ON THE ORDER OF $a$ MODULO $n$, ON AVERAGE

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ABSTRACT. Let $a > 1$ be an integer. Denote by $l_a(n)$ the multiplicative order of $a$ modulo integer $n \geq 1$. We prove that there is a positive constant $\delta$ such that if $x^{\delta} = o(y)$, then

$$\frac{1}{y} \sum_{y < n < x} \sum_{a < n < x, (a,n) = 1} l_a(n) = \frac{x}{\log x} \exp \left( B \frac{\log \log x}{\log \log \log x} (1 + o(1)) \right)$$

where

$$B = e^{-\gamma} \prod_p \left( 1 - \frac{1}{(p-1)^2(p+1)} \right).$$

It was known for $y = x$ in [KP, Page 3] in which they refer to [LS].

1. INTRODUCTION

Let $a > 1$ be an integer. If $n$ be coprime to $a$, we write $d = l_a(n)$ if $d$ is the multiplicative order of $a$ modulo $n$. Then $d$ is the smallest positive integer in the congruence $a^d \equiv 1 \pmod{n}$.

The Carmichael’s lambda function $\lambda(n)$ is defined by the exponent of the group $(\mathbb{Z}/n\mathbb{Z})^\times$. It was known in [EPS] that

$$\frac{1}{x} \sum_{n < x} \lambda(n) = \frac{x}{\log x} \exp \left( B \frac{\log \log x}{\log \log \log x} (1 + o(1)) \right).$$

Assuming GRH for Kummer extensions $\mathbb{Q}(\zeta_d, a^{1/d})$, P. Kurlberg and C. Pomerance [KP] showed that

$$\frac{1}{x} \sum_{n < x} l_a(n) = \frac{x}{\log x} \exp \left( B \frac{\log \log x}{\log \log \log x} (1 + o(1)) \right)$$

with $B = e^{-\gamma} \prod_p \left( 1 - \frac{1}{(p-1)^2(p+1)} \right)$. The upper bound implicit is unconditional because $l_a(n) \leq \lambda(n)$.

An unconditional average result over all possible nonzero residue classes is obtained by F. Luca and I. Shparlinski [LS]:

$$\frac{1}{x} \sum_{n < x} \frac{1}{\phi(n)} \sum_{a < n} l_a(n) = \frac{x}{\log x} \exp \left( B \frac{\log \log x}{\log \log \log x} (1 + o(1)) \right).$$

As pointed out in [KP], by partial summation, we have the following statistics on average order:

$$\frac{1}{x^2} \sum_{a < x} \sum_{a < n < x} l_a(n) = \frac{x}{\log x} \exp \left( B \frac{\log \log x}{\log \log \log x} (1 + o(1)) \right).$$

For fixed $a$, it seems that it is very difficult to remove GRH in P. Kurlberg and C. Pomerance’s result with current knowledge. However, we expect that averaging over $a$ would give some information. So, we take average over $a < y$, but we do not want to have too large $y$ such as $y > x$. For all the average results in this paper, we assume that $y < x$, and try to obtain $y$ as small as possible. By applying a deep result on exponential sums by Bourgain [B], we prove the unconditional average result on a shorter interval.
Theorem 1.1. There is a positive constant $\delta$ such that, if $x^{1-\delta} = o(y)$, then

$$\frac{1}{y} \sum_{a<y} \frac{1}{x} \sum_{a<n<x, (a,n)=1} l_a(n) = \frac{x}{\log x} \exp \left( B \frac{\log \log x}{\log \log \log x} (1 + o(1)) \right)$$

where

$$B = e^{-\gamma} \prod_p \left( 1 - \frac{1}{(p-1)^2(p+1)} \right).$$

2. Backgrounds

2.1. Equidistribution. A sequence $\{a_n\}$ of real numbers are said to be equidistributed modulo 1 if the following is satisfied:

Definition 2.1. Let $0 \leq a < b \leq 1$. Suppose that

$$\lim_{N \to \infty} \frac{1}{N} \left| \left\{ n \leq N : a_n \in (a,b) \mod 1 \right\} \right| = b - a.$$ 

Then we say that $\{a_n\}$ is equidistributed modulo 1.

A well-known criterion by Weyl [W] is

Theorem 2.1. For any integer $k \neq 0$, suppose that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} e^{2\pi i ka_n} = 0.$$ 

Then the sequence $\{a_n\}$ is equidistributed modulo 1.

