LOCAL PROPERTIES OF SELF-DUAL HARMONIC 2-FORMS ON A 4-MANIFOLD

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Abstract. We prove a Moser-type theorem for self-dual harmonic 2-forms on closed 4-manifolds, and use it to classify local forms on neighborhoods of singular circles on which the 2-form vanishes. Removing neighborhoods of the circles, we obtain a symplectic manifold with contact boundary – we show that the contact form on each $S^1 \times S^2$, after a slight modification, must be one of two possibilities.

1. Introduction

This paper is a study of generic self-dual (SD) harmonic 2-forms near their zero sets. Let M^4 be a closed, oriented 4-manifold with $b_2^+(M) > 0$. Then one can show that, for a pair (ω, g) consisting of a generic metric g and a generic self-dual harmonic 2-form ω with respect to g, ω is a transverse section of $\bigwedge_g^+ \rightarrow M$ (i.e., ω is transverse to the zero section). Here $\bigwedge_g^+$ is the self-dual subbundle of $\bigwedge^2 T^*M \rightarrow M$ whose fiber over a point $p \in M$ is $\bigwedge_g^+(p) = \{\omega \in \bigwedge_p^2 T^*M | *_g \omega = \omega\}$. For a generic (ω, g), the zero set C of ω is therefore a disjoint union of embedded circles. Now, since $\omega \wedge \omega = \omega \wedge *\omega$, ω is nondegenerate at p if and only if $\omega(p) \neq 0$. That is, ω is *symplectic* away from C – a union of circles – and is identically 0 on C. For more details, consult [3] or [2].

The interest in self-dual harmonic 2-forms on closed 4-manifolds comes, to a large extent, from our attempt to understand which closed 4-manifolds have symplectic structures. We therefore view the zero set C of ω as an obstruction to the existence of a symplectic structure on M, and will sometimes refer to the self-dual harmonic forms as *singular symplectic forms*.

We briefly outline the contents of the paper. In Section 2, we introduce an almost complex structure J which is naturally associated to our singular symplectic form ω and metric g. Section 3 is devoted to a discussion of a version of Moser's theorem (Theorem 2) which applies to our singular symplectic forms. In Section 4, we use the Moser-type theorem to classify local normal forms for the singular symplectic forms near an S^1, with an eye towards global results, and in the last section we discuss the induced contact structures on the boundaries of $N(S^1)$.

It turns out that most of the theorems described in paper were already known to various researchers, but never published. I hope this manuscript fills a gap in the literature, especially for readers of Taubes' papers [6, 7, 8, 9], which contain an analysis of J-holomorphic

Date: Revised January 13, 2004.
1991 *Mathematics Subject Classification.* Primary 57R17; Secondary 57R57.
Key words and phrases. harmonic form, symplectic, 4-manifold.

curves with boundary on C, and the relationship to the nonvanishing of the Seiberg-Witten invariants.

2. Almost complex structures

Observe first that we can define an almost complex structure J on $M - C$, where C is the zero set of ω.

Proposition 1. *If ω is a self-dual harmonic 2-form which is nondegenerate on $M - C$, then there exists a unique almost complex structure J compatible with ω and $\tilde{g}$ on $M - C$, where $\tilde{g}$ is conformally equivalent to g.*

Proof. Given a metric g, any 2-form ω can be written pointwise as

$$\omega = c_1 e_1 e_2 + c_2 e_3 e_4,$$

where $(e_1, ..., e_4)$ is an oriented orthonormal basis for $T_p^* M$ at a point $p \in M$. Since ω is self-dual, $c_1 = c_2$. Hence,

$$\omega = c(e_1 e_2 + e_3 e_4).$$

We claim this c is well-defined and smooth on $M - C$, up to sign. Since $\frac{1}{2}\omega \wedge \omega = c^2 e_1 ... e_4 = c^2 dv_g$, with dv_g the volume form for g, we find that c^2 is determined by ω and g. Taking advantage of $M - C$ being connected, we may fix c on all of $M - C$ so that $c > 0$.

