NOTES FOR MATH 635: TOPOLOGICAL QUANTUM FIELD THEORY

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The goal of this course is to define invariants of 3-manifolds and knots and representations of the mapping class group, using quantum field theory. We will follow Kohno, *Conformal Field Theory and Topology*, supplementing it with additional material to make it more accessible.

The amount of mathematics that goes into defining these invariants is rather substantial (especially for the geometric approach that we will be taking), and we will spend a considerable amount of time on the preliminaries.

HW will denote "homework", whereas **FS** means "further study", indicating that one can spend some time learning this topic.

1. Lie groups and Lie algebras

1.1. **Lie groups.** In this course, manifolds are assumed to be smooth, unless indicated otherwise.

Definition 1.1. *A* Lie group *is a manifold equipped with smooth maps $\mu : G \times G \to G$ (multiplication) and $i : G \to G$ (inverse) which give it the structure of a group.*

Examples: Let $M_n(K)$ be the space of $n \times n$ matrices with entries in the base field $K = \mathbf{R}$ or $\mathbf{C}$.

(1) $GL(n, K) = \{A \in M_n(K) | \det A \neq 0\}$.
(2) $GL(V) = \{K$-linear isomorphisms $V \xrightarrow{\sim} V\}$, where V is a vector space over K.
(3) $SL(n, K) = \{A \in M_n(K) | \det A = 1\}$.
(4) $U(n) = \{A \in M_n(\mathbf{C}) | AA^* = id\}$. Here the *adjoint* A^* is $(\overline{A})^T$ (the conjugate transpose of A).
(5) $SU(n) = U(n) \cap SL(n, \mathbf{C})$.

Example: $U(n)$. If we write $A = (a_{ij})$ and write out $AA^* = id$, then $\sum_j a_{ij}\overline{a_{kj}} = \delta_{ik}$, and hence the row vectors form a *unitary basis* for $\mathbf{C}^n$.

Example: $SU(2)$. Let us write out $AA^* = id$. Here $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$AA^* = \begin{pmatrix} a\overline{a} + b\overline{b} & a\overline{c} + b\overline{d} \\ c\overline{a} + d\overline{b} & c\overline{c} + d\overline{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1)$$

In addition, we have $ad - bc = 1$.

HW: Prove that $SU(2)$ is diffeomorphic to S^3.

Definition 1.2. *A* Lie subgroup *of G is a subgroup H which is at the same time a* submanifold *such that H is a Lie group with respect to the induced smooth structure.*

Definition 1.3. *A* Lie group homomorphism *is a group homomorphism $\phi : G \to H$ which is also a smooth map of the underlying manifolds.*

Definition 1.4. *Let V be a vector space over $K = \mathbf{R}$ or $\mathbf{C}$, and let G be a Lie group. Then a* Lie group representation $\rho : G \to GL(V)$ *is a Lie group homomorphism, i.e.,* $\rho(gh) = \rho(g)\rho(h)$.

Zen: We can pretend that every Lie group is a matrix group. Every Lie group admits a representation with a 0-dimensional kernel.

1.2. **Left-invariant vector fields and 1-forms.** A Lie group G has a left action and a right action onto itself: Let $g \in G$. Then

$$L_g : G \to G, g' \mapsto gg'.$$
$$R_g : G \to G, g' \mapsto g'g.$$

Definition 1.5. *A vector field X (defined globally) on G is* left-invariant *if $(L_g)_*X = X$ for all $g \in G$. A 1-form ω on G is* left-invariant *if $L_g^*\omega = \omega$ for all $g \in G$.*

We denote the vector space of left-invariant vector fields by $\mathfrak{X}_G$ and the vector space of left-invariant 1-forms by Ω^1_G.

Proposition 1.6. *$\mathfrak{X}_G \simeq T_eG$ as vector spaces. Hence* $\dim \mathfrak{X}_G = \dim G$.

Proof. Let $e \in G$ be the identity. We propagate $v \in T_eG$ using L_g, $g \in G$. Recall that a tangent vector $v \in T_eG$ corresponds to an equivalence class of smooth arcs $\gamma(t)$, $t \in (-\varepsilon, \varepsilon)$, $\gamma(0) = e$. Then $(L_g)_*v$ corresponds to $g\gamma(t)$. We therefore define the vector field:

$$X_v(g) = g\gamma(t).$$

Then clearly $((L_g)_*X_v)(g') = g(g^{-1}g'\gamma(t)) = g'\gamma(t)$. Hence,

$$\dim \mathfrak{X}_G = \dim T_eG = \dim G.$$

□

Example: $O(n)$. Then $T_IO(n)$ is the set of skew-symmetric matrices. We write $\gamma(t) \in T_IO(n)$ as: $\gamma(t) = I + At$, where we do all the computations modulo t^2. Then:

$$\begin{aligned} I &= \gamma\gamma^T = (I + At)(I + A^Tt) \\ &= I + (A + A^T)t. \end{aligned}$$

Hence $A = -A^T$. Since $\dim O(n) = \frac{n(n-1)}{2}$ and $\dim$ of the set of skew-symmetric matrices $= \frac{n(n-1)}{2}$, $T_IO(n)$ is indeed the set of skew-symmetric matrices. $\mathfrak{X}_{O(n)} = \{X_A | A \in \text{skew-symmetric matrices}\}$, where $X_A(B) = BA$, $B \in O(n)$.

Example: $SL(n, \mathbf{R})$. Then $T_ISL(n, \mathbf{R}) = \{\text{traceless matrices}\}$.

Similarly, we have $\Omega^1_G \simeq T_e^*G$.

1.3. Lie algebras.

Definition 1.7. *A* Lie algebra $\mathfrak{g}$ *over* $K = \mathbf{R}$ *or* $\mathbf{C}$ *is a* K*-vector space together with a* Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying the following:*

(1) $[,]$ *is bilinear,*
(2) *(skew-symmetric)* $[X,Y] + [Y,X] = 0$*,*
(3) *(Jacobi identity)* $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$*.*

Example: Let M be a manifold and $\mathfrak{X}(M)$ be the C^∞ vector fields on M. The Lie bracket $[X,Y]$ makes $\mathfrak{X}(M)$ into an infinite-dimensional Lie algebra.

We now define the Lie algebra $\mathfrak{g}$ associated to a Lie group G. As a vector space, $\mathfrak{g} \simeq T_eG \simeq \mathfrak{X}_G$. The Lie bracket on $\mathfrak{X}_G$ is inherited from that of $\mathfrak{X}(G)$ (Lie bracket of vector fields). We need to verify the following:

Lemma 1.8. $[,] : \mathfrak{X}_G \times \mathfrak{X}_G \to \mathfrak{X}_G$*, i.e., if* $X, Y \in \mathfrak{X}_G$*, then* $[X,Y] \in \mathfrak{X}_G$*.*

Proof. We use the fact that $\phi_*[X,Y] = [\phi_*X, \phi_*Y]$, where $\phi : M \to M$ is a diffeomorphism and $X, Y \in \mathfrak{X}(M)$. (Check this!)

Then, $(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y]$. □

Remark: We will often write $\mathfrak{g} = Lie(G)$.

For matrix groups, i.e., $G \subset GL(V)$, we have $\mathfrak{X}_G = \{X_A | A \in T_eG\}$, where $X_A(g) = gA$. Therefore,

$$[X_A, X_B](I) = \lim_{s,t\to 0} \frac{(I+sA)(I+tB) - (I+tB)(I+sA)}{st} = AB - BA.$$

Examples: In the following, the Lie bracket is always $[A,B] = AB - BA$.

Lie group	Lie algebra
$GL(n,K)$	$\mathfrak{gl}(n,K) = End(K^n)$, $K = \mathbf{R}$ or $\mathbf{C}$
$O(n)$	$\mathfrak{o}(n)$= skew-symmetric matrices
$U(n)$	$\mathfrak{u}(n)$= skew-hermitian matrices
$SL(n,\mathbf{R})$	$\mathfrak{sl}(n,\mathbf{R})$= traceless matrices

Example: An *abelian* lie algebra $\mathfrak{t}^n$ is K^n with bracket $[X,Y] = 0$ for all $X, Y \in \mathfrak{t}^n$.

Definition 1.9. *A* Lie subalgebra $\mathfrak{h}$ *of a Lie algebra* $\mathfrak{g}$ *is a vector subspace which is closed under* $[,]$. *A* Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ *is a bracket-preserving linear map, i.e.,* $\phi([X,Y]) = [\phi(X), \phi(Y)]$. *A* Lie algebra representation *is a Lie algebra homomorphism* $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$*.*

1.4. Adjoint representation.

We define a Lie group representation $Ad : G \to GL(\mathfrak{g})$, where $\mathfrak{g} = Lie(G)$, as follows: Think of $\mathfrak{g} \simeq T_eG$. Then, for $a \in G$, $Ad(a) = (R_{a^{-1}} \circ L_a)_* : T_eG \to T_eG$.

We must show that $Ad(a)$ is indeed in $GL(\mathfrak{g})$. This is immediate, since $Ad(a^{-1})$ is the inverse of $Ad(a)$.

Remark: Here we are viewing $\mathfrak{g}$ simply as a vector space.

Example: $G = GL(n, \mathbf{R})$. $Ad(A) : T_I G \to T_I G$ is given by

$$I + tX \mapsto A(I + tX)A^{-1} = I + tAXA^{-1},$$

where we are viewing $X \in T_e G$ as an arc in G through I. In other words, $X \mapsto AXA^{-1}$.

We can differentiate any Lie group homomorphism at the identity to get a Lie algebra homomorphism. Therefore, there is also an infinitesimal version of $Ad : G \to GL(\mathfrak{g})$, that is, $ad \stackrel{def}{=} Ad_*(e)$. On the Lie algebra level, we have:

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

Example: Let G be a matrix group. Then we claim that

$$Ad : G \to GL(\mathfrak{g}),$$
$$A \mapsto [X \mapsto AXA^{-1}].$$

If we write $A = I + tY$, then $Ad(A)$ maps (up to first order in t):

$$X \mapsto (I + tY)X(I + tY)^{-1} = (I + tY)X(I - tY) = X + t[Y, X].$$

Taking derivatives, we get $Y \mapsto [Y, X]$. Therefore,

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$
$$Y \mapsto [X \mapsto [Y, X]].$$

REFERENCES

[1] Fulton-Harris, *Representation Theory*. (Good for the first several lectures on representations of $\mathfrak{sl}(2, \mathbf{C})$ and $\mathfrak{sl}(3, \mathbf{C})$.)

2. Representations of $\mathfrak{sl}(2, \mathbf{C})$

Today's goal is to work out the (finite-dimensional) irreducible representations of $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is *irreducible* if it has no nontrivial ($\neq 0$ or itself) subrepresentations $W \subset V$ (i.e., subspaces which are invariant under $\mathfrak{g}$). We will be working over the complex numbers.

Take a basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Observe that H is diagonal, E is strictly upper triangular, and F is strictly lower triangular. Then we have the equations:

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H. \tag{2}$$

2.1. **The adjoint representation.** We first study the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. $ad : X \mapsto ad(X)$, where $ad(X) : Y \mapsto [X, Y]$.

HW: Verify that ad is a Lie algebra representation, i.e., $ad([X, Y]) = [ad(X), ad(Y)]$. Hint: this follows from the Jacobi identity.

In the expression $\mathfrak{gl}(\mathfrak{g})$, it's best to view $V = \mathfrak{g}$ as $V_{-2} \oplus V_0 \oplus V_2$, where $V_{-2} = \mathbf{C}F$, $V_0 = \mathbf{C}H$, and $V_2 = \mathbf{C}E$. The structure equations imply that all the V_i are eigenspaces of $ad(H)$, since $ad(H)(E) = [H, E] = 2E$, $ad(H)(H) = [H, H] = 0$, and $ad(H)(F) = [H, F] = -2F$.

Also note that $ad(E)$ isomorphically maps $V_{-2} \xrightarrow{\sim} V_0$, $V_0 \xrightarrow{\sim} V_2$. Similarly, $ad(F)$ isomorphically maps $V_2 \xrightarrow{\sim} V_0$, $V_0 \xrightarrow{\sim} V_{-2}$.

Lemma 2.1. *The adjoint representation is irreducible.*

Proof. Let $v \in V$. Then we can write $v = aF + bH + cE$. If $a \neq 0$, then $ad(E)(v) = aH - 2bE$ and $(ad(E))^2(v) = -2aE$. These three vectors clearly span all of V. If $a = 0$, then we need to use $ad(F)$'s as well, but the proof is similar. □

2.2. **General case.** Let $\rho : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(V)$ be a (finite-dimensional) irreducible representation. We will extensively use Equation 2. If $v \in V$ and $X \in \mathfrak{g}$, then we will write Xv to mean $\rho(X)(v)$. This way we're thinking of V as a left $\mathfrak{g}$-module.

Let $v \in V$ be an eigenvector of H with eigenvalue λ. (Every endomorphism of V has at least one eigenvector.)