There was a series of efforts to obtain a quantitative form of the equidistribution theorem. Erdős and Turán [ET] succeeded in obtaining the following result:

Theorem 2.2. Let $\{a_n\}$ be a sequence of real numbers. Then for some positive constants $c_1$ and $c_2$,

$$\sup_{0 \leq a < b \leq 1} \left| \left\{ n \leq N : a_n \in (a,b) \mod 1 \right\} \right| - (b-a)N \leq c_1 \frac{N}{M+1} + c_2 \sum_{m=1}^{M} \frac{1}{m} \left| \sum_{n \leq N} e^{2\pi i ma_n} \right|.$$ 

H. Montgomery [M] obtained $c_1 = 1$, $c_2 = 3$. C. Mauduit, J. Rivat, A. Sárközy [MRS] obtained $c_1 = c_2 = 1$. Thus, we have a quantitative upper bound of discrepancy when we have good upper bounds for exponential sums.

2.2. Exponential Sums in $\mathbb{Z}_n^*$. We define arithmetic functions $a_n(d)$ and $b_n(d)$ for $1 \leq d | \lambda(n)$ as follows:

$$a_n(d) = |\{0 < a < n : l_a(n) = d\}|,$$

$$b_n(d) = |\{0 < a < n : a^d \equiv 1 \mod n\}|.$$ 

Then

$$a_n(d) = \sum_{d'|d} \mu \left( \frac{d}{d'} \right) b_n(d').$$

We give some algebraic remarks about the function $b_n(d)$. First, we see that

$$H_{n,d} := \{0 < a < n : a^d \equiv 1 \mod n\}$$

forms a subgroup of $\mathbb{Z}_n^*$ of order $b_n(d)$. The following proposition is from elementary group theory:
Proposition 2.1. Let $H_{n,d}$ and $b_n(d)$ be defined as above. For any $k|n$, denote by $\pi_k$ the reduction modulo $n/k$. Then we have

$$\pi_k : H_{n,d} \rightarrow H_{n/k,d}$$

where $\pi_k$ is a group homomorphism with kernel

$$K = \{0 < a < n : a^d \equiv 1(n), a \equiv 1(n/k)\}.$$

By the First Isomorphism Theorem, we have

$$|K| = \frac{b_n(d)}{|\pi_k(H_{n,d})|} \leq k.$$ 

Note that the map $\pi_k$ restricted to $H_{n,d}$ is not always surjective. To see this, let $p > 2$ prime number, and $a = p + 1$, $d = p$, $n/k = p^2$, $n = p^3$. Then

$$a^p \equiv p^2 + 1 \pmod{p^3}.$$ 

Thus,

$$a^p \equiv 1 \pmod{p^2}.$$ 

But for any $a' \equiv a \pmod{p^2}$, so that $a' = p^2j + p + 1$ for some integer $j$, we have

$$(a')^p \equiv (p + 1)^p \pmod{p^3} \equiv p^2 + 1 \pmod{p^3}.$$ 

From this, we see that the element $a = p + 1 \in H_{n/k}$ is not a preimage of $\pi_k$. The proof of $|K| \leq k$ is clear by $a \equiv 1(n/k)$.

J. Bourgain [B] proved a nontrivial exponential sum result when a subgroup $H$ of $\mathbb{Z}_n^*$ has order greater than $n^\epsilon$ for $\epsilon > 0$.

Theorem 2.3. Let $n \geq 1$. For any $\epsilon > 0$, there exist a constant $\delta = \delta(\epsilon) > 0$ such that for any subgroup $H$ of $\mathbb{Z}_n^*$ with $|H| > n^\epsilon$,

$$\max_{(m,n)=1} \left| \sum_{a \in H} e^{2\pi i m n a} \right| < n^{-\delta}|H|.$$ 

Corollary 2.1. Let $\epsilon > 0$ be arbitrary, and let $y \geq 1$. Assume that $d|\lambda(n)$ and $b_n(d) > n^\epsilon$. Then there exists $\delta = \delta(\epsilon) > 0$ such that

$$\sum_{a < y, a^d \equiv 1(n)} 1 = \frac{y}{n} b_n(d) + O(b_n(d)n^{-\delta}).$$ 

If $d|\lambda(n)$, the congruence $a^d \equiv 1$ yields $b_n(d)$ roots in $\mathbb{Z}_n$. Thus, we need to count $a < y$ satisfying those $b_n(d)$ congruences modulo $n$. Considering $\frac{y}{n} = \left\lfloor \frac{y}{n} \right\rfloor + \frac{y}{n} - \left\lfloor \frac{y}{n} \right\rfloor$, it is enough to prove the result for $y < n$.