We then set $J : e_1 \mapsto e_2, e_2 \mapsto -e_1, e_3 \mapsto e_4, e_4 \mapsto -e_3$. This definition is equivalent to the following: Let $\tilde{g} = cg$, and define J such that $\tilde{g}(x, y) = \omega(x, Jy)$. Hence we see that if there is a J compatible with ω and $\tilde{g}$, it must be unique. Thus J is compatible with ω and $\tilde{g} = cg$ on $M - C$. □

Observe that ω is defined on all of M and is zero on C, $\tilde{g}$ can be defined on all of M and is zero on C, but is not smooth on C, while J is defined only on $M - C$.

3. Moser argument for self-dual harmonic 2-forms

Let $\{\omega_t\}$, $t \in [0, 1]$, be a family of self-dual harmonic 2-forms on M, where ω_t is a transverse section of the corresponding $\bigwedge^+_{g_t}$ for each t, and the following hold:

(i) $[\omega_t] \in H^2(M; \mathbf{R})$ is constant.
(ii) The sets $C_t = \{x \in M | \omega_t(x) = 0\}$ are all S^1's; hence via an isotopy, we may assume that $C = C_t$ is a fixed S^1.
(iii) $[\omega_t] \in H^2(M, C; \mathbf{R})$ does not vary with t.

For simplicity, we have assumed in (ii) that the zero set C is a single circle. The result also holds if there are more circles, provided their number remains constant with t. If C is contractible, then (iii) is equivalent to the following:

(iii′) Let Ω be an oriented surface with $\partial\Omega = C$. Then $\int_\Omega \omega_t$ does not vary with t.

Then we have the following:

Theorem 2. *There exists a 1-parameter family of C^0-homeomorphisms of M, which is smooth away from C and takes $(M - C, \omega_0) \xrightarrow{\sim} (M - C, \omega_1)$ symplectically.*

This generalizes the classical Moser's theorem:

Theorem 3 (Moser). *Let $\{\omega_t\}$ be a family of symplectic forms on a closed manifold M. Provided $[\omega_t] \in H^2(M;\mathbf{R})$ is fixed, there is a 1-parameter family of diffeomorphisms ϕ_t such that $\phi_t^*\omega_t = \omega_0$.*

Proof of Theorem 3. Let η_t be a 1-parameter family of 1-forms such that $\frac{d\omega_t}{dt} = d\eta_t$. (The nontrivial part of this proof is to construct a smooth family η_t using Hodge theory.) Thus, if we define X_t such that $i_{X_t}\omega_t = -\eta_t$, then $\mathcal{L}_{X_t}\omega_t = (i_{X_t} \circ d + d \circ i_{X_t})\omega = -d\eta_t$, which, integrated, gives a 1-parameter family ϕ_t such that $\phi_t^*\omega_t = \omega_0$. □

Proof of Theorem 2. The point here is to find a suitable η_t satisfying $\frac{d\omega_t}{dt} = d\eta_t$, and $\eta_t|_C = 0$ "up to first order" near C.

Claim. *There exists a 1-form $\tilde\eta_t$ satisfying (1) $\frac{d\omega_t}{dt} = d\tilde\eta_t$ and (2) $i^*\tilde\eta_t$ is exact, where $i : C \to M$ is the inclusion.*

Proof of Claim. This follows from condition (iii) and the relative cohomology sequence:

$$H^1(M;\mathbf{R}) \to H^1(C;\mathbf{R}) \xrightarrow{\delta} H^2(M,C;\mathbf{R}) \to H^2(M;\mathbf{R}).$$

In the de Rham setting, δ is given as follows: Given a class $[\alpha] \in H^1(C;\mathbf{R})$, represented by a closed 1-form α on C, we extend α to (a not necessarily closed) 1-form $\widetilde{\alpha}$ on M. Then let $\delta[\alpha] = [d\widetilde{\alpha}] \in H^2(M,C;\mathbf{R})$.

