Statement of research results and future plans

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Extremal combinatorics is one of the central themes of discrete mathematics. It deals with the problems of finding the maximum or minimum possible cardinality of a collection of finite objects satisfying certain restrictions. These problems are often related to other areas including number theory, analysis, geometry, computer science and information theory. This branch of mathematics has developed spectacularly in the past several decades and many interesting open problems arose from it. My research focuses on various problems in extremal combinatorics, as well as some other related areas. The following is a brief list of my main contributions and plans for future research.

Chromatic number and biclique partition: Graph coloring is an assignment of colors to all vertices such that no two adjacent vertices share the same color. We call the minimum number needed for such a coloring the chromatic number of the graph. It has many practical applications including scheduling, register relocation and pattern matching, as well as theoretical challenges. One particular example is a theorem of Graham and Pollack in 1972, motivated by a communication problem that arose in connection with data transmission. They proved that the edge set of a complete graph on k vertices cannot be partitioned into edge disjoint union of less than $k-1$ complete bipartite graphs. A natural generalization is to ask whether the same argument holds for all the graphs with chromatic number k. This question was first raised in the early 90s by Alon, Saks, and Seymour and has been open over the past two decades despite attempts by various researchers to attack it. Together with Sudakov [5], we constructed an infinite family of graphs with polynomial gap between the minimum size of partition of the graph G into complete bipartite subgraphs and its chromatic number, thereby disproving this long-standing conjecture. Our work also has other interesting applications in combinatorial geometry and complexity theory. Based on our construction, we were able to give the first non-trivial lower bound on the non-deterministic communication complexity of "clique v.s. independent set" problem in theoretical computer science.

Quasi-random properties: The random graph $G(n, p)$ is a graph obtained by starting with a set of n vertices and adding edges randomly and independently with probability p. Quasi-random graphs can be informally described as graphs whose edge distribution closely resembles that of a truly random graph of the same edge density. Chung, Graham and Wilson established a fundamental theorem in this area by showing that many natural properties satisfied by random graphs force a graph to be quasi-random. Lots of research has been done on finding what kind of graph property is quasi-random. In particular, much attention has been focused on the property of having a correct number of certain graph H with vertices from different parts of a cut in G, where the parts have fixed sizes. Chung and

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Graham showed that for every fixed parameter $\alpha \neq 1/2$, if in any partition of graph G to two parts of size αn and $(1-\alpha)n$, the number of edges with exactly one end from each part is similar to what is expected in random graph $G(n,p)$, then G is quasi-random. They also constructed a non-quasi-random graph satisfying this cut property when $\alpha = 1/2$. Shapira-Yuster generalized this result to hypergraphs, and posed the following open problem: if for any partition of the graphs G into $k \geq 3$ equal parts, the number of complete graphs K_k with one vertex in each part is close to what we can expect in a random graph, is this graph always quasi-random? This question was also asked independently by Janson. Together with Lee [3], we were able to use probabilistic and analytical techniques to give a complete and positive answer to this problem.

Nonnegative k-sum, fractional covers and probability of small deviations: More than twenty years ago, Manickam, Miklós and Singhi posed the following conjecture, which was motivated from their study of distribution invariants of Johnson scheme. Given a sequence of n numbers $x_1, \cdots, x_n$ with a nonnegative sum, it is not hard to see that at least half of the subsets must have a nonnegative sum by pairing each subset with its complement. A natural question one can ask is: what happens if we restrict ourselves only to the subsets of fixed size k? In other words, how many k-subsets of nonnegative sums can we guarantee, over all possible choices of n numbers with nonnegative sum? Setting $x_1 = n-1, x_2 = \cdots = x_n = -1$ yields $\binom{n-1}{k-1}$ k-subsets with nonnegative sum. There are better examples when n is very small, but Manickam, Miklós and Singhi conjectured that $\binom{n-1}{k-1}$ is an answer when $n \geq 4k$. For a long time this conjecture was only known when n is exponential in k. We were able to find a connection of this problem to hypergraph theory, in particular, to an old conjecture of Erdős dated back to 60s of bounding the number of edges in a graph if the maximum number of disjoint edges is given. By looking at its fractional relaxation and using probabilistic techniques, the original problem can be reduced to bounding the probability that the sum of independent random variables deviates from sum of their expectations, which is another long-standing open question asked by Feige and Samuels, as a generalization of Markov's inequality. With Alon and Sudakov [2], we used this connection to prove that Manickam-Miklós-Singhi conjecture holds for the range $n \geq 33k^2$, which gives the first polynomial range on n for this conjecture, and substantially improves all previous known bounds.