Lemma 2.2. *If $Hv = \lambda v$, then $H(Ev) = (\lambda + 2)(Ev)$ and $H(Fv) = (\lambda - 2)(Fv)$, i.e., Ev and Fv are also eigenvectors of H with eigenvalues $\lambda + 2$ and $\lambda - 2$, respectively.*

Proof. By Equation 2,

$$H(Ev) = EHv + 2Ev = E(\lambda v) + 2Ev = (\lambda + 2)(Ev).$$

The expression for $H(Fv)$ is similar. □

Let v be the eigenvector of H with the largest eigenvalue. Such an eigenvector v is called the *highest weight vector*. Then $Ev = 0$, since Ev, if nonzero, would have a larger eigenvalue. Starting with $V_\lambda = \mathbf{C}v$, we take $V_{\lambda-2i} = \mathbf{C}F^i v$. ($F^i v$ has eigenvalue $\lambda - 2i$.) Note that $V_{\lambda-2k} = 0$ for some k. Let $W = \oplus_{i=0}^{k} V_{\lambda-2i}$.

Lemma 2.3. *W is a subrepresentation of V.*

Proof. It suffices to show that $E : W \to W$, since F and H clearly map W to itself. We have the following:

$$Ev = 0,$$
$$E(Fv) = FEv + Hv = \lambda v,$$
$$E(F^2 v) = FE(Fv) + H(Fv) = F(\lambda v) + (\lambda - 2)Fv = [(\lambda) + (\lambda - 2)]Fv.$$

In general,

$$E(F^i v) = \{(\lambda) + (\lambda - 2) + \cdots + (\lambda - 2(i-1))\} F^{i-1} v = (\lambda - i + 1) i F^{i-1} v. \tag{3}$$

□

Since V is irreducible, it follows that $V = W = \oplus_{i=0}^{k-1} V_{\lambda-2i}$.

Also, observe that $E(Fv) = \lambda v$ implies that $Fv \neq 0$ unless $\lambda = 0$; $E(F^2 v) = (\lambda + (\lambda - 2))Fv$ implies that $F^2 v \neq 0$ unless $\lambda = 1$; etc. In particular:

(1) *λ must be a positive integer for V to be finite-dimensional.*
(2) *Moreover, the only opportunity for V to be finite-dimensional is if $F^{\lambda+1} v = 0$.*

Putting these together, we have the following theorem:

Theorem 2.4. *The irreducible representations of $\mathfrak{sl}(2, \mathbf{R})$ are parametrized by a positive integer $k \in \mathbf{Z}$. For each k, the representation $V \simeq \mathbf{C}^k$ decomposes into 1-dimensional eigenspaces V_λ of H, and $V = V_{1-k} \oplus V_{3-k} \oplus \cdots \oplus V_{k-3} \oplus V_{k-1}$.*

Remark: We still haven't shown that these representations really exist....

2.3. **Tensor products and duals.** Given representations $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho_W : \mathfrak{g} \to \mathfrak{gl}(W)$, we can construct their *tensor product* as follows:

$$\rho_{V\otimes W} : \mathfrak{g} \to \mathfrak{gl}(V \otimes W),$$
$$\rho_{V\otimes W}(X) : v \otimes w \mapsto (\rho_V(X)(v)) \otimes w + v \otimes (\rho_W(X)(w)).$$

Since the tensor product of Lie group representations acts diagonally and the Lie algebra representations are derivatives of those, the Leibniz rule is in effect.

Given the representation $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$, we define the *dual* representation as follows: Let $V^* = Hom_{\mathbf{C}}(V, \mathbf{C})$ and $\langle , \rangle : V^* \times V \to \mathbf{C}$ be the natural pairing. If $X \in \mathfrak{g}$, $\xi \in V^*$, $\eta \in V$, then we define a right-$\mathfrak{g}$ action by $\langle \xi X, \eta \rangle = \langle \xi, X\eta \rangle$. Then we set

$$\rho^*(X)(\xi) = -\xi X.$$

HW: Verify that this indeed gives a Lie algebra representation.

HW: Prove that the dual representation to $\rho : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(V)$ is isomorphic to ρ itself.

Notation Change: From now on, V_λ will be the finite-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbf{C})$ with highest weight λ. Let $V = V_1$, the standard representation $\rho : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(2, \mathbf{C})$. This is irreducible. Then

$$V \otimes V = V_2 \oplus V_0,$$
$$V \otimes V \otimes V = V_2 \oplus V_1 \oplus V_1,$$
$$V \otimes V \otimes V \otimes V = V_4 \oplus 3V_2 \oplus 2V_0.$$

HW: Decompose $V^{\otimes n}$ in general.

In particular, the representations V_λ for $\lambda = 0, 1, 2, \ldots$ are all constructed as subrepresentations of $V^{\otimes n}$.

3. Day 3

3.1. **Clebsch-Gordan rule.** Let V, W be $\mathfrak{g}$-modules. Then $Hom_{\mathfrak{g}}(V,W)$ denotes the $\mathfrak{g}$-linear homomorphisms $\phi : V \to W$. This means that ϕ is a **C**-linear map and $\phi(Xv) = X\phi(v)$ for all $X \in \mathfrak{g}$.

Lemma 3.1 (Schur's Lemma)**.** *Given finite-dimensional irreducible $\mathfrak{g}$-modules V and W,*

$$Hom_{\mathfrak{g}}(V,W) \simeq \mathbf{C}$$

iff $V \simeq W$ as $\mathfrak{g}$-modules. Otherwise, $Hom_{\mathfrak{g}}(V,W) \simeq 0$.

Proof. Given a nontrivial $\phi : V \to W$, both $\ker \phi$ and $\phi(V)$ are $\mathfrak{g}$-modules. This is not possible unless $\ker \phi = 0$ and ϕ is onto, since V and W are irreducible. Hence ϕ is an isomorphism.

We will now show that there is only one $\mathfrak{g}$-linear isomorphism $\phi : V \to V$, namely a multiple of the identity. Since V is finite-dimensional, there is a nonzero vector $v \in V$ satisfying $\phi(v) = \lambda v$. Now, $\phi - \lambda \cdot id$ has nontrivial kernel, since v is in it. Since V is irreducible, $V = \ker(\phi - \lambda \cdot id)$ and $\phi = \lambda \cdot id$. □

Now consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$.

Theorem 3.2 (Clebsch-Gordan rule)**.** $Hom_{\mathfrak{g}}(V_i \otimes V_j \otimes V_k, \mathbf{C}) \simeq \mathbf{C}$ *iff the following hold:*

(1) $i + j + k$ *is even;*
(2) $i \leq j + k; j \leq k + i; k \leq i + j.$

Observe that, since $V_i^* = V_i$ for $\mathfrak{sl}(2, \mathbf{C})$,

$$Hom_{\mathfrak{g}}(V_i \otimes V_j \otimes V_k, \mathbf{C}) \simeq Hom_{\mathfrak{g}}(V_i \otimes V_j, V_k^*) \simeq Hom_{\mathfrak{g}}(V_i \otimes V_j, V_k).$$

In other words, we are asking whether there is a unique factor of V_k inside the tensor product $V_i \otimes V_j$.

Illustrative Example: $V_5 \otimes V_7 \simeq V_{12} \oplus V_{10} \oplus V_8 \oplus V_6 \oplus V_4 \oplus V_2$. Hence

$$Hom_{\mathfrak{g}}(V_5 \otimes V_7 \otimes V_k, \mathbf{C}) \simeq \mathbf{C}$$

iff $k = 2, 4, 6, 8, 10, 12$, which is consistent with the Clebsch-Gordan rule.

Suggestive Notation: We draw a trivalent (directed) graph with one vertex and three edges. Two of the edges (labeled i and j) are incoming and one edge (labeled k) is outgoing. It is supposed to suggest particle interaction.

HW: Do the same for $Hom_{\mathfrak{g}}(V_i \otimes V_j \otimes V_k \otimes V_l, \mathbf{C})$.

3.2. $SL(3, \mathbf{C})$**.** We will now study the finite-dimensional irreducible representations of $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$. Let E_{ij} be the $n \times n$ matrix with 1 in the ij-th position and 0 elsewhere.

We first examine the adjoint representation.

Decompose $\mathfrak{sl}(3, \mathbf{C})$ into $\mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where

$$\mathfrak{h} = \mathbf{C}\{E_{11} - E_{22}, E_{22} - E_{33}\},$$
$$\mathfrak{n}_+ = \mathbf{C}\{E_{12}, E_{23}, E_{13}\},$$
$$\mathfrak{n}_- = \mathbf{C}\{E_{21}, E_{32}, E_{31}\}.$$

Here $\mathfrak{h}$ consists of the diagonal matrices, $\mathfrak{n}_+$ consists of the strictly upper triangular matrices, and $\mathfrak{n}_-$ consists of the strictly lower triangular matrices.

Consider the action of $\mathfrak{h}$ on $\mathfrak{sl}(3, \mathbf{C})$ via the adjoint action. $\mathfrak{h}$ is killed by $ad(\mathfrak{h})$ and $\mathfrak{h}$ acts (simultaneously) diagonally on $\mathfrak{sl}(3, \mathbf{C})$.

We compute that

$$ad(E_{11} - E_{22}) : \mathfrak{h} \mapsto 0, E_{12} \mapsto 2E_{12}, E_{23} \mapsto -E_{23}, E_{13} \mapsto E_{13},$$

and

$$ad(E_{22} - E_{33}) : \mathfrak{h} \mapsto 0, E_{12} \mapsto -E_{12}, E_{23} \mapsto 2E_{23}, E_{13} \mapsto E_{13}.$$

(The calculations for $\mathfrak{n}_-$ are similar.)

Now let $\mathfrak{h}^*$ be the dual of $\mathfrak{h}$. If L_i maps the diagonal matrix $diag(a_1, a_2, a_3)$ to a_i, then $\mathfrak{h}^* = \mathbf{C}\{L_1, L_2\} = \mathbf{C}\{L_1, L_2, L_3\}$.

We verify that, on $\mathbf{C}\{E_{12}\}$,

$$ad(H)(E_{12}) = (L_1 - L_2)(H) \cdot E_{12},$$

for all $H \in \mathfrak{h}^*$. In other words, $\mathbf{C}\{E_{12}\}$ is the one-dimensional eigenspace on which $\mathfrak{h}$ acts by $L_1 - L_2 \in \mathfrak{h}^*$. We write $\mathfrak{g}_{L_1 - L_2}$ for $\mathbf{C}\{E_{12}\}$. Therefore, $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{L_i - L_j} \mathfrak{g}_{L_i - L_j}),$$

where the sum is over all $i \neq j$ and $\mathfrak{g}_{L_i - L_j} = \mathbf{C}\{E_{ij}\}$.

Diagram for the roots: We can draw a diagram which represents the configuration of roots in $\mathfrak{h}^* = \mathbf{R}^2$. Usually, we take L_1, L_2, L_3 to be at the third roots of unity (in $\mathbf{R}^2 \simeq \mathbf{C}$). (See, for example, Fulton-Harris for pretty diagrams.)

$\mathfrak{h}$ is called the *Cartan subalgebra*. In general, for a semisimple $\mathfrak{g}$, $\mathfrak{h}$ is the maximal abelian subalgebra consisting of semisimple elements (X semisimple = $ad(X) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable). The elements $L_i - L_j \in \mathfrak{h}^*$ are called *roots*. $\mathfrak{g}_{L_i - L_j}$ is the *root space* corresponding to the root $L_i - L_j$.

4. More on $\mathfrak{sl}(3, \mathbf{C})$-representations

Let V be a finite-dimensional, irreducible $\mathfrak{sl}(3, \mathbf{C})$-module. Much of what we say will generalize readily to other semisimple Lie algebras $\mathfrak{g}$.

Fact (without proof): V can be simultaneously diagonalized under the action of $\mathfrak{h}$.

We write $V = \oplus V_\lambda$, where V_λ is the eigenspace for which $v \in V_\lambda$ satisfies $Hv = \lambda(H) \cdot v$ for all $H \in \mathfrak{h}$, and λ runs over a finite subset of $\mathfrak{h}^*$. The λ for which $V_\lambda \neq 0$ are called the *weights*. The corresponding V_λ are the *weight spaces*.

Lemma 4.1. *$\mathfrak{g}_\alpha$ maps V_λ to $V_{\lambda+\alpha}$.*

Proof. If $v \in V_\lambda$, then for example we have:

$$HE_{12}v = E_{12}Hv + [H, E_{12}]v = \lambda(H)E_{12}v + (L_1 - L_2)(H)(E_{12}v) = (\lambda + (L_1 - L_2))(H)(E_{12}v).$$

□

Since V is finite-dimensional, by successively applying elements of $\mathfrak{n}_+ = \mathbf{C}\{E_{12}, E_{23}, E_{13}\}$, we eventually obtain a nontrivial V_λ which is annihilated by $\mathfrak{n}_+$. (Remark: If v is annihilated by the first two, then it is also annihilated by the last.)

λ is then the *highest weight* and $v \in V_\lambda$ is the highest weight vector.