Now choose any smooth family $\tilde\eta_t$ satisfying $\frac{d\omega_t}{dt} = d\tilde\eta_t$. Since $\left[\frac{d\omega_t}{dt}\right] = 0 \in H^2(M,C;\mathbf{R})$, if $[i^*\tilde\eta_t] \neq 0 \in H^1(C;\mathbf{R})$, we can kill this class by adding to $\tilde\eta_t$ a closed 1-form on M which represents a class in $H^1(M;\mathbf{R})$ and which maps to $[i^*\tilde\eta_t] \in H^1(C;\mathbf{R})$. This modification can be performed smoothly with respect to t, using Hodge theory. □

Now fix an $\tilde\eta_t$ as in the Claim. Then there exists a family f_t of functions on C such that $i^*\tilde\eta_t = df_t$. Our goal is to extend f_t to a function on M such that $\tilde\eta_t = df_t$ "up to first order" near C.

In order to extend f_t to a neighborhood $N(C)$ of C, first observe that there is only one orientable rank 3 bundle over S^1 ($\pi_1(BSO(3)) = 0$ implies $S^1 \to BSO(3)$ is homotopically trivial) and hence $N(C) \simeq C \times D^3$. Choose coordinates (θ, x_1, x_2, x_3) such that $d\theta, dx_1, dx_2, dx_3$ at $(\theta, 0)$ are orthonormal.

Setting

$$f_t(\theta, x_1, x_2, x_3) = f_t(\theta, 0) + \sum_i \tilde\eta_i(\theta, 0)x_i + \frac{1}{2}\sum_{i,j} \frac{\partial\tilde\eta_i}{\partial x_j}(\theta, 0)x_i x_j$$

on $N(C)$, where $\tilde\eta_t = \tilde\eta_\theta d\theta + \sum_i \tilde\eta_i dx_i$, we have

$$\begin{aligned} df_t(\theta, x_1, x_2, x_3) &= \frac{\partial f_t}{\partial\theta}(\theta, 0)d\theta + \sum_i \frac{\partial\tilde\eta_i}{\partial\theta}(\theta, 0)x_i d\theta \\ &\quad + \sum_i \tilde\eta_i(\theta, 0)dx_i + \frac{1}{2}\sum_{i,j} \frac{\partial\tilde\eta_i}{\partial x_j}(\theta, 0)(x_i dx_j + x_j dx_i) \end{aligned}$$

up to first order in the x_i's. Now observing that

$$\frac{\partial f}{\partial \theta}(\theta,0)=\tilde{\eta}_\theta(\theta,0), \tag{1}$$

$$d\tilde{\eta}_t(\theta,0)=0, \tag{2}$$

and that Equation 2 gives

$$\frac{\partial \tilde{\eta}_\theta}{\partial x_i}(\theta,0)=\frac{\partial \tilde{\eta}_i}{\partial \theta}(\theta,0),$$
$$\frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta,0)=\frac{\partial \tilde{\eta}_j}{\partial x_i}(\theta,0),$$

we obtain

$$\begin{aligned} df_t(\theta,x) &= \left(\tilde{\eta}_\theta(\theta,0)+\sum_i \frac{\partial \tilde{\eta}_\theta}{\partial x_i}(\theta,0)x_i\right)d\theta \\ &\quad +\sum_i\left(\tilde{\eta}_i(\theta,0)+\sum_j \frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta,0)x_j\right)dx_i \\ &= \tilde{\eta}_\theta(\theta,x)d\theta+\sum_i \tilde{\eta}_i(\theta,x)dx_i \end{aligned}$$

up to first order in x.

Damping f_t out to 0 outside $N(C)$, we arrive at $\eta_t=\tilde{\eta}_t-df_t$. Finally, we obtain the vector field X_t such that $i_{X_t}\omega_t=-\eta_t$. X_t will then give rise to a 1-parameter family of symplectomorphisms, away from C, once we establish that $X_t\to 0$ rapidly enough as $p\in M$ approaches C.

