ must be.

Solution 13. (B) Let h be the height of the fence and ℓ the length. Then we want to maximise $A(h, \ell) = h \cdot \ell$ under $2h + \ell = x$. This gives us $\ell = x - 2h$, so $A(h) = h(x - 2h)$. $A'(h) = x - 4h$, which gives us a maximum at $h = x/4$. This makes $\ell = x/2$, so the area all told is $x^2/8$.

Solution 14. (D) There is a pattern in the units digits:

$$7^1 = 7, \quad 7^2 = 49, \quad 7^3 = 343, \quad 7^4 = 2401, \quad 7^5 = \cdots 7.$$

You only need only know the units digit for all of those, but the point is that the units digit of 7^n is determined by the equivalence class of $n \mod 4$. Since $25 \mod 4 = 1$, the units digit should be 7.

Solution 15. (E) Continuous functions on compact intervals are very nice indeed, but they don't have to be differentiable. (E) says that $f'(0)$ exists, when it needn't by any means. For instance, let $f(x) = |x|$.

Solution 16. (D) The volume of this solid is given by

$$\int_0^\infty \pi \cdot f(x)^2 \, dx.$$

The calculation is fairly straightforward:

$$\pi \int_0^{\infty} \frac{dx}{1+x^2} = \pi \cdot \lim_{R\to\infty} \arctan x \Big|_0^R = \pi \cdot \lim_{R\to\infty} \arctan R = \frac{\pi \cdot \pi}{2}$$

Solution 17. (B) An odd degree polynomial has an odd number of real roots, giving us only 1, 3, or 5. Rather than try to find the roots directly, let us look at $f'(x) = 10x^4 + 8$. This is strictly positive, so $f(x)$ is a strictly increasing function. Therefore it could only have one root.

Solution 18. (A) We see that $\dim V = 6$ and $\dim W = 4$. Since $\dim \operatorname{im} T = \dim W = 4$, we must have $\dim \ker T = 6 - 4 = 2$.

Solution 19. (C) There is no reason that $f(x) > g(x)$, or that $f''(x) > g''(x)$. But we do know that

$$\int_0^x f'(t)\,dt > \int_0^x g'(t)\,dt \implies f(x) - f(0) > g(x) - g(0).$$

This is precisely an answer.

Solution 20. (D) f is continuous at 0, but it won't be continuous anywhere else. The difference between $x/2$ and $x/3$ is always substantial enough for discontinuities when $x \neq 0$.

Solution 21. (C) If m, n are two numbers which differ by one prime factor, then $P_m \cap P_n$ will have some element in common. Then we can see that $12 \cdot 5 = 20 \cdot 3$, so $P_{12} \cap P_{20} \neq \varnothing$.

Solution 22. (B) A subspace must be closed under addition, scalar multiplication, and contain the zero element. Right away, the function $0(x)$ does not satisfy the premise of III, so that is out. We then need only to check II given the answers that are left. Since the derivative is linear, we should have no problem. Let $f(x), g(x)$ satisfy the condition of II. Then

$$(f+g)'' = f'' + g'' = 3f' + 3g' = 3(f' + g') = 3(f+g)'$$

Scalar multiplication follows easily as well.

Solution 23. (A) There are two parts to this question. First, we must get $y = 10x$ to intersect the graph of $y = e^{bx}$. Second, it must be tangent at that point. That means that we have (x_0, y_0) such that $10x_0 = e^{bx_0}$ and $10 = be^{bx_0}$. A little bit of algebra shows that $x_0 = 1/b$ and so $b = 10/e$.

Solution 24. (E) We can actually just integrate this, and not worry about differentiation under the integral.

$$\int_0^{x^2} e^{x+t}\,dt = e^x \int_0^{x^2} e^t\,dt = e^x(e^{x^2} - 1) = e^{x^2+x} - e^x.$$

Then deriving that,

$$h'(x) = (2x+1)e^{x^2+x} - e^x,$$

whence our result follows immediately.

Solution 25. (A) Let us write down some of the elements.

$$a_1 = 1, \quad a_2 = \frac{3}{1}, \quad a_3 = \frac{4}{2} \cdot \frac{3}{1}, \quad a_4 = \frac{5}{3} \cdot \frac{4}{2} \cdot \frac{3}{1}$$

Given the cancellation we are seeing, $a_n = (n+1)(n)/2$. That makes $a_{30} = (31)(30)/2$.