We apply the Erdős-Turán inequality to the set $\{ \frac{a}{n} : 0 < a < n, a^d \equiv 1(n) \}$. Then

$$\left| \{0 < a < n : a^d \equiv 1(n), \frac{a}{n} \in (0, \frac{y}{n} \pmod{1})\} - \frac{y}{n} b_n(d) \right| \leq \frac{b_n(d)}{n} + \sum_{m=1}^{n-1} \frac{1}{m} \sum_{a \in \mathbb{Z}_n, a^d \equiv 1(n)} e^{2\pi im \frac{a}{n}}.$$ 

Unlike the prime modulus case, we immediately encounter a problem. The exponential sum result (Theorem 2.3) is only for $(m,n) = 1$, but the sum takes all $1 \leq m < n$. Then we have too many terms with $(m,n) \neq 1$. Therefore, we need some modification in applying the Erdős-Turán inequality. A starting point is to observe that we can take $M$ arbitrary in the Erdős-Turán inequality.

Proof of Corollary 2.1)

Assuming that $k|n$ and $b_n(d) > n^\epsilon$, we have

$$n^\epsilon < b_n(d) \leq k|\pi_k(H_{n,d})|.$$ 

Then

$$\frac{n^\epsilon}{k} < |\pi_k(H_{n,d})|.$$ 

If we can assume that

$$\left( \frac{n}{k} \right)^{\epsilon'} < \frac{n^\epsilon}{k},$$ 

then

$$\frac{n^\epsilon}{k} < |\pi_k(H_{n,d})|.$$
for some positive \( \epsilon'' < \epsilon \), then we can use Theorem 2.3 with \( \epsilon'' \) and \( \delta'' = \delta(\epsilon'') \). This is achieved by

\[
k < n^{1-\epsilon''}.
\]

Let \( \epsilon' = \frac{\epsilon''}{1-\epsilon''} \) and we take \( M + 1 = \lfloor n^{\epsilon'} \rfloor \) in the Erdős-Turán inequality. Then we have reduced the number of terms appearing in the sum on the right side. We rewrite the sum by substituting \( (m,n) = k, \frac{m}{k} = j \) and apply Theorem 2.3 to the exponential sums inside. This is possible due to

\[
\left( \frac{n}{k} \right)^{\epsilon''} < |\pi_k(H_{n,d})|
\]

and \( \pi_k(H_{n,d}) \) being a subgroup of \( \mathbb{Z}_n^\times/k \). The sum on the right becomes

\[
\sum_{m<n'} \frac{1}{m} \sum_{a \in \mathbb{Z}_n} e^{2\pi ina/m} \leq \sum_{k|m} \frac{1}{k} \sum_{j=1}^{(k,\frac{n}{k})} \frac{1}{j} \sum_{a \in \mathbb{Z}_n} e^{2\pi ij \frac{n}{n/k}}
\]

\[
= \sum_{k|m} \frac{1}{k} \sum_{j=1}^{(k,\frac{n}{k})} \frac{1}{j} \left| \pi_k(H_{n,d}) \right| \sum_{a \in \pi_k(H_{n,d})} e^{2\pi ij \frac{n}{n/k}}
\]

\[
\leq \sum_{k|m} \frac{1}{k} \sum_{j=1}^{(k,\frac{n}{k})} \frac{1}{j} \left| \pi_k(H_{n,d}) \right| \left( \frac{n}{k} \right)^{-\delta''}
\]

\[
\leq n^{-\delta''(1-\epsilon')} b_n(d)(1 + \log n)^2.
\]

Thus, the Erdős-Turán inequality gives

\[
\left| \left\{ 0 < a < n : a^d \equiv 1(n), \frac{a}{n} \in (0, \frac{y}{n}) \mod 1 \right\} - \frac{y}{n} b_n(d) \right| \leq \frac{b_n(d)}{n^\epsilon'} + b_n(d)n^{-\delta''(1-\epsilon')}(1 + \log n)^2.
\]

Therefore we can take \( 0 < \delta < \min(\epsilon', \delta''(1 - \epsilon')) \). This completes the proof of Corollary 2.1.