Fact: A highest weight vector v generates an irreducible representation by successively applying elements in $\mathfrak{n}_- = \mathbf{C}\{E_{21}, E_{32}, E_{31}\}$. (You don't need elements in $\mathfrak{n}_+$.)

Proof. One needs to generalize the following argument: $E_{12}v = 0$ and

$$E_{12}E_{21}v = E_{21}E_{12}v + [E_{12}, E_{21}]v = 0 + H_{12}v = \lambda(H_{12})v,$$

where $H_{12} = E_{11} - E_{22} = [E_{12}, E_{21}]$. In general, given a word W in $\mathfrak{n}_-$ and $E_{12} \in \mathfrak{n}_+$, say, we use the commutation relations to prove that $E_{12}Wv$ can be written as $W'v$ for some word W' in $\mathfrak{n}_-$ by induction. □

Claim. *The distribution of λ's in $\mathfrak{h}^*$ corresponding to weights is symmetric about the lines (=hyperplanes) $\Omega_{12} = \{\alpha \in \mathfrak{h}^* | \ \alpha(H_{12}) = 0\}$, $\Omega_{23} = \{\alpha \in \mathfrak{h}^* | \ \alpha(H_{23}) = 0\}$, and $\Omega_{13} = \{\alpha \in \mathfrak{h}^* | \ \alpha(H_{13}) = 0\}$.*

Proof. As usual, we will treat a special case. Start with a highest weight vector $v \in V_\lambda$ and successively apply E_{21}. Observe that E_{12}, E_{21} and H_{12} generate an $\mathfrak{sl}(2, \mathbf{C}) \hookrightarrow \mathfrak{sl}(3, \mathbf{C})$. Hence, the string

$$V_\lambda \oplus V_{\lambda-(L_1-L_2)} \oplus V_{\lambda-2(L_1-L_2)} \oplus \dots$$

must be an $\mathfrak{sl}(2, \mathbf{C})$-representation and the values

$$\{\lambda(H_{12}), (\lambda - (L_1 - L_2))(H_{12}) = \lambda(H_{12}) - 2, \dots\}$$

must be a set of integers which are symmetric about 0.

Also observe that

$$E_{23}E_{21}v = E_{21}E_{23}v + [E_{23}, E_{21}]v = 0,$$

since $E_{23}v = 0$ by the highest weight condition. Similarly, $E_{23}^i E_{21}v = 0$. This implies that $v, E_{21}v, E_{21}^2 v$, etc. form an edge of a polygon P which delineates the weights of V. $\square$

Remark: Ω_{12} is spanned by $L_1 + L_2$. We can verify that $L_1 + L_2$ is orthogonal to $L_1 - L_2$ with respect to the Killing form, described below. Hence, reflections about $L_1 + L_2$, $L_2 + L_3$, and $L_1 + L_3$ are all symmetries of the set of weights of V. These involutions generate the *Weyl group* of the Lie algebra $\mathfrak{g}$.

Remark: Also note that $\lambda(H_{12})$ and $\lambda(H_{23})$ must be integers (using what we know about $\mathfrak{sl}(2, \mathbf{C})$-representations). For $\mathfrak{sl}(3, \mathbf{C})$, the set of all $\alpha \in \mathfrak{h}^*$ which are integer-valued on H_{12} and H_{23} is spanned by L_1 and L_2. Hence all weights λ lie on the *weight lattice* Λ_W generated by the L_i.

Remark: See Fulton-Harris for pictures of P.

FS: The multiplicities (= dimension) of the V_α on the edge of the polygon P are all one. However, the multiplicities in the interior are not always one, and require further study. See Fulton-Harris, for example.

Define a *Weyl chamber* $\mathcal{W}$ to be the closure of a connected component of $\mathfrak{h}^* - \Omega_{12} - \Omega_{23} - \Omega_{13}$.

Theorem 4.2. *There is a 1-1 correspondence between finite-dimensional irreducible representations of* $\mathfrak{sl}(3, \mathbf{C})$ *and points* α *in* $\Lambda_W \cap \mathcal{W}$. *The representation* V *corresponding to* α *has a highest weight vector with weight* α *and* $\dim V_\alpha = 1$.

4.1. **The Killing form.** We conclude this lecture by discussing a symmetric bilinear form on $\mathfrak{g}$, called the *Killing form*. The observant student/reader may have already noticed some sort of inner product lurking in $\mathfrak{h}^*$.

Definition 4.3. *The* Killing form *on* $\mathfrak{g}$ *is a bilinear form on* $\mathfrak{g}$ *given by* $\langle X, Y\rangle \stackrel{def}{=} Tr(ad(X) \circ ad(Y))$.

HW: Prove that the Killing form is symmetric.

$\mathfrak{g}$ is said to be *semisimple* if the Killing form is nondegenerate.

HW: Prove that, on $\mathfrak{sl}(n, \mathbf{C})$, $\langle X, Y\rangle = 2nTr(XY)$. (Observe that this is much easier to calculate directly. In particular, on $\mathfrak{h}$, the Killing form is, up to a scaling constant, inherited from the standard inner product.)

HW: Prove that the Killing form is nondegenerate on $\mathfrak{sl}(n, \mathbf{C})$.

To do the above HW, it helps to understand the Killing form with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha)$. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, and $\alpha + \beta \neq 0$, then $ad(X) \circ ad(Y)$ maps $\mathfrak{g}_\gamma$ to

$\mathfrak{g}_{\gamma+\alpha+\beta} \neq \mathfrak{g}_\gamma$. Hence if we take the trace, we get zero! The Killing form is a (potentially) nonzero pairing only on $\mathfrak{h} \times \mathfrak{h}$ and $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$. (If the Killing form is nondegenerate, the above pairings are also nondegenerate.)

Finally, the nondegenerate pairing on $\mathfrak{h}$ induces a nondegenerate pairing on $\mathfrak{h}^*$ via the natural isomorphism:

$$\mathfrak{h} \to \mathfrak{h}^*,$$
$$X \mapsto \alpha : \alpha(Y) = \langle X, Y \rangle \ \forall Y.$$

HW: Verify that L_1, L_2, L_3 in $\mathfrak{h}^*$ have equal lengths and the angle between L_1 and L_2 is $\frac{2\pi}{3}$ with respect to the Killing form.

5. AFFINE LIE ALGEBRAS

The following fact will play an important role today:

Theorem 5.1. *The Killing form on a semisimple Lie algebra $\mathfrak{g}$ is, up to a scaling constant, the unique bilinear form which is Ad-invariant. In particular, this automatically implies that the bilinear form is* symmetric *as well.*

The Ad-invariance can be translated into:

$$\langle [X,Z],Y\rangle = \langle X,[Z,Y]\rangle.$$

Beginning of proof: Suppose $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, and $Z \in \mathfrak{h}$. Then

$$\langle -\alpha(Z)X, Y\rangle = \langle X, \beta(Z)Y\rangle.$$

If $\langle X,Y\rangle \neq 0$, then $\alpha = -\beta$. Therefore, $\langle,\rangle$ is nonzero only for $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{C}$ and $\mathfrak{h}\times\mathfrak{h} \to \mathbf{C}$. (Recall this is also the case for the Killing form.) □

5.1. Central extensions of a Lie algebra.

Let $\mathfrak{g}$ be a complex Lie algebra. We study *central extensions* of $\mathfrak{g}$:

$$0 \to \mathbf{C}c \to \mathfrak{g}' \to \mathfrak{g} \to 0.$$

As a vector space, $\mathfrak{g}' = \mathfrak{g} \oplus \mathbf{C}c$. Define $[,]$ on $\mathfrak{g}'$ so that:

(i) $[c,X]=0$ for all $X \in \mathfrak{g}'$ (i.e., c is a central element).
(ii) $[X+\alpha c, Y+\beta c] = [X,Y] + \omega(X,Y)c$,

if $X,Y \in \mathfrak{g}$, $\alpha,\beta \in \mathbf{C}$, and $\omega : \mathfrak{g}\times\mathfrak{g} \to \mathbf{C}$ is a bilinear form.

Claim. $[,]$ *is a Lie bracket iff*

(1) ω *is skew-symmetric, and*
(2) $\omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) = 0.$

(2) is called the *2-cocyle condition*.

We now explain the classification of central extensions via Lie algebra cohomology.

Given a Lie algebra $\mathfrak{g}$, define the p-th cochain group:

$$C^p(\mathfrak{g},\mathbf{C}) = Hom_{\mathbf{C}}(\wedge^p \mathfrak{g}, \mathbf{C}).$$

The coboundary $d_p : C^p(\mathfrak{g},\mathbf{C}) \to C^{p+1}(\mathfrak{g},\mathbf{C})$ is given by:

$$d_p\omega(X_1,\dots,X_p) = \sum_{i<j} (-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1}).$$

The p-th Lie algebra cohomology group is $H^p(\mathfrak{g},\mathbf{C}) = \ker d_p/\text{im } d_p$.

HW: Verify that $d_p \circ d_{p-1} = 0$.

Remark: The coboundary map d_p coincides with the exterior derivative (Cartan's formula?). In the context of left-invariant forms and vector fields, terms of the form $X_i\omega(X_1,\dots,\hat{X}_i,\dots,X_{p+1})$ vanish.

Theorem 5.2. *There is a 1-1 correspondence between isomorphism classes of central extensions of* $\mathfrak{g}$ *and elements of* $H^2(\mathfrak{g},\mathbf{C})$.

Proof. Any $\omega\in C^2(\mathfrak{g},\mathbf{C})$ with $d\omega=0$ satisfies (1), (2) above.

If $\eta\in C^1(\mathfrak{g},\mathbf{C})$, then $d\eta(X,Y)=\eta([X,Y])$.

Consider a Lie algebra isomorphism ϕ of central extensions. As a vector space isomorphism, $\phi:\mathfrak{g}\oplus\mathbf{C}c\xrightarrow{\sim}\mathfrak{g}\oplus\mathbf{C}c$ maps $(0,1)\mapsto(0,1)$ (i.e., $c\mapsto c$), and $(X,0)\mapsto(X,\eta(X))$ for some linear functional $\eta:\mathfrak{g}\mapsto\mathbf{C}$.

If $[(X,0),(Y,0)]=([X,Y],\omega(X,Y))$ for the source and $[(X,0),(Y,0)]=([X,Y],\omega'(X,Y))$ for the target, then

$$\phi([(X,0),(Y,0)])=[(X,\eta(X)),(Y,\eta(Y))]=([X,Y],\omega'(X,Y)),$$

whereas

$$\phi([(X,0),(Y,0)])=\phi(([X,Y],\omega(X,Y)))=([X,Y],\omega(X,Y)+\eta([X,Y])).$$

Hence, $\omega'(X,Y)=\omega(X,Y)+\eta([X,Y])$. □

5.2. **The loop algebra.** Let G be a Lie group and $\mathfrak{g}$ be its (complexified) Lie algebra. We define the *loop group* LG as the space of smooth maps from S^1 to G, equipped with a group structure as follows: given $\gamma_1,\gamma_2:S^1\to G$, define $(\gamma_1\gamma_2)(t)=\gamma_1(t)\cdot\gamma_2(t)$. (This is not to be confused with the product/concatenation of paths.) Question: What is the identity element?

Remark: At this point, we will not be concerned with topologies on LG.

Next $L\mathfrak{g}$ is the tangent space $T_e(LG)$. By appealing to Fourier series, we define $L\mathfrak{g}=\mathfrak{g}\otimes\mathbf{C}((t))$, i.e., the Laurent series with values in $\mathfrak{g}$. Here $\mathbf{C}((t))$ consists of elements of the form $\sum_{i=n}^{\infty}a_it^i$ for some $n\in\mathbf{Z}$. The Lie bracket is

$$[X\otimes f,Y\otimes g]=[X,Y]\otimes fg.$$

Remark: We are thinking of $S^1=\{|t|=1\}\subset\mathbf{C}$, i.e., $t=e^{i\theta}$. Hence $\mathbf{C}((t))$ is a reasonable class of meromorphic functions on $\mathbf{C}$, and we are taking the restriction to S^1, which is effectively a Fourier series.

Theorem 5.3. *Suppose* G *is a connected, compact Lie group, with corresponding Lie algebra* $\mathfrak{g}$. *Then* $H^2(L\mathfrak{g},\mathbf{C})\simeq\mathbf{C}$, *and a nontrivial element is given by*

$$\omega(X\otimes f,Y\otimes g)=\langle X,Y\rangle Res_{t=0}(df\cdot g),$$

where $Res_{t=0}(\sum c_it^i)=c_{-1}$.

Another way of writing ω is:

$$\omega(X \otimes t^m, Y \otimes t^n) = m\delta_{m,-n}\langle X, Y\rangle.$$

Here, $\delta_{a,b} = 1$ if $a = b$ and 0 if $a \neq b$.

Let us denote by $\widetilde{L\mathfrak{g}}$ the central extension given by ω. Then:

$$[X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes t^{m+n} + \langle X, Y\rangle m\delta_{m,-n}c.$$

Proof. .