Solution 26. (A) We are concerned about its extrema, we should find some partial derivatives.

$$f_x = 2x - 2y, \quad f_y = -2x + 3y^2.$$

We would like to know when they are both zero. The first equation gives us $x = y$ and the second gives us $2x = 3y^2$, so that

$$2y = 3y^2 \implies (3y-2)y = 0 \implies y = 0, 2/3.$$

Therefore our solutions are $(0,0)$ and $(2/3, 2/3)$. Indeed, our relative extrema are all on the line $x = y$. To do some more checking (which you should not do on the actual test),

$$f_{xx} = 2, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = -2.$$

Then the determinant of the Hessian is $12y - 4$. This shows that $(0,0)$ is a saddle point. There is no reason that $(2/3, 2/3)$ is an absolute minimum without further verification, and $(1,1)$ needn't be an extreme point.

Solution 27. (D) First, we know that the intersection of two planes in $\mathbb{R}^3$ should be either a plane or a line. In our case, the two planes are definitely not the same, so we will obtain a line. The slope of the line can be found by taking the cross product of the normal vectors of the two planes in question.

$$(1,3,-2) \times (2,1,-3) = \det \begin{bmatrix} i & j & k \\ 1 & 3 & -2 \\ 2 & 1 & -3 \end{bmatrix} = (-7,-1,-5).$$

The only solution corresponding to this slope is (D), as the coefficients of t in (x,y,z) are $(7,1,5)$.

Solution 28. (D) The easiest way to do this is to play around, I think. One, two, or three edges don't seem to be enough, but removing four edges gives us the straight graph on six vertices.

Solution 29. (C) We know that continuous functions commute with limits. That makes (A) and (B) true. The relation is clearly symmetric as well, and we can verify that

$$\lim_{x\to\infty} \frac{f(x)+g(x)}{2g(x)} = \frac{1}{2}\left(\lim_{x\to\infty}\frac{f(x)}{g(x)} + \lim_{x\to\infty}\frac{g(x)}{g(x)}\right) = 1.$$

That leaves only (C). This makes sense: suppose that $f(x) = x$ and $g(x) = x - 1$. Then clearly $f \sim g$, but

$$\frac{e^x}{e^{x-1}} = e.$$

Therefore $e^f \not\sim e^g$.

Solution 30. (C) To write things out, P is the definition of a function, Q is onto, and R is the negation of one-to-one. This makes the answer easy. The negation of 'both one-to-one and onto' is 'either not one-to-one or not onto', i.e. not Q or R.

Solution 31. (A) The slope of the graph is always positive, and $y'(0) = 1$. Moreover, as y tends to $\pm\infty$, y' gets very large. There is only one graph fitting that description.

Solution 32. (D) The basic description resembles a (noncommutative) ring, though lacking identity elements or the distributive property. Since the multiplication isn't commutative, I needn't hold. II is true because addition is commutative. III will also hold by induction.

Solution 33. (D) The first value for r is $273 \mod 110$, which is 53. The second value for r is $110 \mod 53$, which is 4. This gives us a solution uniquely.

Solution 34. (E) The minimal distance will be on a straight line between the centres of the spheres. Specifically, it is that distance minus the distances of the two radii. It is straightforward to calculate that the distance between $(2, 1, 3)$ and $(-3, 2, 4)$ is $3\sqrt{3}$. We then must subtract the final $2 + 1$ from the radii.

Solution 35. (E) There are 15! total arrangements of all people in the row of chairs. To ensure that all men are sitting together, we can treat the 15 people as 9 women and 1 clump of men. There are 10! arrangements of these entities, and then a further 6! arrangements of men within the clump. That gives us our answer.

Solution 36. (A) Most of these are criteria for invertibility. Unfortunately, pairwise linear independence of the column vectors is not one of them. Take, for example,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Pairwise each of these are linearly independent, but the fifth column is the sum of the first four.

Solution 37. (D) Let $z = a + bi$. Then

$$z^2 = a^2 - b^2 + 2abi, \quad |z|^2 = a^2 + b^2.$$

To have an equality between these two, we need $a = 0$ or $b = 0$ for the imaginary part, and more specifically we need $b = 0$ for the real part. Therefore the graph is precisely the real axis, so a line.