The size of hypergraph and its matching number: The study of matchings in a graph or hypergraph is one of the classical areas in extremal combinatorics, and has many applications in various different branches of mathematics, computer science, and even computational chemistry. Yet although the graph matching problem is fairly well-understood, and solvable in polynomial time, most of the problems related to hypergraph matching tend to be very difficult and remain unsolved. Indeed, the hypergraph matching problem is known to be NP-hard even for 3-uniform hypergraphs, without any good approximation algorithm.

One of the most basic open questions in hypergraph matching is the following conjecture raised by Erdős in 1965. Given a k-uniform hypergraph H, we denote by $\nu(H)$ the size of the largest matching, i.e. the maximum number of disjoint edges in H. Erdős asked to determine the maximum possible number of edges that can appear in any k-uniform hypergraph with matching number $\nu(H) < t \leq \frac{n}{k}$. He conjectured that this problem has only two extremal examples. The first one is a clique consisting of all the k-subsets on $kt-1$ vertices, which obviously has matching number $t-1$. The second example is a k-uniform hypergraph on n vertices containing all the edges intersecting a fixed set of $t-1$ vertices, which also forces the matching number to be at most $t-1$. Neither construction is uniformly better than the other across the entire parameter space, so the conjectured bound is the maximum of these two possibilities. In particular the case $t=2$ is the well-known Erdős-Ko-Rado theorem, and the graph case ($k=2$) was separately verfied by Erdős and Gallai. Although this appears to be a basic instance of the hypergraph Turán problem (with a t-edge matching as the excluded hypergraph), progress on this question has remained elusive. With Loh and Sudakov [4], we gave a solution of Erdős' question for $t < \frac{n}{3k^2}$. This improves upon the best previously known range $t = O(\frac{n}{k^3})$, which dated back to the work by Bollobás, Daykin and Erdős in 70's.

Existence of perfect matchings in hypergraph: The previous conjecture by Erdős can be restated as finding the least number of edges in a hypergraph which guarantees the existence of a matching of certain size. Various other conditions on the existence of a perfect matching have also been studied. Problems of this type have a long history going back to Dirac who in 1952 proved that minimum degree $n/2$ implies the existence of a Hamiltonian cycle (thus also a perfect matching when n is even) in graphs. It is natural to consider the following generalization of this result to hypergraph. For a k-uniform hypergraph H and for $1 \le d \le k$, let $\delta_d(H)$ denote the minimum number of edges containing a subset of d vertices of H, where the minimum is taken over all such subsets. For k that divides n, let $m_d(k,n)(f_d(k,n))$ denote the smallest number so that any k-uniform hypergraph H on n vertices with $\delta_d(H)$ above this threshold contains a perfect (fractional) matching. With Alon, Frankl, Rödl, Ruciński and Sudakov [1] we prove that if $f_d(k,n) \sim c\binom{n-d}{k-d}$, then $m_d(k,n) \sim \max\{\frac{1}{2}, c\}\binom{n-d}{k-d}$. Moreover, using the probabilistic approach developed in [2], we reduce this problem to the conjectures of Erdős and Samuels, which we are able to solve in some special cases. In particular, we asymptotically determine the values of $m_1(5,n)$, which was not known before. As it turns out, our method also has interesting applications in information theory on finding optimal data allocations in distributed storage systems.

Future plans: In the near future I plan to continue to work on several problems in Extremal combinatorics, Ramsey theory and graph coloring. I intend to work on a 40-year-old conjecture by Erdős, Faber and Lovász, which says that the union of n pairwise edge-disjoint complete graphs with n vertices has chromatic number n. This problem can be viewed as an analogue of Alon-Saks-Seymour conjecture for cliques and it might also have connections with problems from theoretical computer science. I also plan to continue my research on extremal problems for graphs and hypergraphs and try to obtain some results on Turán numbers. Finally, I am interested in developing spectral techniques for the study of hypergraphs, in particular I will seek to solve the Erdős conjecture on hypergraph matching in full generality.

References

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