Step 1: *Represent any* $[\omega] \in H^2(L\mathfrak{g}, \mathbf{C})$ *by a 2-cocycle which is invariant under conjugation by* G.

Writing $g = I + tZ$, and taking $X, Y \in L\mathfrak{g}$, we compute:

$$\begin{aligned} \omega(g^{-1}Xg, g^{-1}Yg) - \omega(X, Y) &= \omega((I - tZ)X(I + tZ), (I - tZ)Y(I + tZ)) - \omega(X, Y) \\ &= t(\omega([X, Z], Y) + \omega(X, [Y, Z])) = t\omega(Z, [X, Y]), \end{aligned}$$

where the last equality uses the 2-cocycle condition. (Notice that in the computation, $\otimes t^m$ are independent of Ad.) If we define $\alpha_Z(V) = \omega(Z, V)$, then $d\alpha_Z(X, Y) = \alpha_Z([X, Y]) = \omega(Z, [X, Y])$, and we see that $\omega_g(X, Y) = \omega(g^{-1}Xg, g^{-1}Yg)$ is cohomologous to ω by integrating. Finally, average by taking $\int_{g\in G} \omega_g dg$. Since G is assumed to be a *compact* Lie group, the resulting 2-cocycle is invariant under $Ad(G)$.

Step 2: Let $\alpha : \wedge^2\mathfrak{g} \to \mathbf{C}$ be a 2-cocycle which is invariant under $Ad(G)$. Define $\alpha_{m,n} : \mathfrak{g}\times\mathfrak{g} \to \mathbf{C}$ by $\alpha_{m,n}(X, Y) = \alpha(X \otimes t^m, X \otimes t^n)$. *Since* $\alpha_{m,n}$ *is* Ad*-invariant,* $\alpha_{m,n}$ *is a multiple of the Killing form and is symmetric!*

We then have $\alpha_{m,n} = -\alpha_{n,m}$ (by the anti-symmetry of α and the symmetry of $\alpha_{m,n}$), and also

$$\alpha_{m+n,p} + \alpha_{n+p,m} + \alpha_{p+m,n} = 0,$$

by the 2-cocycle condition:

$$\alpha([X \otimes t^m, Y \otimes t^n], Z \otimes t^p) + \alpha([Y \otimes t^n, Z \otimes t^p], X \otimes t^m) + \alpha([Z \otimes t^p, X \otimes t^m], Y \otimes t^n) = 0.$$

Specializing at various values (i) $n = p = 0$, (ii) $p = -m - n$, (iii) $p = q - m - n$, eventually gives us that $\alpha_{m,n} = m\delta_{m,-n}\alpha_{1,-1}$. Finally observe that $\alpha_{1,-1}$ is the Killing form up to a constant multiple.

Step 3: (Nontriviality) Let $\omega(X \otimes t^m, Y \otimes t^n) = \langle X, Y\rangle m\delta_{m,-n}$. If $\omega = d\alpha$, then

$$\omega(H \otimes t, H \otimes t^{-1}) = \alpha([H \otimes t, H \otimes t^{-1}]) = \alpha([H, H]) = 0,$$

whereas

$$\omega(H \otimes t, H \otimes t^{-1}) = \langle H, H\rangle,$$

and there are $H \in \mathfrak{h}$ with $\langle H, H\rangle \neq 0$. $\square$

5.3. **The Virasoro algebra.** We discussed this material on the next day, but this material is probably better placed here. Let Diff(S^1) be the group of diffeomorphisms of the unit circle $S^1 = \{|z| = 1\} \subset \mathbf{C}$. One possible tangent space to Diff(S^1) at the identity is the Lie algebra $A = \{f(z)\frac{d}{dz} |\ f(z) \in \mathbf{C}[z, z^{-1}]\}$ of Laurent polynomial vector fields.

Write $L_m = -z^{m+1}\frac{d}{dz}$. Then

$$\begin{aligned} [L_m, L_n] &= \left[-z^{m+1}\frac{d}{dz}, -z^{n+1}\frac{d}{dz}\right] \\ &= ((n+1)z^{m+n+1} - (m+1)z^{m+n+1})\frac{d}{dz} \\ &= -(m-n)z^{m+n+1}\frac{d}{dz} = (m-n)L_{m+n}. \end{aligned}$$

Just as in the case of the loop algebra $L\mathfrak{g}$, we have:

Theorem 5.4. $H^2(A;\mathbf{C}) \simeq \mathbf{C}$ *and a representative of a nonzero class is:* $\omega(L_m, L_n) = \frac{m^3-m}{12}\delta_{m,-n}$.

For details of the proof which is very similar to Theorem 5.3, see Kohno. The *Virasoro algebra*, as a vector space, is $Vir = A \oplus \mathbf{C}c$, and the Lie bracket is given by:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}c.$$

References

[1] T. Kohno, *Conformal field theory and topology*. (We closely followed his presentation in this lecture.)

6. Affine Lie algebras, Day II

6.1. **The affine Lie algebra.** Last time we started with the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbf{C}((t))$ and constructed the (essentially unique central extension $\widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{C}c$ with Lie bracket

$$[X \otimes t^m, Y \otimes t^n] = [X,Y] \otimes t^{m+n} + m\delta_{m,-n}\langle X, Y\rangle c.$$

We will (slightly) enlarge $\widetilde{L\mathfrak{g}}$ to the *affine Lie algebra* $\hat{\mathfrak{g}}$. Consider the derivation d on $\widetilde{L\mathfrak{g}}$ given by $d(X \otimes t^m) = X \otimes mt^m$ and $d(c) = 0$. (In other words, $d = t\frac{d}{dt}$.)

HW: Show that $\frac{d}{dt}$ is not a derivation of $\widetilde{L\mathfrak{g}}$, but $t\frac{d}{dt}$ is. (A *derivation*, by definition, satisfies $d[\xi,\eta] = [d\xi,\eta] + [\xi, d\eta]$.)

As a vector space, the *affine Lie algebra* $\hat{\mathfrak{g}}$ is given by:

$$\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \oplus \mathbf{C}d = (\mathfrak{g} \otimes \mathbf{C}((t))) \oplus \mathbf{C}c \oplus \mathbf{C}d.$$

The Lie bracket extends $[,]$ for $\widetilde{L\mathfrak{g}}$ via $[c,d] = 0$, and $[d, X \otimes t^m] = mX \otimes t^m$. (In other words, $\hat{\mathfrak{g}}$ is the *semidirect product* of $\widetilde{L\mathfrak{g}}$ and $\mathbf{C}d$.)

Remark: The definition of $\hat{\mathfrak{g}}$ is different from that of Kohno. He calls $\widetilde{L\mathfrak{g}}$ the affine Lie algebra. Observe that $\widetilde{L\mathfrak{g}}$ is $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$.

6.2. **Root space decomposition.** An important reason for extending to $\hat{\mathfrak{g}}$ is the following: Define $\hat{\mathfrak{h}} = (\mathfrak{h} \otimes 1) \oplus \mathbf{C}c \oplus \mathbf{C}d$. Then we have:

Lemma 6.1. *$\hat{h}$ is a maximal abelian Lie subalgebra.*

Proof. Take an element $X \otimes t^m$ which commutes with $\hat{\mathfrak{h}}$. Then $[d, X \otimes t^m] = X \otimes mt^m = 0$ implies $m = 0$, and $[X \otimes 1, \mathfrak{h} \otimes 1] = 0$ implies $X \in \mathfrak{h}$. □

We now decompose $\hat{\mathfrak{g}}$ into root spaces via the action of $\hat{\mathfrak{h}}$. Let Δ be the set of roots for $\mathfrak{g}$. Define $\gamma, \delta : \hat{\mathfrak{h}} \to \mathbf{C}$ by:

$$\gamma(\mathfrak{h} \otimes 1) = 0, \gamma(c) = 1, \gamma(d) = 0,$$
$$\delta(\mathfrak{h} \otimes 1) = 0, \delta(c) = 0, \delta(d) = 1.$$

Then

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus (\oplus_{\beta \in \Delta_{\text{aff}}} \hat{\mathfrak{g}}_\beta),$$

where the set $\Delta_{\text{aff}} \subset \hat{\mathfrak{h}}^*$ of *affine roots* is:

$$\Delta_{\text{aff}} = \{\alpha + n\delta |\ \alpha \in \Delta, n \in \mathbf{Z}\}.$$

The corresponding root spaces $\hat{\mathfrak{g}}_{\alpha+n\delta}$ are $\mathbf{C}\{X_\alpha \otimes t^n\}$, where $X_\alpha \in \mathfrak{g}_\alpha$.

Remark: A good way to picture the root space decomposition for $\widehat{\mathfrak{sl}(2,\mathbf{C})}$ is to place α corresponding to E on the x-axis and δ on the y-axis. (γ is in the z-direction.) The roots are all of the form $\alpha + n\delta$, $n\delta$, or $-\alpha + n\delta$.

As before, define

$$\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus (\mathfrak{g} \otimes t) \oplus (\mathfrak{g} \otimes t^2) \oplus ...$$
$$\hat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus (\mathfrak{g} \otimes t^{-1}) \oplus (\mathfrak{g} \otimes t^{-2}) \oplus ....$$

For $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, $\hat{\mathfrak{n}}_+$ is generated by the roots $\alpha_0 = -\alpha + \delta$ and $\alpha_1 = \alpha$.

6.3. **The invariant bilinear form.** We now define an invariant bilinear form $\langle , \rangle$ on $\hat{\mathfrak{g}}$. Let

$$\langle X \otimes t^m, Y \otimes t^n \rangle = \langle X, Y \rangle \delta_{m,-n},$$
$$\langle c, c \rangle = \langle d, d \rangle = 0, \langle c, d \rangle = 1.$$

Here $X, Y \in \mathfrak{g}$ and $\langle X, Y \rangle$ is the Killing form on $\mathfrak{g}$. (Observe that the first definition makes sense because we want to pair $\hat{\mathfrak{g}}_{\alpha+n\delta}$ with $\hat{\mathfrak{g}}_{-\alpha-n\delta}$.)

Lemma 6.2. *The invariant bilinear form on $\hat{\mathfrak{g}}$ is invariant.*

Proof. We will work out one case.

$$\begin{aligned} \langle [X \otimes t^m, Y \otimes t^n], d \rangle &= \langle [X, Y] \otimes t^{m+n} + \langle X, Y \rangle m \delta_{m,-n} c, d \rangle \\ &= \langle X, Y \rangle m \delta_{m,-n}, \end{aligned}$$

whereas

$$\begin{aligned} \langle X \otimes t^m, [Y \otimes t^n, d] \rangle &= \langle X \otimes t^m, -Y \otimes n t^n \rangle \\ &= \langle X, Y \rangle (-n) \delta_{m,-n} = \langle X, Y \rangle m \delta_{m,-n} \end{aligned}$$

□

Remark: For $\mathfrak{sl}(2, \mathbf{C})$ and $\mathfrak{sl}(3, \mathbf{C})$, the Killing form on $\mathfrak{h}$ was positive definite, and the Weyl group was a subgroup of $O(n)$. For $\hat{\mathfrak{h}}$, the invariant bilinear form is no longer positive definite, and the corresponding affine Weyl group is a subgroup of $O(n, 1)$.

7. Coroots, Weyl group, etc.

Before we explain the representation theory of affine Lie algebras, we need to present some more theory.

7.1. **Coroots.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\langle , \rangle$ be the Killing form, and $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$ be the root space decomposition.

Let $E_\alpha \in \mathfrak{g}_\alpha, E_{-\alpha} \in \mathfrak{g}_{-\alpha}$, and $H \in \mathfrak{h}$. (Assume $E_\alpha, E_{-\alpha} \neq 0$.) Then we have:

$$\begin{aligned} \langle [E_\alpha, E_{-\alpha}], H\rangle &= \langle E_\alpha, [E_{-\alpha}, H]\rangle \\ &= \langle E_\alpha, \alpha(H)E_{-\alpha}\rangle \\ &= \alpha(H)\langle E_\alpha, E_{-\alpha}\rangle. \end{aligned}$$

Since the root spaces α are 1-dimensional (one needs to verify this for general semisimple $\mathfrak{g}$) and the Killing form is nondegenerate on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$, it follows that $[E_\alpha, E_{-\alpha}] \neq 0$.

Normalize so that $\langle E_\alpha, E_{-\alpha}\rangle = 1$. Then define $H_\alpha = [E_\alpha, E_{-\alpha}]$. H_α then satisfies:

$$\langle H_\alpha, H\rangle = \alpha(H),$$

so the natural map $\mathfrak{h} \to \mathfrak{h}^*$ given by the Killing form sends $H_\alpha \mapsto \alpha$.

HW: $E_\alpha, E_{-\alpha}, H_\alpha$ generate a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbf{C})$ inside $\mathfrak{g}$.