On $N(C)$,

$$\begin{aligned} \omega_t &= L_1(\theta,x)(d\theta dx_1+dx_2dx_3) \\ &\quad + L_2(\theta,x)(d\theta dx_2+dx_3dx_1) \\ &\quad + L_3(\theta,x)(d\theta dx_3+dx_1dx_2) \\ &\quad + Q, \end{aligned} \tag{3}$$

where $L_i(\theta,x)=\sum_j L_{ij}(\theta)x_j$ and Q consists of forms in $d\theta$ and dx_i, whose coefficients are quadratic or higher in the x_i. In terms of matrices, ω_t corresponds to

$$A=\begin{pmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & L_3 & -L_2 \\ -L_2 & -L_3 & 0 & L_1 \\ -L_3 & L_2 & -L_1 & 0 \end{pmatrix}+\widetilde{Q},$$

where $\widetilde{Q}$ has quadratic or higher terms in the x_i and the matrix is with respect to the basis $\{d\theta,dx_1,dx_2,dx_3\}$. $i_{X_t}\omega_t=-\eta_t$ then becomes

$$(a_\theta\ a_1\ a_2\ a_3)A=-(\eta_\theta\ \eta_1\ \eta_2\ \eta_3)$$

with $X_t = a_\theta d\theta + \sum_i a_i dx_i$. Thus,

$$\begin{aligned}(a_\theta \ a_1 \ a_2 \ a_3) &= -(\eta_\theta \ \eta_1 \ \eta_2 \ \eta_3)A^{-1} \\ &= \frac{(\eta_\theta \ \eta_1 \ \eta_2 \ \eta_3)}{L_1^2+L_2^2+L_3^2}\begin{pmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & L_3 & -L_2 \\ -L_2 & -L_3 & 0 & L_1 \\ -L_3 & L_2 & -L_1 & 0 \end{pmatrix}\end{aligned}$$

up to first order in x. This means that $|X_t| < k|x|$ near C; hence, as $x \to 0$, $|\phi_1(\theta, x) - \phi_0(\theta, x)| \to 0$, where ϕ_t is the flow such that $\frac{d\phi_t}{dt} = X_t$. This concludes our proof. □

4. Local normal forms

On a neighborhood $N(C) = C \times D^3$ of C, ω can be written as in Equation 3. If ω is generic, then it is transverse to the zero section of $\bigwedge_g^+$, and $(L_{ij}(\theta))$ is nondegenerate for all θ.

Lemma 4. *$(L_{ij}(\theta))$ is symmetric and traceless.*

Proof. By comparing zeroth order terms in the x_i, $d\omega = 0$ implies

$$\frac{\partial L_1}{\partial x_1} + \frac{\partial L_2}{\partial x_2} + \frac{\partial L_3}{\partial x_3} = 0,$$

$$\frac{\partial L_2}{\partial x_3} - \frac{\partial L_3}{\partial x_2} = 0, \frac{\partial L_3}{\partial x_1} - \frac{\partial L_1}{\partial x_3} = 0, \frac{\partial L_1}{\partial x_2} - \frac{\partial L_2}{\partial x_1} = 0.$$

□

The traceless symmetric matrix $(L_{ij}(\theta))$ thus has a basis $\{v_1(\theta), v_2(\theta), v_3(\theta)\}$ of eigenvectors for each θ (though the v_i are not necessarily continuous in θ). Since $(L_{ij}(\theta))$ is traceless, either two of the eigenvalues are positive and the remaining is negative for all θ, or vice versa. Hence, $(L_{ij}(\theta))$ gives rise to a splitting of $\mathbf{R}^3 \times S^1 \to S^1$ into a real line bundle over S^1 and a rank 2 vector bundle over S^1. Such splittings are classified by homotopy classes of maps from S^1 to $\mathbf{RP}^2$, and $\pi_1(\mathbf{RP}^2) = \mathbf{Z}/2\mathbf{Z}$. Hence, we have the well-known:

Proposition 5. *There exist two homotopy classes of splittings of $\mathbf{R}^3 \times S^1 \to S^1$, the oriented one and the unoriented one.*

What is rather remarkable is the following:

Theorem 6. *For either of the two splitting types, there exists a SD harmonic 2-form for a flat metric on $S^1 \times D^3$ whose zero set is $C = S^1 \times \{0\}$ and which has the given splitting type.*

Proof. We give representatives of both types.

