Recall. p82 theorem A (Existence and Uniqueness)

Let $P, Q, R$ be continuous functions on a closed interval $[a, b]$. If $x_0$ is any point in $[a, b]$, and if $y_0$ and $y_0'$ are any numbers, then

$$y'' + P(x)y' + Q(x)y = 0$$

has one and only one solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

Corollary: If $y(x_0) = 0$ then $y'(x_0) \neq 0$ for any nontrivial solution $y$.

Recall. p89 Lemma 2

If $y_1$, $y_2$ are two solutions of (1) on $[a, b]$, then they are linearly dependent on this interval if and only if their Wronskian

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$

is identically zero.

Corollary: If $y_1, y_2$ are linearly independent solutions of (1) on $[a, b]$, then

$W > 0$ on $[a, b]$

or $W < 0$ on $[a, b]$, i.e., sign of $W$ is fixed.
Theorem A

If \( y_1, y_2 \) are two linearly independent solutions of \( y'' + P(x) y' + Q(x) y = 0 \) \( (1) \) then the zeros of these functions are distinct and occur alternatively.

Proof) Suppose that \( y_1(x_1) = 0 \), then \( y_2(x_1) \neq 0 \) otherwise \( W(x_1) = y_1(x_1) y_2'(x_1) - y_1'(x_1) y_2(x_1) \) becomes zero.

Thus, the zeros are distinct.

Suppose that \( y_1 \) has successive zeros \( x_1, x_2 \).

Then \( W(x_1) = y_1(x_1) y_2'(x_1) - y_1'(x_1) y_2(x_1) \)

\( W(x_2) = y_1(x_2) y_2'(x_2) - y_1'(x_2) y_2(x_2) \)

have the same sign.

Therefore \( -y_1'(x_1) y_2(x_1) \) and \( -y_1'(x_2) y_2(x_2) \) have the same sign.

Note that \( y_1'(x_1) \) and \( y_1'(x_2) \) have different signs (see Figure (1))

Thus, \( y_2(x_1) \) and \( y_2(x_2) \) have different signs. Therefore there is \( x_3 \in (x_1, x_2) \) such \( y_2(x_3) = 0 \).
Standard Form
\[ y'' + P(x)y' + Q(x)y = 0 \] (1)

Normal Form
\[ u'' + q(x)u = 0. \]

(Standard) \(\rightarrow\) (Normal).

Let \( y = uv, \quad y' = uv' + u'v \)
\[ y'' = u''v + 2u'v' + u'v' \]

By putting in (1),
\[ v u'' + (2v' + Pu') u' + (v'' + Pu' + Qv)u = 0 \]

We require \( 2v' + Pu' = 0 \), then
\[ \frac{1}{2} \int P \, dx \]
\[ v = e \]

Then \( q(x) = Q(x) - \frac{1}{4} P(x)^2 - \frac{1}{2} P'(x) \) (check)

Theorem B
If \( q(x) < 0 \), and if \( u \) is a nontrivial solution of \( u'' + q(x)u = 0 \), then \( u(x) \) has at most one zero.

Proof. Let \( u(x_0) = 0 \), then \( u'(x_0) \neq 0 \).

WLOG, let \( u'(x_0) > 0 \). We have
\[ u''(x) = -q(x)u(x) \]

Since \( -q(x) > 0 \), and \( u(x) > 0 \) over some interval to the right of \( x_0 \) (see Figure (2)),
\[ u''(x) > 0 \] over some interval to the right of \( x_0 \).

Then \( u' \) is increasing, \( u \) cannot have zero
Figure 3

\[ u' = f(u) \]

Then, if \( u(x) = -g(x) u(x) \), \( u(0) = 0 \) for some \( x > B \), (see Figure 3) (contradiction)

since \( u(x) \) is decreasing. Therefore, \( u(x) < 0 \) for some \( x > B \), and

\[ u(x) = \int_{x}^{\infty} e^{x} f(x) dx \]

for \( x > B \) can become

Then \( V(x) = \frac{u(x)}{u(x)^2} \) for all \( x > B \) considered.

This shows that \( u(x) > 0 \) for

Assume that \( u(x) \) is bounded, i.e., \( u(x) = 0 \) for \( x > B \), then \( u(x) \) has infinitely many zeros, which is upper bounded for all \( x > 0 \), therefore, there are infinitely many zeros for all \( x > 0 \).

Next, let \( u' = 0 \) be any non-trivial solution to the right of \( x_0 \). Similarly, let \( u \) be any non-trivial solution to the left of \( x_0 \).
Bessel Function Zeros.

\[ x^2 y'' + x y' + (x^2 - \rho^2) y = 0 \]  \hspace{1cm} (Standard)

\[ u'' + \left( 1 + \frac{1-4\rho^2}{4x^2} \right) u = 0 \]  \hspace{1cm} (Normal)

Note that Theorem C holds if we assume \( q(x) > 0 \) for all \( x > x_0 \), and
\[ \int_a^\infty q(x) \, dx = \infty \]  \text{where} \( a > x_0 \).

In the case of Bessel functions,
\[ q(x) = 1 + \frac{1-4\rho^2}{4x^2} \]  \text{satisfies}
the above two conditions.

Therefore, the Bessel function \( J_\rho \)
has infinitely many zeros. Indeed,
zeros \( \rightarrow \infty \).