Solution 38. (A) Neither of the equalities should hold – these are in fact nonsense statements, as one side lies in A and the other in B. To unravel the remaining two sets,

$$f^{-1}(f(C)) = \{x \in A : f(x) \in f(C)\}, \quad f(f^{-1}(D)) = f\left(\{y \in A : f(y) \in D\}\right)$$

Clearly the second set must always be contained in D, but not the other way around. Similarly the first set certainly contains all $c \in C$ (as $f(c) \in f(C)$) but not the other way around.

Solution 39. (B) By the law of cosines, we know that

$$s^2 = r^2 + 1 - 2r\cos 110^\circ \implies s = \sqrt{r^2 - 2r\cos 110^\circ + 1}$$

We can use this substitution for the limit.

$$\begin{aligned} \lim_{s,r\to\infty}(s - r) &= \lim_{r\to\infty} \sqrt{r^2 - 2r\cos 110^\circ + 1} - r \\ &= \lim_{r\to\infty} \frac{(r^2 - 2r\cos 110^\circ + 1) - r^2}{\sqrt{r^2 - 2r\cos 110^\circ + 1} + r} \\ &= \lim_{r\to\infty} \frac{1 - 2r\cos 110^\circ}{\sqrt{r^2 - 2r\cos 110^\circ + 1} + r} \end{aligned}$$

Dividing everything through by r on top and bottom,

$$= \lim_{r\to\infty} \frac{1/r - 2\cos 110^\circ}{\sqrt{1-(2\cos 110^\circ)/r + 1/r^2} + 1}$$
$$= \frac{-2\cos 110^\circ}{2} = -\cos 110^\circ = \cos 70^\circ.$$

This gives us a positive number less than 1.

Alternatively, we can draw the height of this triangle and so split it into two different triangles. We can call the obtuse angle the vertical makes with the hypotenuse θ. Then as $s, r \to \infty$, $\theta \to 90^\circ$, as a limit larger than 90° would imply that the lengths of s, r are finite. Hence the smaller lefthand triangle, which includes $s - r$ as a side, becomes a right triangle with hypotenuse 1. Thus the side $s - r$ must be strictly less than 1, but positive.

Solution 40. (C) Each of these is an integral domain except for (C), which is also the most exotic. We can take two continuous functions which are supported on disjoint subsets of $[0, 1]$. Each function will not be identically zero, but their product will be.

Solution 41. (E) This is a classic Green's theorem problem.

$$\oint_{\partial D} L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy.$$

In our case,

$$\oint_C (2x-y)\,dx + (x+3y)\,dy = \iint_D (1+1)\,dx\,dy = 2A,$$

where A is the area of the unit circle, i.e. π.

Solution 42. (B) The probability that $X > 3$ is equal to $1 - P(X = 1) - P(X = 2) - P(X = 3)$, which is easier to calculate. Specifically, $P(X > 3) = 1 - 1/2 - 1/4 - 1/8 = 1/8$. Since $P(Y > 3) = 1/8$ as well, we just need to use that

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

When A and B are independent, $P(A \text{ and } B) = P(A) \cdot P(B)$. Therefore we get $1/8 + 1/8 - 1/64$.

Solution 43. (E) We can use that $z^5 = 1$ to make this a bit easier. That replacement makes our sum

$$1 + z + z^2 + z^3 + 5z^4 + 4 + 4z + 4z^2 + 4z^3 + 5z^4 = 5(1 + z + z^2 + z^3 + z^4) + 5z^4.$$

That sum in parentheses is zero, as it is the sum of all the fifth roots of unity. Using the final fact that $e^{\pi i} = -1$, we get $5e^{8\pi i/5} = 5 \cdot -e^{3\pi i/5}$.

Solution 44. (D) We can do some quick calculations to find $P(H = n)$ for various n. This is done with the binomial theorem:

$$P(H = n) = \binom{100}{n} \cdot (1/2)^n \cdot (1/2)^{100-n} = \binom{100}{n} \cdot (1/2)^{100}.$$

But this is probably not the most helpful way to think about this. The highest probability is going to be around the centre of the distribution ($n = 50$). However, it is also better to take a wider range, such as (D), which is $n = 48, 49, 50, 51, 52$. It might be tempting to take a larger region of the distribution, like $n < 40$, but these will not amount to as much.