Given a root α, we can define its *coroot* $\check{\alpha} = \frac{2H_\alpha}{\langle\alpha,\alpha\rangle}$. Given a set of simple roots $\alpha_1, \dots, \alpha_n$ (a minimal set which generates $\mathfrak{n}_+$), we have the corresponding coroots $\check{\alpha}_1, \dots, \check{\alpha}_n$. For example, for $\mathfrak{sl}(3, \mathbf{C})$ we had $\alpha_1 = L_1 - L_2, \alpha_2 = L_2 - L_3$ and $\check{\alpha}_1 = E_{11} - E_{22}$, $\check{\alpha}_2 = E_{22} - E_{33}$. We can define the *Cartan matrix*

$$(C_{ij}) = (\alpha_j(\check{\alpha}_i)).$$

FS: (1) The Cartan matrix encodes all of the structure of a semisimple Lie algebra. (2) Cartan matrices can be classified for $\langle , \rangle$ positive definite on $\mathfrak{h}$.

7.2. **The Weyl group.** The *Weyl group* is the group of reflections of $\mathfrak{h}^*$ (i.e., elements of $O(\mathfrak{h}^*)$), generated by:

$$W_\alpha : \lambda \mapsto \lambda - \lambda(\check{\alpha})\alpha.$$

W_α preserves the hyperplane $\{\lambda|\ \lambda(\check{\alpha}) = 0\}$ and maps $\alpha \mapsto -\alpha$. Since $\lambda(\check{\alpha}) = \lambda(\frac{2H_\alpha}{\langle\alpha,\alpha\rangle}) = \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$, we see that the hyperplane is orthogonal (with respect to the Killing form) to α. Also note that α gets mapped to $-\alpha$.

Recall that if we have a string of weights

$$\lambda, \lambda - \alpha, \lambda - 2\alpha, \dots,$$

for a $\mathfrak{g}$-module V, then

$$\lambda(\check{\alpha}), (\lambda - \alpha)(\check{\alpha}), (\lambda - 2\alpha)(\check{\alpha}), \dots$$

is symmetric about 0 (since the direct sum of the corresponding weight spaces is an $\mathfrak{sl}(2,\mathbf{C})$-representation). Here we are assuming that $\lambda+\alpha$ is not a weight, i.e., λ is extremal. Hence the set of weights of V is invariant under the Weyl group.

7.3. **The affine Lie algebra $\widehat{\mathfrak{sl}(2,\mathbf{C})}$.** Let $\mathfrak{g}=\mathfrak{sl}(2,\mathbf{C})$. Then the set of simple roots for $\hat{\mathfrak{g}}$ was $\alpha_0=\delta-\alpha$, $\alpha_1=\alpha$, where α was the simple root for $\mathfrak{g}$.

Take $E_{\alpha_0}=F\otimes t$, $E_{-\alpha_0}=E\otimes t^{-1}$. Then

$$\langle F\otimes t, E\otimes t^{-1}\rangle=\langle F,E\rangle=1,$$

and

$$\check{\alpha}_0=H_{\alpha_0}=[F\otimes t,E\otimes t^{-1}]=-H\otimes 1+\langle F,E\rangle c=-H+c.$$

Similarly, $\check{\alpha}_1=H_{\alpha_1}=H$.

HW: Compute the Cartan matrix for $\mathfrak{sl}(3,\mathbf{C})$ and $\widehat{\mathfrak{sl}(2,\mathbf{C})}$.

The Weyl group for $\hat{\mathfrak{g}}$ is called the *affine Weyl group*, and is generated by: $W_0=W_{\alpha_0}$ and $W_1=W_{\alpha_1}$. If we write $x_0\alpha+x_1\gamma+x_2\delta$ by (x_0,x_1,x_2), then we compute that:

$$W_0(x_0,x_1,x_2)=(-x_0+x_1,x_1,2x_0-x_1+x_2),$$

$$W_1(x_0,x_1,x_2)=(-x_0,x_1,x_2).$$

W_1 is just a simple reflection which switches x_0 and $-x_0$, while W_0 is harder to understand.

As a symmetry on the set of roots, we have $x_1=0$, so W_0 maps $(x_0,0,x_2)\mapsto(-x_0,0,2x_0+x_2)$. Composing with W_1, we get $(x_0,0,x_2)\mapsto(x_0,0,2x_0+x_2)$, which we can visualize on the root space. It certainly is not an element of finite order, unlike elements of the Weyl group for $\mathfrak{g}$ finite-dimensional and semisimple.

7.4. **The universal enveloping algebra.** Let $\mathfrak{g}$ be a Lie algebra. Then its *universal enveloping algebra* $U(\mathfrak{g})$ is a $\mathbf{C}$-algebra (with unit), constructed as follows. Take the tensor algebra $T(\mathfrak{g})=\oplus_{i=0}^{\infty}\mathfrak{g}^{\otimes i}$, where $\mathfrak{g}^{\otimes 0}$ is defined to be $\mathbf{C}$. If I is the double-sided ideal generated by $xy-yx-[x,y]$ for all $x,y\in\mathfrak{g}$, then

$$U(\mathfrak{g})\overset{def}{=}T(\mathfrak{g})/I.$$

There is a natural inclusion $i:\mathfrak{g}\to U(\mathfrak{g})$.

Given an associative algebra A, denote by $Lie(A)$ the Lie algebra which is A as a vector space and has Lie bracket $[a,b]=ab-ba$. The universal enveloping algebra has the following universal property: Given a Lie algebra homomorphism $\phi:\mathfrak{g}\to Lie(A)$, there is a unique algebra homomorphism $\Phi:U(\mathfrak{g})\to A$ so that $\phi=\Phi\circ i$.

Remark: Previously, when we had expressions such as $EF(v)$, we meant it to be $\rho(E)\rho(F)(v)$ for some representation ρ. We can instead think of them as $\rho(EF)$, where EF is an element of the universal enveloping algebra.

7.5. **Integrable highest weight representations of $\mathfrak{g}$.** A $\mathfrak{g}$-module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there is $v \neq 0 \in V$ such that $\mathfrak{n}_+ v = 0$, $Hv = \lambda(H)v$ for all $H \in \mathfrak{h}$, and $V = U(\mathfrak{g})v$. It is not hard to see that this implies that $V = U(\mathfrak{n}_-)v$. (Note that $\mathfrak{n}_\pm$ is a Lie subalgebra of $\mathfrak{g}$.)

First we construct a *Verma module* $M(\lambda) = U(\mathfrak{g})/I$, where I is a left ideal in $U(\mathfrak{g})$ generated by $\mathfrak{n}_+$ and $H - \lambda(H)$ for all $H \in \mathfrak{h}$. $M(\lambda)$ is an $U(\mathfrak{g})$-module and $1 \in U(\mathfrak{g})/I$ is the highest weight vector with weight λ, since $H \cdot 1 = H = \lambda(H) \cdot 1$. After modding out by I, every element of $M(\lambda)$ can be represented by an element of $U(\mathfrak{n}_-)$ (and uniquely so)! Hence the Verma module is a free $U(\mathfrak{n}_-)$-module with highest weight vector λ.

The Verma module is infinite-dimensional, and to obtain finite-dimensional representations, we need to quotient by $U(\mathfrak{g})$-submodules.

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. Then $M(\lambda) \simeq U(\mathfrak{n}_-) = \mathbf{C}\{1, F, F^2, F^3, \dots\}$, where $H \cdot 1 = \lambda \cdot 1$, and 1 is the highest weight vector with weight λ. If we recall the formula

$$EF^i v = (\lambda - i + i)iF^{i-1}v,$$

we notice that $EF^i v = 0$ if $i = \lambda + 1$. Therefore, there is an $U(\mathfrak{g})$-submodule I generated by $F^{\lambda+1}$ which is $\mathbf{C}\{F^{\lambda+1}, F^{\lambda+2}, \dots\}$. (This is provided λ is a nonnegative integer.) The quotient $M(\lambda)/I$ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ.

Remark: Somehow by doing some abstract nonsense, we have managed to construct all the finite dimensional irreducible representations of a finite dimensional semisimple Lie algebra.

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[2] V. Kac, *Infinite dimensional Lie algebras*.

8. Integrable highest weight representations of $\hat{\mathfrak{g}}$

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Recall that the Verma module $M = M(\lambda)$ is a rank 1 free $U(\mathfrak{n}_-)$-module generated by the highest weight vector v. Suppose $\lambda \in \mathfrak{h}^*$ is in the *weight lattice*, i.e., $\lambda(\check{\alpha}_i) \in \mathbf{Z}$ for all coroots $\check{\alpha}_i$. Moreover, assume $\lambda(\check{\alpha}_i) \geq 0$. Then we have the following:

Theorem 8.1 (Harish-Chandra)**.** *The (unique) maximal ideal I is generated by $E_{-\alpha_i}^{\lambda(\check{\alpha}_i)+1}$ for all i.*

The quotient module M/I is an irreducible $\mathfrak{g}$-module with highest weight λ. Since the Weyl group acts on M/I, it is easy to see that M/I is finite-dimensional. This way, we have abstractly created all finite-dimensional irreducible representations of $\mathfrak{g}$.

Example: Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$. In this case, the maximal ideal is generated by two elements $E_{-\alpha_i}^{\lambda(\check{\alpha}_i)+1}$, $i = 1, 2$, where $\alpha_1 = L_1 - L_2$ and $\alpha_2 = L_2 - L_3$. Then the strings

$$\lambda, \lambda - \alpha_i, \lambda - 2\alpha_i, \dots, \lambda - \lambda(\check{\alpha}_i)\alpha_i,$$

for $i = 1, 2$, are two edges of the boundary of the set P of weights of an irreducible representation. (Note that the edges may be degenerate, i.e., a point.) Now, using the Weyl group, we may reflect these edges and obtain the boundary of P. Observe that if μ is a weight of $M(\lambda)$ which does not occur in I, then the multiplicity of that weight in the irreducible finite-dimensional representation $M(\lambda)/I$ is simply that of $M(\lambda)$. We just have to calculate multiplicities in $U(\mathfrak{n}_-)$. (Here, the *multiplicity* of a weight μ is the dimension of the weight space V_μ.)

Theorem 8.1 holds for affine Lie algebras as well. For $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}(2, \mathbf{C})}$, suppose we have a Verma module M with highest weight vector v satisfying:

$$\hat{\mathfrak{n}}_+ v = 0,\ cv = kv,\ dv = \mu v,\ Hv = \lambda v.$$

Since c is central, $cw = kw$ for all w in the irreducible representation M/I.

Recall that the affine Weyl group for $\hat{\mathfrak{g}}$ is generated by

$$W_0(x_0, x_1, x_2) = (-x_0, x_1, 2x_0 - x_1 + x_2),$$
$$W_1(x_0, x_1, x_2) = (-x_0, x_1, x_2),$$

where $\hat{\mathfrak{h}}^* = \{x_0\alpha + x_1\gamma + x_2\delta\}$. The highest weight vector v is in the weight space $(\frac{\lambda}{2}, k, \mu)$. The factor of two in $\frac{\lambda}{2}$ comes from the fact that $\alpha(H) = 2$. Since the second coordinate k is invariant under reflections, we will only write the first and third coordinates.

If $0 \leq \lambda \leq k$ and $\lambda, k \in \mathbf{Z}$, then the maximal ideal I of M is generated by $(F \otimes 1)^{\lambda+1}$ and $(E \otimes t^{-1})^{k-\lambda+1}$. These give two edges of the weight space configuration of M/I.

Starting with $x = (\frac{\lambda}{2}, \mu)$, we use W_0 and W_1 to obtain:

$$W_1(x) = \left(-\frac{\lambda}{2}, \mu\right),$$

$$W_0(x) = \left(k - \frac{\lambda}{2}, -(k-\lambda) + \mu\right) = x + (k-\lambda)(1,-1),$$
$$W_0W_1(x) = W_0(x) + \lambda(1,-2),$$
$$W_0W_1W_0(x) = W_0W_1(x) + \left(k - \frac{\lambda}{2}\right)(1,-3),$$

and so on. Note that the boundary fits inside an upside-down parabola. (From the point of view of physics, one might want to use lowest weight vectors instead. Then the row $x_2 = \mu$ is the ground state and is precisely the irreducible weight λ representation of $\mathfrak{sl}(2,\mathbf{C})$.)

8.1. **The Casimir element.** Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Let $\{I_i\}, \{I^i\}$ be *dual bases* for $\mathfrak{g}$ with respect to the Killing form $\langle , \rangle$, i.e., $\langle I_i, I^j\rangle = \delta_{ij}$. Then we define the *Casimir element* to be an element of $U(\mathfrak{g})$ given by:

$$C = \sum_i I_i I^i.$$

HW: Show that C is independent of the choice of dual bases $\{I_i\}$, $\{I^i\}$. (Hint: If $\{J_i\}$, $\{J^j\}$ is another dual basis, write $J_i = a_{ij}I_j$ and $J^i = b_{ij}I^j$, where the summation is omitted. Then $\langle J_i, J_j\rangle = \delta_{ij} = \langle a_{ik}I_k, b_{jl}I^l\rangle = a_{ik}b_{jk}$. Hence $a_{ik}b_{jk} = \delta_{ij}$.)

