In the next homework: 15.2
Textbook problems: 5.4, 33, 5.5, 34, 5.5, 36, 35.2, 22, 5.2, 32, 5.2, 44, 5.3, 89, 60, 76

No extra problems.

Differentiability and Tangent Planes

$f(x,y)$ is differentiable if it is locally linear, i.e., behavior of $f$ around $P = (a, b, f(a,b))$ is like a linear function.

What's linear function?

In 2D: $y = ax + b$, a line
In 3D: $z = ax + by + c$, a plane.

Linearization at $(a, b)$

$L(x, y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$

$y = b; \quad z = L(x, b)$ is an equation of tangent line to the vertical trace $z = f(x, b)$ at $P$

$x = a; \quad z = L(a, y)$ \quad \therefore \quad z = f(a, y)$ \quad at $P$. 
Definition of Differentiability of $f(x,y)$

Assume that $f(x,y)$ is defined in a disk $D$ containing $(a,b)$ and that $f_x(a,b)$ and $f_y(a,b)$ exist.

- $f(x,y)$ is differentiable at $(a,b)$ if it is locally linear—that is if

$$f(x,y) = L(x,y) + e(x,y)$$

where $e(x,y)$ satisfies

$$\lim_{(x,y) \to (a,b)} \frac{e(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

- In this case, the tangent plane to the graph at $(a,b, f(a,b))$ is the plane with equation $z = L(x,y)$. Explicitly,

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Theorem 1: Criterion for Differentiability

If $f_x(x,y)$, $f_y(x,y)$ exist, and continuous on an open disk $D$, then $f$ is differentiable on $D$. Important assumption.
Check if the following functions are differentiable and find an equation of tangent planes at each point.

1. \( f(x, y) = 5x + 4y^2 \), \((a, b) = (2, 1)\)

2. \( h(x, y) = \sqrt{x^2 + y^2} \), \((a, b) = \sqrt{r} \neq (0, 0)\)
   shape: circular cone

3. \( f(x, y) = xy^3 + x^2 \) at \((2, -2)\)

In 2 at \((0, 0)\), the partial derivatives do not exist at \((0, 0)\).

Linear Approximation

\[ \Delta f = f(x, y) - f(a, b), \quad \Delta x = x - a, \quad \Delta y = y - b \]

\[ \Delta f \approx f_x(a, b) \Delta x + f_y(a, b) \Delta y \]

In terms of differentials

\[ df = f_x \, dx + f_y \, dy \]

\[ \Delta f \approx df \]
In 3 variables (or more)
\[ df = f_x \, dx + f_y \, dy + f_z \, dz \]

\textbf{Ex) Use the linear approximation to estimate}

\[ 3.99^3 \, 1.01^4 \, 1.98^{-1} \]

\textbf{Solution) Use } \( f(x,y,z) = x^3 \, y^4 \, z^{-1} \), and

\[ \Delta f \approx df = f_x \, dx + f_y \, dy + f_z \, dz \]

\[ \approx f_x \, \Delta x + f_y \, \Delta y + f_z \, \Delta z \]

\textbf{so that}

\[ \Delta f \approx f_x \, \Delta x + f_y \, \Delta y + f_z \, \Delta z \]

\[ A^+ (4,1,2), \quad f(4,1,2) = 4^3 \, 1^4 \, 2^{-1} = 32 \]

\[ f_x (x,y,z) = 3x^2 \, y^4 \, z^{-1}, \quad f_x (4,1,2) = 24 \]

\[ f_y (x,y,z) = 4x^3 \, y^3 \, z^{-1}, \quad f_y (4,1,2) = 128 \]

\[ f_z (x,y,z) = -x^2 \, y^4 \, z^{-2}, \quad f_z (4,1,2) = -16 \]

\[ \Delta x = -0.01, \quad \Delta y = 0.01, \quad \Delta z = -0.02 \]

\[ \Delta f \approx 24(-0.01) + 128(0.01) - 16(-0.02) \]

\[ = 1.36 \]

\[ \Delta f = f(3.99, 1.01, 1.98) - f(4,1,2) \]

\textbf{Therefore}

\[ f(3.99, 1.01, 1.98) \approx 33.36 \]