(A) Corresponding to the oriented splitting, we have:

$$\begin{aligned}\omega_A &= x_1(d\theta dx_1 + dx_2 dx_3) \\ &\quad + x_2(d\theta dx_2 + dx_3 dx_1) \\ &\quad - 2x_3(d\theta dx_3 + dx_1 dx_2) \\ &= *_3\mu + d\theta \wedge \mu,\end{aligned}$$

where $\mu = d(\frac{1}{2}(x_1^2+x_2^2)-x_3^2)$, and $*_3$ is the $*$-operator for the flat metric on D^3. Here, $(L_{ij}(\theta)) = \mathrm{diag}(1,1,-2)$, with fixed positive and negative eigenspaces. Note that ω_A is S^1-invariant.

(B) We construct ω_B, corresponding to the unoriented splitting, as follows. Starting with

$$\begin{aligned}\Omega &= x_1(d\theta dx_1 + dx_2dx_3)\\ &\quad + x_2(d\theta dx_2 + dx_3dx_1)\\ &\quad - 2x_3(d\theta dx_3 + dx_1dx_2)\end{aligned}$$

on $[0,2\pi]\times D^3$, we glue $\phi : \{2\pi\}\times D^3 \to \{0\}\times D^3$ by sending

$$\begin{aligned}\theta &\mapsto \theta - 2\pi\\ x_1 &\mapsto x_1\\ x_2 &\mapsto -x_2\\ x_3 &\mapsto -x_3.\end{aligned}$$

One easily verifies that $\phi^*\Omega = \Omega$. $\square$

Theorem 7. *Given a generic SD harmonic 2-form ω on M, there exists a 1-parameter family of closed 2-forms ω_t, $t\in[0,1]$, on M which satisfy the following:*

1. $\omega_0 = \omega$.
2. *The ω_t are symplectic away from their common zero set C.*
3. $\omega_t = \omega$ *except on a neighborhood $N(C)$ of C.*
4. *On each connected component of $N(C)$, ω_1 is (up to sign) one of the two local forms ω_A or ω_B of Theorem 6.*
5. $[\omega_t] \in H^2(M;\mathbf{R})$ *is independent of t.*

Proof. Suppose for simplicity that the zero set C of the SD 2-form ω is a single circle. Assume that ω on $N(C)$ gives rise to an oriented splitting, i.e., we are in case (A). (Case (B) is identical.) After an orthonormal change of frame, we may write

$$\begin{aligned}\omega &= (L_{11}(\theta)x_1 + L_{12}(\theta)x_2)(d\theta dx_1 + dx_2dx_3)\\ &\quad + (L_{21}(\theta)x_1 + L_{22}(\theta)x_2)(d\theta dx_2 + dx_3dx_1)\\ &\quad + \lambda_3(\theta)x_3(d\theta dx_3 + dx_1dx_2)\\ &\quad + Q,\end{aligned}$$

with, say, $(L_{ij}(\theta))_{1\le i,j\le 2}$ positive definite and $\lambda_3(\theta) < 0$. Here, the $L_{ij}(\theta)$ and $\lambda_3(\theta)$ are differentiable in θ.

Now, take a 1-parameter family $\beta_t = (1-t)\omega + t\omega_A$ on $N(C)$. After shrinking $N(C)$ if necessary, β_t is symplectic on $N(C) - C$. Using a local version of our Moser argument (see the proof of Theorem 2), we see that there exist homeomorphisms

$$\phi_t : (N_0(C), \beta_0 = \omega) \xrightarrow{\sim} (N_t(C), \beta_t),$$

where $N_t(C)$ are small neighborhoods of C whose boundary depend smoothly on t, $\phi_t = id$ on C, and ϕ_t is smooth away from C. The ϕ_t allow us to remove $(N_0(C),\omega)$ and graft on $(N_t(C),\beta_t)$. Hence there exists a global family ω_t on M with $\omega_0 = \omega$ and $\omega_1|_{N(C)} = \omega_A$, after further shrinking $N(C)$. Moreover, the perturbation can be performed in an arbitrarily small neighborhood of C and without altering the cohomology class. □

In essence, Theorem 7 tells us that, in studying the singular circles of ω from a global perspective, we may assume that the zeros are either (A) or (B).