If $\{I_i\}$ is an *orthonormal* basis, i.e., $\langle I_i, I_j\rangle = \delta_{ij}$, then $\{I_i\}$ is dual to itself, and $C = \sum_i I_i \cdot I_i$.

Lemma 8.2. $[C, X] = 0$ *for all* $X \in \mathfrak{g}$.

Proof. It is useful to use the (easily verified) identity:$[ab, c] = [a, c]b + a[b, c]$. Then, with respect to an orthonormal basis $\{I_i\}$,

$$[C, X] = \sum_i [I_i, X]I_i + \sum_i I_i[I_i, X].$$

If we write $[I_i, I_j] = a_{ij}^k I_k$, then $\langle [I_i, I_j], I_k\rangle = \langle I_i, [I_j, I_k]\rangle$ implies that $a_{ij}^k = a_{jk}^i$, namely a_{ij}^k is invariant under cyclic permutation. Also observe that $a_{ij}^k = -a_{ji}^k$. Now,

$$\sum_i [I_i, I_j]I_i = \sum_{i,k} a_{ij}^k I_k I_i,$$

and

$$\sum_i I_i[I_i, I_j] = \sum_{i,k} a_{ij}^k I_i I_k = \sum_{i,k} a_{kj}^i I_k I_i = \sum_{i,k} -a_{ij}^k I_k I_i.$$

□

Since $[C, X] = 0$ for all $X \in \mathfrak{g}$, it follows that C acts by a constant on an irreducible representation V.

Example: If $\mathfrak{g} = \mathfrak{sl}(2,\mathbf{C})$, then we take an orthonormal basis $\frac{1}{\sqrt{2}}H$, $\frac{1}{\sqrt{2}}(E+F)$, $\frac{i}{\sqrt{2}}(E-F)$. We compute that $C = \frac{1}{2}H^2 + EF + FE$. (Remark: With respect to this orthonormal basis, the identification with $\mathfrak{so}(3, \mathbf{C})$ is clear.) We could also have taken dual bases $\{\frac{1}{\sqrt{2}}H, E, F\}$ and $\{\frac{1}{\sqrt{2}}, F, E\}$.

Next let C act on the irreducible representation V_λ with highest weight λ. Since we can evaluate C on any vector and get the same answer, let us use the highest weight vector v.

$$Cv = \left(\frac{1}{2}H^2 + EF + FE\right)v = \left(\frac{1}{2}H^2 + [E, F] + 2FE\right)v = \left(\frac{1}{2}H^2 + H\right)v = \left(\frac{1}{2}\lambda^2 + \lambda\right)v,$$

noting that $Ev = 0$. Now, writing $j = \frac{\lambda}{2}$, we have $Cv = 2j(j + 1)v$.

FS: The Casimir C is used to give *character formulas*, i.e., formulas that give the multiplicities (i.e., dimensions) of weight spaces V_μ. (Key useful property: C has the same eigenvalue for the highest weight vector as well as any $w \in V_\mu$.) There are character formulas such as Freudenthal's and Weyl's.

REFERENCES

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9. Sugawara operator; Introduction to symplectic geometry

9.1. **The Sugawara operators.** As before, let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra (say $\mathfrak{sl}(2,\mathbf{C})$) and $\{I_i\}$, $\{I^i\}$ be dual bases for $\mathfrak{g}$ with respect to the Killing form.

Notation: We will denote elements $X \otimes t^n$ in the affine Lie algebra $\hat{\mathfrak{g}}$ by $X^{(n)}$. (Here $X \in \mathfrak{g}$.)

Define the *Sugawara operators* as formal sums:

$$L_0 = \sum_i I_i I^i + 2\sum_i \sum_{n=1}^{\infty} I_i^{(-n)} I^{i(n)},$$

$$L_m = \sum_i \sum_{n=-\infty}^{\infty} I_i^{(-n)} I^{i(m+n)},$$

for $m \neq 0$. Note that L_0 is almost a Casimir for $U(\hat{\mathfrak{g}})$, although, strictly speaking, it is not in $U(\hat{\mathfrak{g}})$ because the sum is infinite and it is missing the terms $cd + dc = 2cd$, where c is the central element and d is the derivation.

L_m are elements of a suitably-defined completion $U_c(\widetilde{L\mathfrak{g}})$ of $U(\widetilde{L\mathfrak{g}})$ consisting of infinite sums that act *finitely* on integral highest weight representations $H_{k,\lambda}$ of $\mathfrak{g}$. (Here k is the level and λ is the highest weight, i.e., $cv = kv$ and $Hv = \lambda v$ for the highest weight vector v.) By acting *finitely*, we mean that for any $w \in H_{k,\lambda}$ we have $X^j w = 0$ for $j >> 0$.

Observe that, if we take sums $2\sum_{n=1}^{\infty} I_i^{(-n)} I^{i(n)}$ then there are only finitely many nonzero terms in $2\sum_{n=1}^{\infty} I_i^{(-n)} I^{i(n)} w$ for any $w \in H_{k,\lambda}$. This is why we don't use $\sum_{n\in\mathbf{Z}} I_i^{(-n)} I^{i(n)}$ instead.

Remark: If $m \neq 0$, then $\sum_i [I_i^{(-n)}, I^{i(m+n)}] = 0$, so we do not need to worry about the order of $I_i^{(-n)}$ and $I^{i(m+n)}$ in the definition of L_m.

Proof: Since $m \neq 0$, $\sum_i [I_i^{(-n)}, I^{i(m+n)}] = \sum_i [I_i, I^i]^{(m)}$. To show that $\sum_i [I_i, I^i] = 0$ we claim that the expression is independent of the choice of dual bases. Write $J_i = a_{ij} I_j$ and $J^i = b_{ij} I^j$ with $a_{ik} b_{jk} = \delta_{ij}$ (see the HW in Section 8.1). Then

$$\sum_i [J_i, J^i] = \sum_{i,j,k} a_{ij} b_{ik} [I_j, I^k] = \sum_{j,k} \delta_{j,k} [I_j, I^k] = \sum_j [I_j, I^k].$$

HW: Prove that the definition of the Sugawara operators does not depend on the choice of dual bases $\{I_i\}$, $\{I^i\}$.

The Sugawara operators satisfy the following commutation relations:

Theorem 9.1.

(1) $[L_m, X^{(n)}] = -2(c+2)nX^{(m+n)}$,
(2) $[L_m, L_n] = 2(c+2)((m-n)L_{m+n} + \frac{m^3-m}{6}\delta_{m+n,0} 3c)$.

Proof. We will do a sample calculation. For the rest, refer to Kac's book. Note that the calculation requires some care – whenever you use a commutation relations, etc., you must make sure that you stay within $U_c(\widetilde{L\mathfrak{g}})$!

Take an orthonormal basis $\{I_i\}$. Using the formula $[ab,c]=a[b,c]+[a,c]b$, we compute:

$$\begin{aligned}[L_0,I_j^{(n)}] &= I_i[I_i,I_j^{(n)}]+[I_i,I_j^{(n)}]I_i+2I_i^{(-m)}[I_i^{(m)},I_j^{(n)}]+2[I_i^{(-m)},I_j^{(n)}]I_i^{(m)}\\ &= I_i[I_i,I_j^{(n)}]+[I_i,I_j^{(n)}]I_i+2I_i^{(-m)}([I_i,I_j]^{(m+n)}+\delta_{ij}(m)\delta_{m+n,0}c)\\ &\quad +2([I_i,I_j]^{(-m+n)}+\delta_{ij}(-m)\delta_{-m+n,0}c)I_i^{(m)}\\ &= -2ncI_j^{(n)}+A,\end{aligned}$$

where

$$A=\sum_i\left(I_i[I_i,I_j]^{(n)}+[I_i,I_j]^{(n)}I_i\right)+\sum_i\sum_{m>0}\left(2I_i^{(-m)}[I_i,I_j]^{(m+n)}+2[I_i,I_j]^{(-m+n)}I_i^{(m)}\right).$$

Next, if we let $[I_i,I_j]=a_{ij}^kI_k$, then

$$\sum_i I_i^{(-m+n)}[I_i,I_j]^{(m)}=\sum_{i,k}a_{ij}^kI_i^{(-m+n)}I_k^{(m)},$$

$$\sum_i [I_i,I_j]^{(-m+n)}I_i^{(m)}=\sum_{i,k}a_{ij}^kI_k^{(-m+n)}I_i^{(m)}.$$

Switching i and k in the second equation and using $a_{ij}^k=-a_{kj}^i$, we see that

$$\sum_i I_i^{(-m+n)}[I_i,I_j]^{(m)}+\sum_i[I_i,I_j]^{(-m+n)}I_i^{(m)}=0.$$

Hence we have

$$A=\sum_i[I_i,[I_i,I_j^{(n)}]]+\sum_i\sum_{m=0}^{n}2[I_i,I_j]^{(-m+n)}I_i^{(m)}.$$

We now compute

$$\begin{aligned}\sum_i\left([I_i,I_j]^{(n-m)}I_i^{(m)}+[I_i,I_j]^{(m)}I_i^{(n-m)}\right) &= \sum_{i,k}\left(a_{ij}^kI_k^{(n-m)}I_i^{(m)}+a_{ij}^kI_k^{(m)}I_i^{(n-m)}\right)\\ &= \sum_{i,k}\left(a_{ij}^kI_k^{(n-m)}I_i^{(m)}-a_{ij}^kI_i^{(m)}I_k^{(n-m)}\right)\\ &= \sum_{i,k}a_{ij}^k[I_k,I_i]^{(n)}=\sum_{i,k,l}a_{ij}^ka_{ki}^lI_l^{(n)}=\sum_i[[I_i,I_j],I_i]^{(n)}\end{aligned}$$

Here we are using the fact that $a_{ij}^k=-a_{kj}^i$. Now, the term $\sum_i[[I_i,X],I_i]$ be interpreted as the minus the Casimir (for $\mathfrak{g}$) acting on X by the adjoint action. Since the Casimir is constant on all the vectors of an irreducible representation,

$$\sum_i [[I_i, X], I_i] = -4X,$$

and

$$A = -4nI_j^{(n)}.$$

Putting the above computation together:

$$[L_0, X^{(n)}] = -2(c+2)nX^{(n)}.$$

□

FS: Read Kac for a proof of the theorem (or prove it yourself!)

If we rescale L_m by multiplying by $\frac{1}{2(k+2)}$ and let the central element c act on $H_{k,\lambda}$ by k, then, as operators on $H_{k,\lambda}$ we have:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\frac{3k}{k+2}.$$

We have a "representation" of the Virasoro algebra Vir to $U_c(\widetilde{L\mathfrak{g}})$ by sending the central element of Vir to $\frac{3k}{k+2}$.

Finally, we decompose $H_{k,\lambda}$ using L_0. If $v \in H_{k,\lambda}$ is the highest weight vector, then $L_0 v = \frac{1}{2(k+2)}(\sum_i I_i I^i)v = \frac{j(j+1)}{k+2}v$, where $j = \frac{\lambda}{2}$. We define $\Delta_\lambda = \frac{j(j+1)}{k+2}$ to be the *conformal weight*. By using the first commutation relation in Theorem 9.1, we have

$$L_0(X^{(n)}v) = (\Delta_\lambda - n)(X^{(n)}v).$$

Recall the weight space diagram from the previous lecture (the upside-down parabola). The top row is isomorphic to the highest weight representation V_λ of $\mathfrak{sl}(2,\mathbf{C})$ and has conformal weight Δ_λ, the next row down has conformal weight $\Delta_\lambda + 1$, etc.

9.2. **Some symplectic geometry.** Now that we are done with the algebraic "preliminaries", we'd like to start from the beginning and explain ideas of quantization.

Definition 9.2. *A* symplectic manifold *is an (even-dimensional) manifold M with a closed nondegenerate 2-form ω.*

Recall that a form ω is closed if $d\omega = 0$, and ω is nondegenerate if $\omega(x) : T_xM \times T_xM \to \mathbf{R}$ is a nondegenerate pairing for all $x \in M$. A (symmetric/skew-symmetric) nondegenerate pairing $\langle , \rangle : V \times V \to \mathbf{R}$ has the property that for any $v \neq 0 \in V$ there exists $w \in V$ such that $\langle v, w\rangle \neq 0$.

HW: Show that ω is nondegenerate iff ω^n is nowhere vanishing, if $\dim M = 2n$.

Example: $(\mathbf{R}^{2n}, \omega = \sum_{i=1}^n dx_i dy_i)$. Note that $d\omega = 0$ and $\omega^n = n! dx_1 dy_1 \dots dx_n dy_n$.

Theorem 9.3 (Darboux)**.** *Every symplectic manifold* (M,ω) *is locally isomorphic to* $(\mathbf{R}^{2n},\omega=\sum dx_i dy_i)$

Example: The cotangent bundle T^*M of M has a natural symplectic structure. Let $\pi: T^*M\to M$ be the standard projection. Then there is a *canonical* 1-form θ on T^*M given by:

$$\theta(x)(v)=x(\pi_*(v)),$$

where $x\in T^*M$ and $v\in T_x(T^*M)$. The symplectic form is $\omega=d\theta$. Let q_i be local coordinates on $U\subset M$. Then $x\in\pi^{-1}(U)$ can be written as $\sum p_i dq_i$, and with respect to coordinates $(q_1,\dots,q_n,p_1,\dots,p_n)$ we have $\theta=\sum p_i dq_i$.