Solution 45. (D) This is a pigeonhole principle problem. Since there are 21 points and 5 sectors, one sector must contain at least 5 points. There is no reason that any sector must have any points, however. But III holds for the same reason that I does.

Solution 46. (E) $G \cong \mathbb{Z}/4$, and is generated by $i \in G$ multiplicatively. I and II are both multiplicative functions, so they are perfectly fine. Furthermore, any endomorphism $f : G \to G$ is completely determined by $f(i)$. $f(i)$ is always of the form i^k for some k (as that is how G is generated). Therefore III is true as well. Complex conjugation may not look like that, but it is the case when $k = 3$.

Solution 47. (C) The formula for this is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot \vec{r}'(t)\, dt$$

For us, we have $\vec{r}'(t) = (1, 2t, 3t^2)$, and $\vec{F}(r(t)) = (-1, 0, 1)$ as it is constant. Thus

$$\int_0^1 -1 + 3t^2\, dt = (-t + t^3)\Big|_0^1 = 0.$$

Solution 48. (A) The only issue here seems to be that (4) implies that $f(x)$ gets very large so long as $f'(c_1)$ is positive. But we know that it is, since f is a strictly increasing function. Therefore everything is satisfactory.

Solution 49. (D) How many abelian groups of order 16 are there anyway? By the classification theorem, they are the products of $\mathbb{Z}/2^n$. Since the elements of our group must have order 4, we must take products of $\mathbb{Z}/4$ and $\mathbb{Z}/2$ only. We may either pick two copies of $\mathbb{Z}/4$, one copy, or none. That gives three different groups.

Solution 50. (B) There is no reason that all the entires of A^2 need to be nonnegative. Its determinant must be nonnegative though: $\det(A^2) = (\det A)^2$. For III, suppose A is the diagonal matrix with entires $\pm\lambda$. Then those are its eigenvalues, and they are distinct so long as $\lambda \neq 0$. But A^2 has only one eigenvalue: λ^2.

Solution 51. (B) This ridiculous integral is expressed better as a sum. For $n \leq x < n + 1$, $\lfloor x \rfloor = n$. Therefore

$$\int_0^\infty \lfloor x \rfloor e^{-x}\, dx = \sum_{n=0}^\infty \int_n^{n+1} \lfloor x \rfloor e^{-x}\, dx = \sum_{n=0}^\infty \int_n^{n+1} n \cdot e^{-x}\, dx.$$

Now we can do some calculations.

$$\int_n^{n+1} n \cdot e^{-x}\, dx = n \cdot (-e^{-x})\Big|_n^{n+1} = n \cdot (e^{-n} - e^{-(n+1)}) = ne^{-n} - ne^{-(n+1)}.$$

This gives us something resembling a telescoping series. The total coefficient of e^{-n} in the final sum is $-(n-1) + n = 1$. Therefore

$$\sum_{n=1}^\infty ne^{-n} - ne^{-(n+1)} = \sum_{n=1}^\infty e^{-n} = \frac{1}{1 - e^{-1}} = \frac{1}{e - 1}$$

by the general formula for an infinite geometric series.

Solution 52. (B) We know that $\mathbb{Q} \subset \mathbb{R}$ is dense, i.e. $\overline{\mathbb{Q}} = \mathbb{R}$. Therefore if $\mathbb{Q} \subset A$ is closed, then $\overline{\mathbb{Q}} = \mathbb{R} \subset A$, so $A = \mathbb{R}$.

Solution 53. (C) This looks like a 3-variable problem, but y is totally irrelevant. Therefore we might as well consider $x^2 + z^2 \leq 2$ and try to minimise $x + 4z$. We can also safely assume our minimum is on the boundary, so let $x^2 + z^2 = 2$. Our constraint gives us $x = \sqrt{2 - z^2}$ and so we would like to minimise $f(z) = \sqrt{2 - z^2} + 4z$. Then we must solve

$$f'(z) = \frac{-z}{\sqrt{2 - z^2}} + 4 = 0 \implies z = 4\sqrt{2 - z^2}$$

$$\implies z^2 = 32 - 16z^2 \implies z = \pm\sqrt{\frac{32}{17}} = \pm 4\sqrt{\frac{2}{17}}.$$