5. Contact structures on the boundaries

In this section we investigate the boundary properties of ω_A and ω_B. More precisely, we have:

Theorem 8. *There exist contact forms λ_A and λ_B on $\partial N(C) = S^1 \times S^2$ such that $d\lambda_A = i^*\omega_A$ and $d\lambda_B = i^*\omega_B$, where i is the inclusion $S^1 \times S^2 \hookrightarrow N(C) = S^1 \times D^3$.*

Proof. (A) Consider the following S^1-invariant 1-form

$$\lambda = -\frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)d\theta + x_2x_3dx_1 - x_1x_3dx_2$$

on $N(C)$. We then compute that $d\lambda = \omega_A$ on $N(C)$ and

$$\sum_i x_idx_i \wedge \lambda \wedge d\lambda = \left(\frac{1}{2}(x_1^2 + x_2^2)(x_1^2 + x_2^2 + 2x_3^2) + 2x_3^4\right) d\theta dx_1dx_2dx_3.$$

Since $S^1 \times S^2 = \{\sum_i x_i^2 = 1\}$ is an integral submanifold of $\sum_i x_idx_i = 0$, $i^*(\lambda \wedge d\lambda) \neq 0$ on $S^1 \times S^2$ if and only if $\lambda \wedge d\lambda \wedge \sum_i x_idx_i \neq 0$ in a neighborhood of $S^1 \times S^2$. Noting that $\lambda \wedge d\lambda \wedge \sum_i x_idx_i = 0$ if and only if $x_1 = x_2 = x_3 = 0$, we conclude that $\lambda_A = i^*\lambda$ is a contact 1-form on $S^1 \times S^2$ and $(M - N(S^1), \omega_A = d\lambda)$ is a symplectic manifold with contact-type boundary.

(B) Consider the 1-form

$$\lambda = -\frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)d\theta + x_2x_3dx_1 - x_1x_3dx_2$$

on $[0, 2\pi] \times D^3$. Using the notation from the proof of Theorem 6, $d\lambda = \Omega$ and $\phi^*\lambda = \lambda$, so we glue together (after pulling back via i) a contact 1-form λ_B such that $d\lambda_B = i^*\omega_B$. The rest is the same as (A). □

Let us now describe the orbits of the Reeb vector fields.

(A) ω_A is compatible with a metric $\tilde{g} = cg$, where g is the standard product metric on $S^1 \times D^3$. We can then write the compatible J satisfying $\tilde{g}(x, y) = \omega(x, Jy)$ as $J = -\frac{1}{c}A$, where

$$A = \begin{pmatrix} 0 & x_1 & x_2 & -2x_3 \\ -x_1 & 0 & -2x_3 & -x_2 \\ -x_2 & 2x_3 & 0 & x_1 \\ 2x_3 & x_2 & -x_1 & 0 \end{pmatrix}$$

represents ω with respect to $\{\theta, x_1, x_2, x_3\}$. Now, the Reeb vector field X for λ_A is given, up to multiple, by

$$J\left(\sum_i x_i \frac{\partial}{\partial x_i}\right) = \frac{-1}{\sqrt{x_1^2+x_2^2+4x_3^2}}\left((x_1^2+x_2^2-2x_3^2)\frac{\partial}{\partial\theta} - 3x_2x_3\frac{\partial}{\partial x_1} + 3x_1x_3\frac{\partial}{\partial x_2}\right).$$

Finally, $\lambda_A(X) = 1$ implies that

$$X = \frac{1}{f}\left[(x_1^2+x_2^2-2x_3^2)\frac{\partial}{\partial\theta} - 3x_2x_3\frac{\partial}{\partial x_1} + 3x_1x_3\frac{\partial}{\partial x_2}\right],$$

with

$$f = -\frac{1}{2}\left[(x_1^2+x_2^2)(x_1^2+x_2^2+2x_3^2)+4x_3^4\right].$$

Solving for the orbits, $x_1^2+x_2^2$ and x_3^2 are fixed for each orbit, and hence,

$$\begin{aligned} x_1 &= \sqrt{1-r^2}\cos R_1(r)t \\ x_2 &= \sqrt{1-r^2}\sin R_1(r)t \\ x_3 &= r \\ \theta &= R_2(r)t + c, \end{aligned}$$

where r is a constant, and R_1 and R_2 are functions of r.

