AN EXERCISE ON FABER POLYNOMIALS

KIM, SUNGJIN

Theorem 1. Let $M'_0 = \mathbb{C}[j]$ be a space of weakly holomorphic modular forms of weight 0. $M'_0$ has a canonical basis $\{j_m(z)\}_{m=0}^{\infty}$, where $j_m(z) = q^{-m} + O(q)$. Then, we have:

$$\sum_{m=0}^{\infty} j_m(\tau)q^m = \frac{E^2_4}{\Delta} \frac{1}{j(z) - j(\tau)}.$$  \hspace{1cm} (1)

Proof. Let $j(z) = q^{-1} + \sum_{m=0}^{\infty} b_m q^m$, and we have $\frac{E^2_4}{\Delta} = -\frac{1}{2\pi i} j'$. Also, we remark that $-\frac{1}{2\pi i} \frac{q^j}{q^j} j'$ has a convergent power series for $|q| < r$ for some $r > 0$. We prove the theorem by showing that both sides are equal in the formal power series ring $\mathbb{C}[[q]]$. For, we need:

$$\left(\sum_{m=0}^{\infty} j_m(\tau)q^m\right) q(j(z) - j(\tau)) = -\frac{1}{2\pi i} q^j. \hspace{1cm} (2)$$

After this, we obtain that $\sum_{m=0}^{\infty} j_m(\tau)q^m$ equals to $(-\frac{1}{2\pi i} q^j)[q(j(z) - j(\tau))^{-1}$ as an element in $\mathbb{C}[[q]]$, which gives Theorem 1. Now, we multiply two power series,

$$\left(\sum_{m=0}^{\infty} j_m(\tau)q^m\right) q(j(z) - j(\tau)) = 1 + \sum_{m=1}^{\infty} \left( j_{m+1}(\tau) + \sum_{m_1 + m_2 = m} j_{m_1}(\tau) b_{m_2} - j_m(\tau) j(\tau) \right) q^{m+1}. \hspace{1cm} (3)$$

We claim that:

$$j_{m+1}(\tau) + \sum_{m_1 + m_2 = m} j_{m_1}(\tau) b_{m_2} - j_m(\tau) j(\tau) = -mb_m. \hspace{1cm} (2)$$

Recall that $j_m$ is a polynomial of $j$ for each $m$, and by multiplying $q$-expansions of the last term, we obtain that the coefficients of negative powers of $q$ in LHS are all 0, thus we have the LHS of (2) is a holomorphic modular form of weight 0, which is a constant. Moreover, (2) is equivalent to $j_m(\tau) = q^{-m} + mbmq + O(q^2)$.

Let

$$j^m = q^{-m} + \sum_{0 \leq n \leq m+1} b_{m,-n} q^{-n} + b_{m,1} q + O(q^2). \hspace{1cm} (3)$$

Then, we have

$$j_m = j^m - \sum_{0 \leq n \leq m-1} b_{m,-n} j_n. \hspace{1cm} (4)$$

Our claim is obvious when $m = 1$, for $j_1 = j - b_0 = q^{-1} + b_1 q + O(q^2)$. Suppose that the claim is true for all $1 \leq m' < m$, i.e. $j_{m'}(z) = q^{-m'} + m'b_{m'}q + O(q^2)$ holds for $1 \leq m' < m$. Differentiating $j$ with respect to $q$ gives,

$$\frac{d}{dq} j = -q^{-2} + \sum_{n=1}^{\infty} nb_n q^{n-1}. \hspace{1cm} (5)$$
Comparing the coefficient of $q^{-1}$ in the series $j^m \frac{d}{dq} j = \frac{1}{m+1} \frac{d}{dq} q^{m+1}$ gives,

$$mb_m + \sum_{1 \leq n \leq m-1} nb_m = 0.$$  

In view of (4), we obtain $j_m = q^{-m} + mb_m q + O(q^2)$, which completes the proof of our claim by induction. Now, Theorem 1 follows by $-\frac{1}{2\pi i} qj' = 1 + \sum_{m=1}^{\infty} (-mb_m)q^{m+1}$. 