Corollary 2.1 plays a key role in proving Theorem 1.1. Note that the upper bound provided in Corollary 2.1 is significantly better than the trivial bound which is:

\[
\sum_{a<y, \ a^d \equiv 1(n)} 1 = \frac{y}{n} b_n(d) + O(b_n(d)).
\]

### 3. PROOF OF THEOREMS

#### 3.1. Proof of Theorem 1.1

We start with the change of order in summation:

\[
\sum_{a<y} \sum_{n<x} l_a(n) = \sum_{d<x} d \sum_{n<x} \sum_{a<y} \sum_{a \equiv d \mod l(n)} 1
\]

\[
= \sum_{d<x} d \sum_{n<x} \sum_{a \equiv d \mod l(n)} \mu \left( \frac{d}{d'} \right) \sum_{a<y} \sum_{a \equiv d' \equiv 1(n)} 1
\]

\[
= \sum_{d<x} d \sum_{n<x} \sum_{a \equiv d \mod l(n)} \mu \left( \frac{d}{d'} \right) \left( \frac{y}{n} b_n(d') + O(b_n(d')) \right)
\]

\[
+ \sum_{d<x} d \sum_{n<x} \sum_{a \equiv d \mod l(n)} \mu \left( \frac{d}{d'} \right) \left( \frac{y}{n} b_n(d') + O(b_n(d')n^{-\delta}) \right)
\]

\[
= \sum_{d<x} d \sum_{n<x} \frac{y}{n} a_n(d) + O(E_1) + O(E_2),
\]

where \( E_1 \) and \( E_2 \) are given by...

...and so forth, completing the proof of Theorem 1.1.
where

$$E_1 = \sum_{d < x} d \sum_{n < x} \sum_{d' \mid d, \lambda(n)} b_n(d') \mu\left(\frac{d}{d'}\right) b_n(d')$$

$$\ll \sum_{d < x} d \sum_{n < x} d' \sum_{n' < x} d' \mid d, \lambda(n) b_n(d') \mu\left(\frac{d}{d'}\right) b_n(d')$$

$$= \sum_{d < x} d \sum_{n < x} d' \mid d, \lambda(n) \tau(d)$$

$$\ll x^{2+\epsilon+o(1)},$$

and

$$E_2 = \sum_{d < x} d \sum_{n < x} \sum_{d' \mid d, \lambda(n)} b_n(d') n^{-\delta}$$

$$\ll \sum_{d < x} d \sum_{n < x} d' \sum_{n' < x} d' \mid d, \lambda(n) b_n(d') n^{-\delta}$$

$$= \sum_{d < x} d \sum_{n < x} d' \mid d, \lambda(n) \tau(d) n^{-\delta}$$

$$\leq \sum_{n < x} n \tau(d) d^{-\delta}$$

$$\ll x^{3-\delta+o(1)}.$$

Now we treat the main term:

$$\sum_{d < x} d \sum_{n < x} \frac{1}{n} a_n(d) = \sum_{n < x} \frac{1}{n} \sum_{d \mid \lambda(n)} d a_n(d).$$

Taking $\delta$ to satisfy $2 + \epsilon \leq 3 - \delta$, we have

$$\sum_{a < y \leq n < x} a(y) = y \sum_{n < x} \frac{1}{n} \sum_{d \mid \lambda(n)} d a_n(d) + O(x^{3-\delta+o(1)}).$$

Let $u(n) = \frac{1}{\phi(n)} \sum_{d \mid \lambda(n)} d a_n(d)$ be the average multiplicative order of the elements of $(\mathbb{Z}/n\mathbb{Z})^*$. The following is proven in [LS, Theorem 6]:

**Theorem 3.1.**

$$\frac{1}{x} \sum_{n < x} u(n) = \frac{x \log \log x}{\log x} \exp\left(\frac{\log \log x}{\log \log \log x} (1 + o(1))\right).$$

What we have for the main term is the middle term in the following inequalities:

$$\frac{1}{\log \log x} \sum_{n < x} u(n) \ll \sum_{n < x} \frac{\phi(n)}{n} u(n) \leq \sum_{n < x} u(n).$$

Since $\log \log x = o\left(\frac{\log \log x}{\log \log \log x}\right)$, it follows that

$$\sum_{n < x} \frac{\phi(n)}{n} u(n) = \frac{x^2}{\log x} \exp\left(\frac{\log \log x}{\log \log \log x} (1 + o(1))\right).$$
Hence, we have

\[
\sum_{a<y} \sum_{n<x} l_a(n) = \frac{yx^2}{\log x} \exp \left( B \frac{\log \log x}{\log \log \log x} (1 + o(1)) \right) + O(x^{3-\delta+o(1)}).
\]

Moreover, if for some \(0 < \delta' < \delta\), and \(x^{1-\delta'} = o(y)\), then the error term can be included in the term with \(o(1)\). The terms that appear when \(n \leq a\), are also included in the term with \(o(1)\). This completes the proof of Theorem 1.1.

REFERENCES