In particular, the noteworthy closed orbits are $S^1\times\{(0,0,1)\}$, $S^1\times\{(0,0,-1)\}$, and $S^1\times\{(x_1,x_2,0)\}$, with $x_1^2+x_2^2=1$ and x_1, x_2 fixed. These correspond to the stable and unstable gradient directions in the Morse theory of $\frac{1}{2}(x_1^2+x_2^2-2x_3^2)$ near $(0,0,0)$. Moreover, the orbit $S^1\times\{(0,0,1)\}$ is *nondegenerate*, and so is the family $S^1\times\{(x_1,x_2,0)\}$. There are other closed orbits, but these do not seem to have any Morse-theoretic significance. The relevance of such periodic orbits is manifest in Taubes' paper [6].

(B) We apply the previous considerations and work on $[0,2\pi]\times S^2/\sim$. There is one orbit $S^1\times\{(0,0,\pm1)\}$, which is a double of the orbits for (A). Since ϕ identifies $(2\pi,(x_1,x_2,0))\sim(0,(x_1,-x_2,0))$, we also have the doubled closed orbits $S^1\times\{(x_1,\pm x_2,0)\}$, with $x_2\neq 0$, and the single closed orbits $S^1\times\{(1,0,0)\}$, $S^1\times\{(-1,0,0)\}$.

Remark. *There is an example of a singularity of type (B) bounding a disk, which can be made to vanish.*

Our final point of investigation is to determine whether the two contact forms represent the same *contact structure* on $S^1\times S^2$. Let ξ_A, ξ_B be the contact structures (i.e., 2-plane distributions) corresponding to the 1-forms λ_A, λ_B. Then we remark the following

Proposition 9. *ξ_A and ξ_B are both overtwisted.*

Proof. A perturbation of the upper hemisphere $\{x_3\geq 0\}\bigcap(\{\text{pt.}\}\times S^2)\subset S^1\times S^2$ is an overtwisted disk. $\square$

Since overtwisted contact structures on a 3-manifold N are classified by the homotopy type of the contact 2-plane distribution on TN (c.f. [1]), it suffices to determine whether ξ_A and ξ_B are homotopic as 2-plane distributions inside $T(S^1\times S^2)$. We have the following:

Theorem 10. *The contact structures ξ_A and ξ_B are overtwisted contact structures for distinct homotopy classes of 2-plane fields on $S^1 \times S^2$.*

Proof. First let us describe the homotopy classes of 2-plane fields of *degree* 1 on $S^1 \times S^2$. A trivialization of $T(S^1 \times S^2)$ and a choice of nonsingular transverse vector field (in our case the Reeb vector field) gives rise to a Gauß map $\psi : S^1 \times S^2 \to S^2$, and the homotopy classes of 2-plane fields are in 1-1 correspondence with homotopy classes $[S^1 \times S^2, S^2]$. We are interested in maps where the induced map $H_2(S^1 \times S^2; \mathbf{Z}) \to H_2(S^2; \mathbf{Z})$ has degree one. The set $[S^1 \times S^2, S^1]_1$ of homotopy classes of maps of degree one is computed as follows:

$$\begin{aligned} [S^1 \times S^2, S^2]_1 &\simeq [S^1, \mathrm{Map}_1(S^2, S^2)] \\ &\simeq [S^1, \mathrm{Diff}^+(S^2)] \\ &\simeq [S^1, SO(3)] \\ &\simeq \pi_1(SO(3)) \simeq \mathbf{Z}/2\mathbf{Z}. \end{aligned}$$

Here, $\mathrm{Map}_1(S^2, S^2)$ is the set of maps of degree 1, and $\mathrm{Diff}^+(S^2)$ is the space of orientation-preserving diffeomorphisms of S^2. The representatives of the two classes are given by

$$\Psi_n : (\theta, x) \mapsto R_{n\theta}(x), \tag{4}$$

with $n = 0, 1$, where R_α is rotation by α about the x_1-axis. (Note we could have chosen representatives to rotate about any axis – however, it is important to remember that R_α is a rotation about the x_1-axis.)