The symplectic form ω (and in fact any nondegenerate pairing) induces a 1-1 correspondence

$$\mathfrak{X}(M)\xrightarrow{\sim}\Omega^1(M),$$
$$X\mapsto i_X\omega,$$

where $\mathfrak{X}(M)$ is the set of smooth vector fields on M and $\Omega^1(M)$ is the set of smooth 1-forms on M. Here the contraction $i_X\omega$ satisfies $i_X\omega(Y)=\omega(X,Y)$.

There exists a special class of vector fields $Ham(M)\subset\mathfrak{X}(M)$ which correspond to exact 1-forms $d\Omega^0(M)\subset\Omega^1(M)$. Given a smooth function $f\in C^\infty(M)$ (often called a *Hamiltonian function*, we define its corresponding *Hamiltonian vector field* X_f as follows: $i_{X_f}\omega=df$.

Example: On $(\mathbf{R}^{2n},\omega)$, $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$ (we often omit subscripts i), and $X_f=\frac{\partial f}{\partial y}\frac{\partial}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial}{\partial y}$.

$C^\infty(M)$ acquires a *Poisson structure* via this correspondence. Define the *Poisson bracket* by $\{f,g\}=-\omega(X_f,X_g)$. (One easily sees that $\{f,g\}=-df(X_g)=dg(X_f)=-X_g(f)=X_f(g)$.) A *Poisson bracket* satisfies the following conditions:

(1) (Jacobi identity) $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$
(2) (Skew-symmetry) $\{f,g\}=-\{g,f\}$.
(3) (Derivation property) $\{fg,h\}=f\{g,h\}+\{f,g\}h$.

In particular, a Poisson bracket is a Lie bracket and $C^\infty(M)$ is a Lie algebra via the Poisson bracket.

10. Towards quantization

10.1. **Poisson structure.** Let (M, ω) be a symplectic manifold. Given $f \in C^\infty(M)$, X_f is the Hamiltonian vector field corresponding to f, given by: $i_{X_f}\omega = df$. We then have a map:

$$C^\infty(M) \to Ham(M),$$
$$f \mapsto X_f,$$

where $Ham(M)$ is the set of Hamiltonian vector fields of M.

Define the Poisson bracket $\{f, g\} = -\omega(X_f, X_g)$.

Example: For $\mathbf{R}^2$ with $\omega = dxdy$, $\{f, g\} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}$.

Lemma 10.1. $[X_f, X_g] = X_{\{f,g\}}$.

Proof.

$$\begin{aligned}
[X_f, X_g] \lrcorner\, \omega &= (\mathcal{L}_{X_f} X_g) \lrcorner\, \omega = \mathcal{L}_{X_f}(X_g \lrcorner\, \omega) - X_g \lrcorner\, (\mathcal{L}_{X_f}\omega) \\
&= \mathcal{L}_{X_f}(dg) - X_g \lrcorner\, d(X_f \lrcorner\, \omega) \\
&= d(X_f \lrcorner\, dg) - X_g \lrcorner\, (d \circ df) \\
&= d\omega(X_g, X_f) = -d(\omega(X_f, X_g)) \\
&= d\{f, g\} = X_{\{f,g\}} \lrcorner\, \omega.
\end{aligned}$$

□

Since a Poisson bracket is a Lie bracket with additional structure, the above lemma implies that the map $f \mapsto X_f$ is a Lie algebra homomorphism! We now prove the properties of the Poisson bracket from last time.

Proof. (2). The skew-symmetry is straightforward.

(1). By the Cartan formula, we have:

$$\begin{aligned}
d\omega(X_f, X_g, X_h) &= X_f\omega(X_g, X_h) - X_g\omega(X_f, X_h) + X_h\omega(X_f, X_g) \\
&\quad -\omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g] - \omega([X_g, X_h], X_f).
\end{aligned}$$

If we call the first row of the above equation A and the second row B, then

$$A = X_f X_h g - X_g X_h f + X_h X_g f = -X_f X_g h + X_g X_f h - X_h X_f g.$$

Adding the two equivalent expressions and dividing by 2, we have:

$$A = -\frac{1}{2}([X_f, X_g]h + [X_g, X_h]f + [X_h, X_f]g).$$

Next,

$$-\omega([X_f, X_g], X_h) = dh([X_f, X_g]) = [X_f, X_g]h,$$

so $A + B = \frac{1}{2}B$, and $B = 0$. Now,

$$\{\{f, g\}, h\} = -\omega(X_{\{f,g\}}, X_h) = -\omega([X_f, X_g], X_h),$$

and we are done.

(3). $-\omega(X_{fg}, X_h) = -d(fg)(X_h) = -fdg(X_h) - gdf(X_h) = f\{g,h\} + g\{f,h\}$. $\square$

HW: Prove the above properties in local coordinates, i.e., for $(\mathbf{R}^{2n}, \omega = \sum_i dx_i dy_i)$.

Remark: We have a *central extension*

$$0 \to \mathbf{R} \to C^\infty(M) \to Ham(M) \to 0.$$

Mantra: Central extensions give rise to quantization.

10.2. **Connections.** Let E be a rank k vector bundle over M and let s be a section of E. (We may take E to be a real or complex vector bundle, but we'll work over **C** today.) s may be local (i.e., in $\Gamma(E,U)$) or global (i.e., in $\Gamma(E,M)$). (Here $U \subset M$ and $\Gamma(E,U)$ is the set of smooth sections of Γ over M.) Also let X be a vector field. We want to differentiate s at $p \in M$ in the direction of $X(p) \in T_pM$.

Definition 10.2. *A* connection *or* covariant derivative ∇ *assigns to every vector field* $X \in \mathfrak{X}(M)$ *a differential operator* $\nabla_X : \Gamma(E) \to \Gamma(E)$ *which satisfies:*

1. $\nabla_X s$ *is* **C***-linear in* s*, i.e.,* $\nabla_X(c_1s_1 + c_2s_2) = c_1\nabla_X s_1 + c_2 \nabla_X s_2$ *if* $c_1, c_2 \in \mathbf{C}$.
2. $\nabla_X s$ *is* $C^\infty(M)$*-linear in* X*, i.e.,* $\nabla_{fX+gY} s = f\nabla_X s + g\nabla_Y s$.
3. *(Leibniz rule)* $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Note: The definition of connection is tensorial in X (condition (2)), so $(\nabla_X s)(p)$ depends on s *near* p but only on X *at* p.

Flat connections: We will now present the first example of a connection. A vector bundle E of rank k is said to be *trivial* or *parallelizable* if there exist sections $s_1, \dots, s_k \in \Gamma(E,M)$ which span E_p at every $p \in M$. (Here $E_p = \pi^{-1}(p)$, where $\pi : E \to M$ is the projection.) Although not every vector bundle is parallelizable, locally every vector bundle is trivial since $E|_U \simeq U \times \mathbf{C}^k$. We will now construct connections on trivial bundles $E|_U \to U$.

Write any section $s \in \Gamma(E,U)$ as $s = \sum_i f_i s_i$, where $f_i \in C^\infty(U)$. Then define

$$\nabla_X s = \sum_i (Xf_i)s_i = (Xf_1)s_1 + \cdots + (Xf_k)s_k \in \Gamma(E).$$

This connection is usually called a *flat connection*.

HW: Check that this satisfies the axioms of a connection.

With respect to the given trivialization, s has coordinates $(f_1, \dots, f_k)^T$ and $\nabla_X s$ has coordinates $(Xf_1, \dots, Xf_k)^T = d(f_1, \dots, f_k)^T(X)$. Hence we can write $\nabla = d$, the exterior derivative.

In general, flat connections do not exist globally on a manifold M, but one can always globally construct (not necessarily flat) connections by patching flat connections in a manner similar to constructing a Riemannian metric.

Difference of connections: Next, given two connections ∇ and ∇', we compute their difference:

$$(\nabla_X - \nabla'_X)(fs) = f(\nabla_X - \nabla'_X)s.$$

Therefore, the difference of two connections is tensorial in s.

Locally, take sections $s_1, \dots, s_k$ which span E. Then $(\nabla_X - \nabla'_X)s_i = \sum_j a_{ij}s_j$, where a_{ij} is a $k \times k$ matrix of functions. We can therefore write

$$\nabla = d + A,$$

where $A = (A_{ij})$ is a $k \times k$ matrix of 1-forms A_{ij}, where $A_{ij}(X) = a_{ij}$.

Gauge change: Suppose that ∇ is written as $d+A$ with respect to the trivialization $\{s_1, \dots, s_k\}$ on $E|_U$. If $\{\overline{s}_1, \dots, \overline{s}_k\}$ is another trivialization (here the $\overline{s}$ does not mean the conjugate of s), then we write $\overline{s}_i = \sum_j g_{ij}s_j$. (Here $g = (g_{ij})$ is a $k \times k$ matrix-valued function on U.) Since $\nabla s_i = A_{ij}s_j$, we compute that:

$$\nabla \overline{s}_i = \nabla g_{ij}s_j = (dg_{ij})s_j + g_{ij}A_{jk}s_k = (dg_{ij})(g^{-1})_{jk}\overline{s}_k + g_{ij}A_{jk}(g^{-1})_{kl}\overline{s}_l,$$

and A transforms to $dg \cdot g^{-1} + gAg^{-1}$. This is usually called *gauge change*.

Curvature: The *curvature* of a connection ∇ is given by:

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

HW: Prove that

$$R(X,Y) = (dA + A \wedge A)(X,Y)$$

in local coordinates. (Here $(A \wedge A)(X,Y) = A(X) \cdot A(Y) - A(Y) \cdot A(X)$.)

HW: Prove that the curvature R transforms to gRg^{-1} under gauge transformation.

References

[1] K. Honda, Math 535a notes, available from my webpage. (These notes were based on R. Bott's lecture notes at Harvard.)

11. Geometric quantization

11.1. **Line bundles and connections.** Suppose L is a complex line bundle over M. Let $\{U_\alpha\}$ be an open cover of M so that $L|_{U_\alpha}$ is trivial. Pick a connection ∇ on L. On U_α let $\nabla = d - 2\pi i A_\alpha$. (Note the minus sign – apparently this is needed to make $c_1(L)$ agree with the usual one.)

Gauge change: Suppose $g : U_\alpha \cap U_\beta \to S^1 = \{|z| = 1\} \subset \mathbf{C}$ is a gauge transformation, i.e., a change of trivialization. Then we write $g(x) = e^{-2\pi i f(x)}$. Under gauge change,

$$-2\pi i A_\alpha \mapsto dgg^{-1} + g(-2\pi i)A_\alpha g^{-1} = -2\pi i(A_\alpha + df).$$

Hence $A_\alpha \mapsto A_\alpha + df$.

Curvature: The curvature is given by $-2\pi i dA_\alpha + (-2\pi i)^2 A_\alpha \wedge A_\alpha = -2\pi i dA_\alpha$, since we're dealing with 1×1 matrices (and they commute)! Moreover, dA_α transforms to $g dA_\alpha g^{-1} = dA_\alpha$, i.e., dA_α is invariant under gauge change. Therefore, $\{dA_\alpha\}$ can be patched into a closed 2-form on M. The cohomology class of the closed 2-form M is called the *first Chern class* of L and is denoted $c_1(L) \in H^2_{dR}(M;\mathbf{R})$. Note that $c_1(L) = \frac{i}{2\pi}[F_A]$.

HW: Prove that the first Chern class of L does not depend on the specific choice of connection.

Remark: $c_1(L)$ is actually an element of $H^2(M;\mathbf{Z}) \subset H^2(M;\mathbf{R})$.

Theorem 11.1. *Let ω be a closed 2-form on M such that $[\omega] \in H^2(M;\mathbf{Z}) \subset H^2(M;\mathbf{R})$. Then there exists a complex line bundle $L \to M$ and a connection ∇ such that $\omega = \frac{i}{2\pi}F_A$. (In particular, this means that $c_1(L) = [\omega]$.)*

Proof. Choose a *good cover* $\{U_\alpha\}$ of M. A *good cover* is a cover for which $U_\alpha \simeq \mathbf{R}^n$, $U_\alpha \cap U_\beta \simeq \mathbf{R}^n$ or $\emptyset$, $U_\alpha \cap U_\beta \cap U_\gamma \simeq \mathbf{R}^n$ or $\emptyset$, etc. Here $\simeq$ means "diffeomorphic to", and $\dim M = n$. Such a good cover exists on any smooth manifold – the usual proof uses a Riemannian metric to construct geodesically convex neighborhoods.

Over U_α, construct the trivial line bundle $U_\alpha \times \mathbf{C} \to U_\alpha$ with connection $-2\pi i A_\alpha$ so that $dA_\alpha = \omega$ on U_α. Here we are using the fact that $U_\alpha \simeq \mathbf{R}^n$ and the Poincaré lemma to find a primitive for ω.