We will take the negative value. Since $z^2 = 32/17$, that makes $x^2 = 2/17$ so $x = -\sqrt{2/17}$. Putting it all together,

$$x + 4z = -\sqrt{\frac{2}{17}} - 4 \cdot 4\sqrt{\frac{2}{17}} = -17\sqrt{\frac{2}{17}} = -\sqrt{2 \cdot 17} = -\sqrt{34}.$$

Solution 54. (E) We can work out what the radius of the smaller circles in the first figure should be. Let R be the radius of one of the smallest circles. Then the total diameter of the large circle is R plus the diagonal of a square of side length $2R$ plus another R. In total, that makes it $(2 + 2\sqrt{2})R$, so the radius of the larger circle is $(1 + \sqrt{2})R$.

The total area of the smaller circles is $16\pi R^2$. The total area of the larger circles is $4\pi \cdot ((1 + \sqrt{2})R)^2$. Hence the ratio is

$$\frac{16\pi R^2}{4\pi \cdot ((1 + \sqrt{2})R)^2} = \frac{4R^2}{(1 + \sqrt{2})^2 R^2} = \left(\frac{2}{1 + \sqrt{2}}\right)^2.$$

Solution 55. (D) The number of zeroes that $k!$ ends in is determined by the number of factors of 2 and 5 in its prime factorisation. There will always be more factors of 2 than 5, so 5 is the limiting factor. We need exactly 99 factors of 5. We can start counting them, recalling that multiples of 25 count double and multiples of 125 count triple. The exact equation is

$$f(n) = \sum_{m \geq 0} \left\lfloor \frac{n}{5^m} \right\rfloor.$$

So we can play around with this to determine when we get $f(n) = 99$, which occurs at $n = 400$ once you do the math.

We can use powers of 5 to chunk this question up somewhat. We see that $f(5) = 1$, $f(25) = 6$, $f(125) = 31$ through brute force. By using chunks of 125, we obtain $f(250) = 62$ and $f(375) = 93$ because we have the same patterns every 125 (until we hit 625). Then the next chunk is $f(400)$, and we know that increasing by 25 (without hitting a factor of 125) increases the count by 6, hence $f(400) = 99$.

This is a multiple-choice type question, for sure. The answer was always going to be zero or five. For instance, there is no factorial that ends in exactly 5 zeroes, since we jump from 4 factors of 5 to 6 factors of 5 at 25!.

Solution 56. (E) The axioms of a metric space are that the metric is symmetric, nonnegative, $d(x, y) = 0$ if and only if $x = y$, and the triangle inequality – $d(x, y) + d(y, z) \geq d(x, z)$.

δ is the discrete metric, so no problem there. Everything with $|x - y|$ in it looks pretty much okay, so let us move to ω first. It satisfies the first three axioms, but does it satisfy the triangle inequality? Suppose that $x = 1$, $y = 0$, and $z = -1$. Then

$$\omega(1,0) + \omega(0,-1) = 2 < 4 = \omega(1,-1).$$

This violates the triangle inequality.

Solution 57. (E) We can use the root test to find the interval of convergence.

$$\sqrt[n]{\frac{n!x^{2n}}{n^n(1+x^{2n})}} = \sqrt[n]{\frac{n!}{n^n} \cdot \frac{x^{2n}}{1+x^{2n}}}.$$

We can use Stirling's approximation to show that

$$\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$$

Furthermore, the fraction $x^{2n}/(1+x^{2n})$ has value in $[0,1)$ for all $x \in \mathbb{R}$, so taking its nth root and the limit as $n \to \infty$ will yield either 0 or 1. Therefore the root test gives 0 or $1/e$ and shows that the sum converges for all $x \in \mathbb{R}$.

Solution 58. (E) Suppose that $A = PBP^{-1}$. $P(B-2I)P^{-1} = PBP^{-1} - 2PP^{-1} = A - 2I$, so I is true. II is true because similar matrices have the same eigenvalues. Finally, if $A = PBP^{-1}$, then $A^{-1} = PB^{-1}P^{-1}$.