Next trivialize $T(S^1 \times S^2) \simeq \mathbf{R}^3 \times (S^1 \times S^2)$ as follows: First view S^2 as the unit sphere $\sum_i x_i^2 = 1$ inside $\mathbf{R}^3$. At the point $(\theta, x_1, x_2, x_3) \in S^1 \times S^2$, the tangent vector $\frac{\partial}{\partial\theta}$ is mapped to the unit vector $(x_1, x_2, x_3) \in \mathbf{R}^3$ and $v \in T_{(x_1,x_2,x_3)}S^2$ to the corresponding tangent vector viewed inside $\mathbf{R}^3$.

Let us consider Case (A). After rescaling, the Reeb vector field is

$$X(\theta, x_1, x_2, x_3) = (-x_1^2 - x_2^2 + 2x_3^2, 3x_2x_3, -3x_1x_3, 0).$$

Let $\psi : S^1 \times S^2 \to S^2$ be the corresponding Gauß map. Since $\psi(\theta, x)$ is independent of t, we write $u(x) = \psi(\theta, x)$. Then u maps $(0, 0, \pm 1) \mapsto (0, 0, \pm 1)$, $(x_1, x_2, 0) \mapsto (-x_1, -x_2, 0)$, and sends $x_3 = \mathit{const}$ to $x_3 = \tau(\mathit{const})$. Here $\tau : [-1, 1] \to [-1, 1]$ sends $-1, 0, 1$ to itself and satisfies $\tau(-x_3) = -\tau(x_3)$. Since τ is homotopic to the identity via a homotopy τ_t which fixes $-1, 0, 1$ and satisfies $\tau_t(-x_3) = -\tau_t(x_3)$, ψ is homotopic to:

$$(\theta, x) \mapsto S_{\pi+\sigma(x_3)}(x),$$

where S_α is the rotation by α along the x_3-axis, and $\sigma(-x_3) = -\sigma(x_3)$. We can then homotop $\sigma = \sigma_0$ to $\sigma_1(x_3) = 0$, via σ_t which satisfies $\sigma_t(-x_3) = -\sigma_t(x_3)$. Therefore, ψ is homotopic to $(\theta, x) \mapsto S_\pi(x)$, which in turn is homotopic to Ψ_0 in Equation 4.

Let us now consider Case (B). Instead of gluing $\{2\pi\} \times S^2$ and $\{0\} \times S^2$ via a twist, we use the diffeomorphism

$$\begin{gathered} \Phi : [0, 2\pi] \times S^2 \xrightarrow{\sim} [0, 2\pi] \times S^2, \\ (\theta, x) \mapsto (\theta, R_{\frac{\theta}{2}} x). \end{gathered}$$

Hence, if we push forward via Φ, we may glue $\{2\pi\} \times S^2 \xrightarrow{\sim} \{0\} \times S^2$ identically. Our Reeb field for (B) is $\Phi_* X$, where $X = (-x_1^2 - x_2^2 + 2x_3^2, 3x_2x_3, -3x_1x_3, 0)$ is defined over $[0, 2\pi] \times S^2$.

The corresponding Gauß map $\psi : S^1 \times S^2 \to S^2$ is (homotopic to one) given on $[0, 2\pi] \times S^2$ by

$$(\theta, x) \mapsto R_{\theta/2} u R^{-1}_{\theta/2}(x),$$

where $u : S^2 \to S^2$ was described in the previous paragraph. Observe that $R_\pi u R_\pi^{-1}(x) = u(x)$, which allows us to glue $\theta = 0$ and $\theta = \pi$. The homotopy u_t from u to S_π which was applied in the analysis of Case (A) also can be applied here, by observing that $R_\pi u_t R_\pi^{-1} = u_t$. Now,

$$R_{\theta/2} S_\pi R^{-1}_{\theta/2}(x) = R_{\theta/2} R_{\theta/2} S_\pi(x) = R_\theta S_\pi(x),$$

and hence ψ is homotopic to Ψ_1.

This shows that ξ_A and ξ_B are not homotopic as 2-plane fields. □

We close with the following question:

Question. *Can the contact homology theory developed by Hofer and Eliashberg (among others), or variations of it, shed any light on the singular symplectic forms?*

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