Isoperimetric Inequality. (E. Schmidt)

\[ l: \text{perimeter} \]
\[ A: \text{area,} \]
\[ A \leq \frac{l^2}{4\pi} \]

Assumptions:
1. The boundary is a simple closed curve.
2. The boundary is parametrized by
   \[ \Gamma(s) = (x(s), y(s)) \text{ for } [-\pi, \pi] \]
   where \( \Gamma \) is continuous, piecewise \( C^1 \).
3. \( \Gamma(s) \) is the arc-length parametrization
   so \( |\Gamma'(s)| = 1 \) for all \( s \).

Then \( l = 2\pi \). We prove \( A \leq \pi \).

Green's Theorem:

\[ \iint_\Omega \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \, dx \, dy = \oint_\gamma L \, dx + M \, dy \]

We have by taking \( M = x \),

\[ A = \int_\pi x \, dy = \int_{-\pi}^{\pi} x(s) y'(s) \, ds \]
Let \( x(s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos ns + b_n \sin ns) \)
(pointwise convergence holds)

Without loss of generality, by shifting \( x \)
if necessary, we can assume that \( a_0 = 0 \).

Similarly, \( y(s) = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos ns + d_n \sin ns) \)
and we can assume that \( c_0 = 0 \). Thus,

\[
x(s) = \sum_{n=1}^{\infty} (a_n \cos ns + b_n \sin ns)
\]

\[
y(s) = \sum_{n=1}^{\infty} (c_n \cos ns + d_n \sin ns)
\]

and pointwise convergence hold for both.

On the other hand, by integration by parts

\[
C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(s) \cos ns \, ds \\
= \frac{1}{\pi} \left[ y(s) \frac{\sin ns}{n} \right]_{-\pi}^{\pi} \quad - \quad \frac{1}{\pi} \int_{-\pi}^{\pi} y'(s) \frac{\sin ns}{n} \, ds
\]

\[
d_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(s) \sin ns \, ds \\
= \frac{1}{\pi} \left[ y(s) \frac{-\cos ns}{n} \right]_{-\pi}^{\pi} \quad + \quad \frac{1}{\pi} \int_{-\pi}^{\pi} y'(s) \frac{\cos ns}{n} \, ds
\]
Since \( y(\pi) = y(-\pi) \),

\[
c_n = -\frac{1}{\pi} \int_{-\pi}^{\pi} y'(s) \frac{\sin ns}{n} ds
\]

\[
d_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y'(s) \frac{\cos ns}{n} ds
\]

Then the Fourier series of \( y'(s) \) is

\[
y'(s) \sim \sum_{n=1}^{\infty} \left( n d_n \cos ns - n c_n \sin ns \right)
\]

\[
(\hat{y}(s)) = \sum_{n=1}^{\infty} (-n a_n)
\]

Note that \( y' \) is only assumed piecewise continuous. So, pointwise convergence is not guaranteed.

1. Since \( y' \) is piecewise continuous,
   it belongs to \( L^2([\pi, \pi]) \).
   We can use the theorems in \( L^2 \) theory such as Bessel's inequality, Parseval's theorem.

**Lemma**

For \( f, g \in L^2([\pi, \pi]) \),

\[
f = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

\[
g = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos nx + d_n \sin nx)
\]

Then \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) dx = \pi \left( \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n) \right) \)
proof. Use \( \langle f, g \rangle = \frac{1}{2} \left( \langle f+g, f+g \rangle - \langle f, f \rangle - \langle g, g \rangle \right) \)

and Parseval's theorem. In fact,

\[
\langle f+g, f+g \rangle = \pi \left( \frac{a_0^2 + c_0^2}{2} + \sum_{n=1}^{\infty} \left( (a_n^2 + c_n^2) + (b_n^2 + d_n^2) \right) \right)
\]

\[
\langle f, f \rangle = \pi \left( \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right)
\]

\[
\langle g, g \rangle = \pi \left( \frac{c_0^2}{2} + \sum_{n=1}^{\infty} (c_n^2 + d_n^2) \right)
\]

Therefore

\[
\langle f, g \rangle = \pi \left( \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n) \right)
\]

**Proposition** (A in terms of Fourier coefficients)

\[ A = \int_{-\pi}^{\pi} x(s) y'(s) \, ds = \pi \sum_{n=1}^{\infty} n (a_n d_n - b_n c_n) \]

By Assumption 3, \( x'(s)^2 + y'(s)^2 = 1 \) for all \( s \in [-\pi, \pi] \). Then

\[
\int_{-\pi}^{\pi} \left( x'(s)^2 + y'(s)^2 \right) \, ds = 2\pi
\]

Then by Lemma,

\[ \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2) = 2\pi \]
Now, \( n \leq n^2 \) (equality only for \( n = 1 \))
and \( \left| a_n d_n - b_n c_n \right| \leq \frac{a_n^2 + d_n^2}{2} + \frac{b_n^2 + c_n^2}{2} \)
by AM-GM inequality.

Then the area \( A \) in proposition 1 is bounded by

\[
A = \pi \sum_{n=1}^{\infty} n (a_n d_n - b_n c_n)
\]
\[
\leq \pi \sum_{n=1}^{\infty} n^2 \left( \frac{a_n^2 + d_n^2}{2} + \frac{b_n^2 + c_n^2}{2} \right) = \frac{2\pi}{2} = \pi.
\]

Therefore, we have proved the isoperimetric inequality 1.

(Equality) Let us find the condition for equality.

If \( \frac{a_n^2 + d_n^2}{2} + \frac{b_n^2 + c_n^2}{2} > 0 \) for some \( n \geq 2 \),
then since \( n < n^2 \), the inequality \( A < \pi \).

become strict. For the equality \( A = \pi \), we must have \( a_n = b_n = c_n = d_n = 0 \) for all \( n \geq 2 \).

Also, \( a_1 = b_1 = c_1 = d_1 = 1 \).

Let \( a_i = \cos \beta , \ b_i = \sin \beta \). Then by trigonometric identity
\[
\chi(s) = \cos (s-\beta) , \ \gamma(s) = \sin (s-\beta)
\]
This is a parametrization of a circle of radius 1.