Solution 59. (A) The Cauchy-Riemann equations tell us that $f_x = g_y$ and $f_y = -g_x$. Hence $g_y = 2$ and $g_x = -3$. That makes a likely candidate for $g(x,y) = 2y - 3x + C$. Since $g(2,3) = 1$, this gives $C = 1$. As such, $g(7,3) = 6 - 21 + 1 = -14$.

Solution 60. (E) A pentagram is just a twisted pentagon. If you like, the edges in a pentagram are the exact opposite choices of those on a pentagon. Therefore their symmetries are the same as well – the dihedral group of order 10.

Solution 61. (C) $\mathbb{R}$ is uncountable, the set of all functions $\mathbb{Z} \to \mathbb{Z}$ is uncountable (it has the same cardinality as $2^{\mathbb{Z}}$), and (D) and (E) are both uncountable as well. $2^{\mathbb{R}}$, the functions $\mathbb{R} \to \{0,1\}$, has a strictly higher cardinality than $\mathbb{R}$, making it the biggest.

Solution 62. (D) I is true because the image of a compact set under a continuous function is compact, hence closed and bounded in the Euclidean space $\mathbb{R}$. For III, there is no reason a compact space is connected – any finite (disjoint) union of compact spaces is compact.

II is a bit trickier. The statement '$f(K)$ is bounded for any continuous $f : \mathbb{R}^n \to \mathbb{R}$' is called pseudocompactness. For a metric space, this is equivalent to compactness. That compactness implies pseudocompactness is trivial, and the converse is proved using the characterisation of compactness as every sequence having a convergent subsequence. We will omit the proof.

Solution 63. (D) Attempting to take a derivative,

$$f'(x) = x \cdot (-2x + 2x^{-3})e^{-x^2-x^{-2}} + e^{-x^2-x^{-2}} = (1 - 2x^2 + 2x^{-2})e^{-x^2-x^{-2}}.$$

Since this righthand term is never going to be zero, we will have to focus on

$$1 - 2x^2 + 2x^{-2} = 0 \implies x^2 - 2x^4 + 2 = 0 \implies 2x^4 - x^2 - 2 = 0.$$

By the quadratic formula, we get

$$x^2 = \frac{1 \pm \sqrt{1+16}}{4} = \frac{1 \pm \sqrt{17}}{4}$$

One of these solutions, however, is negative, so we can't take its square root. That makes 2 real roots for our polynomial, hence two points of horizontal tangent line. However, we should see what the derivative looks like at $x = 0$, where this doesn't technically hold.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} e^{-x-x^{-2}} = 0.$$

That gives us three total points of horizontal tangent.

Solution 64. (D) This function certainly converges pointwise. The sequence converges to 0 at each point $x \in [0, 1)$ and converges to $1/2$ at $x = 1$. That limit function is not continuous, so f_n cannot converge uniformly to it. To check III, we recall the dominated convergence theorem: if we have an integrable function $g(x)$ on $[0, 1]$ such that $|f_n(x)| \le g(x)$ for all n and all x, and further that f_n converge to a limit function f pointwise, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

Since it is easy to dominate $f_n(x)$ (take $g(x) = 1/2$), III holds.

Solution 65. (B) We can define a surjection $(0, 1) \to [0, 1]$ by constructing a homeomorphism $f : [a, b] \to [0, 1]$ (for $0 < a < b < 1$) and letting $f(x) = 0$ for $x < a$ and $f(x) = 1$ for $x > b$. For $[0, 1] \to (0, 1)$, the image of a compact set is compact, but $(0, 1)$ is not compact. Finally, suppose there is a continuous bijection $f : (0, 1) \to [0, 1]$. Then let $f(x) = 1$. Since f is continuous, there are two intervals $[a, x]$ and $[x, b]$ that map to $[\alpha, 1]$ and $[\beta, 1]$ respectively. But this contradicts injectivity.

Solution 66. (B) First, for any ring R, both 0 and R itself are right ideals. Therefore these are the only ones. Hence the principal right ideal generated by any $0 \ne r \in R$ is all of R, so there would exist some $s \in R$ such that $rs = 1_R$. This makes every nonzero element invertible. However, there is no reason R should be commutative or infinite. For example, the quaternions $\mathbb{H}$ are a noncommutative division ring and $\mathbb{F}_2$ is a finite field (which only has two ideals).