Next, on overlaps $U_\alpha \cap U_\beta \simeq \mathbf{R}^n$, $A_\alpha - A_\beta = df_{\alpha\beta}$ since $dA_\alpha = dA_\beta = \omega$. Again, we are using the Poincaré lemma. Observe that the choice of $f_{\alpha\beta}$ is unique up to the choice of a constant function.

We now use $g_{\alpha\beta} = e^{-2\pi i f_{\alpha\beta}}$ to patch the $U_\alpha \times \mathbf{C}$'s. Namely, we glue $(U_\alpha \cap U_\beta) \times \mathbf{C} \subset U_\beta \times \mathbf{C}$ to $(U_\alpha \cap U_\beta) \times \mathbf{C} \subset U_\alpha \times \mathbf{C}$ by sending (x,z) to $(x, g(x)z)$.

In order to make sure that the gluing is consistent, we need to verify the following on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$:

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma},$$

or, equivalently,

$$f_{\alpha\beta} + f_{\beta\gamma} = f_{\alpha\gamma} \text{ mod } \mathbf{Z}. \tag{4}$$

We ask whether it is possible to choose complex numbers $a_{\alpha\beta}$ so that $\overline{f}_{\alpha\beta} = f_{\alpha\beta} + a_{\alpha\beta}$ satisfies Equation 8. To answer this, consider the simplicial complex for M corresponding to the good cover $\{U_\alpha\}$: To each U_α, assign a vertex (0-simplex) v_α. To each nontrivial $U_\alpha \cap U_\beta$, assign an edge (1-simplex) between v_α and v_β. To each nontrivial $U_\alpha \cap U_\beta \cap U_\gamma$, place a 2-simplex with vertices v_α, v_β, v_γ. With respect to this simplicial decomposition of M, $\delta f = \{f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\beta}\}$ is a 2-cocycle with values in **C**. Now, the question can be rephrased as follows: Is there a 1-cochain $a = \{a_{\alpha\beta}\}$ with values in **C** such that $f - \delta a$ has values in **Z**? This is precisely the same as asking for $[\omega]$ to be in $H^2(M; \mathbf{Z})$. ☐

FS: Study the isomorphism between the de Rham cohomology and $H^i_{dR}(M; \mathbf{R})$ and simplicial cohomology $H^i(M; \mathbf{R})$ to verify the last statement.

11.2. **Geometric quantization.** Given a symplectic manifold (M, ω), construct a complex line bundle $L \to M$ and a connection ∇ such that the curvature is $-2\pi i\omega$. Let $C^\infty(M)$ be the Poisson algebra of C^∞-functions on (M, ω), and let $\Gamma(L)$ be the smooth sections of L.

By ***(geometric) quantization*** we mean a Lie algebra representation of $C^\infty(M)$ on $\Gamma(L)$, i.e., a Lie algebra homomorphism from $C^\infty(M)$ to $End(\Gamma(L))$. (Usually the operators are unbounded.) In the case at hand, assign:

$$f \mapsto \nabla_{X_f} - 2\pi i f.$$

The assignment is a Lie algebra homomorphism:

$$\begin{aligned}
\{f, g\} &\mapsto [\nabla_{X_f} - 2\pi i f, \nabla_{X_g} - 2\pi i g] \\
&= (\nabla_{[X_f, X_g]} - 2\pi i\omega(X_f, X_g)) - 2\pi i X_f g - 2\pi i X_g f \\
&= \nabla_{X_{\{f,g\}}} + 2\pi i\omega(X_f, X_g) \\
&= \nabla_{X_{\{f,g\}}} - 2\pi i\{f, g\}
\end{aligned}$$

Primordial Example: Consider $(\mathbf{R}^{2n}, \omega = dpdq)$ with coordinates (p, q). (Here, p stands for momentum and q for position.) Construct the trivial line bundle $\mathbf{R}^{2n} \times \mathbf{C}$ with connection $\nabla = d - \frac{i}{\hbar}A$. Then quantize by sending

$$f \mapsto \nabla_{X_f} - \frac{i}{\hbar}f.$$

We compute that $X_p = -\frac{\partial}{\partial q}$ and $X_q = \frac{\partial}{\partial p}$. Hence, upon quantizing:

$$p \mapsto \nabla_{-\frac{\partial}{\partial q}} - \frac{i}{\hbar}p = -\frac{\partial}{\partial q} - \frac{i}{\hbar}pdq\left(-\frac{\partial}{\partial q}\right) - \frac{i}{\hbar}p = -\frac{\partial}{\partial q},$$

$$q \mapsto \nabla_{\frac{\partial}{\partial p}} - \frac{i}{\hbar}q = \frac{\partial}{\partial p} - \frac{i}{\hbar}pdq\left(\frac{\partial}{\partial p}\right) - \frac{i}{\hbar}q = \frac{\partial}{\partial p} - \frac{i}{\hbar}q.$$

This looks a bit different from what we usually learn in quantum mechanics (i.e., $p \mapsto \frac{\partial}{\partial q}$ and $q \mapsto q$. If we restrict to sections that are functions only in q, then the above (more or less) reduces to the familiar quantization rules.

12. Path integrals

12.1. **Sigma models.** We study maps $u : X \to M$ between Riemannian manifolds. Let $Map(X, M)$ be the set of smooth maps from X to M. Then given $u \in Map(X, M)$ we define the *energy functional*:

$$S_X : Map(X, M) \to \mathbf{R},$$
$$u \mapsto \int_X |du|^2 dvol_X.$$

More precisely, at $x \in X$, take an orthonormal basis $e_1, \dots, e_n$ of T_xX. (Here $\dim X = n$.) Then $|du|^2$ means $\sum_{i=1}^n \langle u_* e_i, u_* e_i \rangle$, where $\langle, \rangle$ is with respect to the Riemannian metric for M.

HW: Prove that $|du|^2$ is independent of the choice of orthonormal frame.

By "functional" we mean a function on some space of functions. A critical point of the energy functional is called a "harmonic map".

HW: Suppose $X = D^n$ (the n-dimensional disk) and $M = \mathbf{R}$. Consider $Map(X, M, f) \subset Map(X, M)$, the set of maps u satisfying the boundary condition $u|_{\partial X} = f$. (This is called a *Dirichlet* boundary condition.) Then a critical point $u \in Map(X, M, f)$ satisfies (by definition)

$$\lim_{t \to 0} \frac{S_X(u + tv) - S_X(u)}{t} = 0,$$

for all $v \in T_u Map(X, M, f)$. Show that a critical u is a *harmonic function,* i.e., $\sum_i \frac{\partial^2 u}{\partial x_i^2} = 0$.

The energy functional (the generic term is "action") has the following obvious properties:

(1) If $f : X' \to X$ is an isometry, then $S_{X'}(u \circ f) = S_X(u)$.
(2) If $-X$ is X with reversed orientation, then $S_{-X}(u) = -S_X(u)$.
(3) If $X = X_1 \sqcup X_2$ (disjoint union), then $S_X(u) = S_{X_1}(u|_{X_1}) + S_{X_2}(u|_{X_2})$.
(4) Suppose $X = X_+ \cup X_-$, where $\partial X_+ = -\partial X_- = Y$ is a codimension 1 submanifold of X. If $u_+ \in Map(X_+, M)$, $u_- \in Map(X_-, M)$, and $u_+|_Y = u_-|_Y$, then $S_X(u) = S_{X_+}(u_+) + S_{X_-}(u_-)$. Here u is defined to be u_+ on X_+ and u_- on X_-.

12.2. **Feynman path integral.** In *classical mechanics*, the trajectory of a particle between two points (say a and b) in configuration space minimizes the action $S(\gamma)$.

In *quantum mechanics*, to each path γ you assign a "probability function" $e^{iS(\gamma)/\hbar}$ and integrate over the space of all paths connecting a and b:

$$\int e^{iS(\gamma)/\hbar} d\mu(\gamma).$$

This is called the *Feynman path integral*. Here $d\mu$ is some measure on the space of paths connecting a and b.

Remark: The Feynman path integral has been rigorously defined only in some cases. (Even in cases where the integral is rigorously defined, I'm not sure if there is a measure μ which satisfies all the properties you'd expect from a measure in finite dimensions.)

When we go from quantum to classical (i.e., in the large $\hbar$ limit), we expect the rapid oscillations of $e^{iS(\gamma)/\hbar}$ to cancel each other, except near the critical points of $S(\gamma)$. Hence the main contributions are the *classical trajectories*.

Sigma model: Let us consider the sigma model. Let $C_X = Map(X, M)$. If X does not have boundary, then the "partition function"

$$Z(X) = \int_{C_X} e^{iS(u)/\hbar} d\mu_X(u)$$

is expected to be a complex number. If $\partial X = Y$, then we can define a function $Z(X)$ on C_Y as follows: Given $\alpha \in C_Y$, let:

$$Z(X)(\alpha) = \int_{C_X(\alpha)} e^{iS(u)/\hbar} d\mu_X(u).$$

Here we are integrating over $C_X(\alpha)$ which is the subset of C_X consisting of maps $u : X \to M$ which restrict to α on $\partial X = Y$.

Plan: Although $Z(X)$ may not be rigorously defined, we can write down expected properties of $Z(X)$ and also of Z_Y, which is some vector subspace of functions on C_Y that $Z(X)$ should live in.

Axioms:

(1) (Orientation) $Z_{-Y} = Z_Y^*$, where Z_Y^* is the dual vector space of Z_Y.
(2) (Multiplication) $Z_{Y_1 \sqcup Y_2} = Z_{Y_1} \otimes Z_{Y_2}$.
(3) (Gluing) $Z(X) = \langle Z(X_+), Z(X_-)\rangle$, where $X = X_+ \cup X_-$, $\partial X_+ = -\partial X_- = Y$, and the pairing is between Z_Y and Z_Y^*.

Explanation: We explain the Gluing Axiom. Using the expected properties of the Feynman path integral (e.g., Fubini's theorem),

$$\begin{aligned} Z(X) &= \int_{C_X} e^{iS_X(u)/\hbar} d\mu_X(u) \\ &= \int_{C_Y} \left(\int_{C_{X_+}(\alpha)} e^{iS_{X_+}(u_+)/\hbar} d\mu_{X_+}(u_+) \cdot \int_{C_{X_-}(\alpha)} e^{iS_{X_-}(u_-)/\hbar} d\mu_{X_-}(u_) \right) d\mu_Y(\alpha) \\ &= \int_{C_Y} Z(X_+)(\alpha) \cdot Z(X_-)(\alpha) d\mu_Y(\alpha) \\ &= \langle Z(X_+), Z(X_-)\rangle. \end{aligned}$$

Here, $u_+ = u|_{X_+}$, $u_- = u|_{X_-}$, and $\alpha = u|_Y$. We are also using $S_X(u) = S_{X_+}(u_+) + S_{X_-}(u_-)$.

12.3. **Topological Quantum Field Theory (TQFT) axioms.** We will now formulate the TQFT axioms in the sense of Atiyah. They are almost the same as the axioms derived for the sigma model above; the only major difference is that we ask the vector spaces to be finite-dimensional.

A TQFT in $(d+1)$-dimensions is a functor Z which:

(1) To a d-dimensional (smooth) manifold Σ without boundary assigns a *finite-dimensional* vector space Z_Σ. (These are the objects.)
(2) To a $(d+1)$-dimensional manifold Y with $\partial Y = \Sigma$ assigns a vector $Z(Y) \in Z_\Sigma$. (These are the morphisms.)

The functor Z satisfies:

A1. $Z_{-\Sigma} = Z_\Sigma^*$.
A2. $Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$.

From A1 and A2 if $\partial Y = -\Sigma_1 \sqcup \Sigma_2$, then

$$Z(Y) \in Z_{-\Sigma_1} \otimes Z_{\Sigma_2} = Z_{\Sigma_1}^* \otimes Z_{\Sigma_2} = Hom_{\mathbf{C}}(Z_{\Sigma_1}, Z_{\Sigma_2}).$$

A3. Given a composition of cobordisms $\partial Y_1 = -\Sigma_1 \sqcup \Sigma_2$, $\partial Y_2 = -\Sigma_2 \sqcup \Sigma_3$, we have $Z(Y_1 \cup Y_2) = Z(Y_2) \circ Z(Y_1)$.
A4. $Z(\emptyset) = \mathbf{C}$.
A5. $Z(\Sigma \times [0,1]) = id : Z_\Sigma \to Z_\Sigma$.

13. Loop groups

13.1. **Maurer-Cartan form.**

Definition 13.1. *Let G be a Lie group. Then the* Maurer-Cartan *form μ is a left-invariant 1-form on G with values in the Lie algebra $\mathfrak{g}$ which satisfies $\mu(e)(A) = A$, where $A \in T_eG = \mathfrak{g}$. (More generally, $\mu(g)(g(I + tA)) = I + tA$, if we write A as $I + tA$.)*

Notation: Often we write $A \in T_eG$ as $I + tA$ or e^{tA} and think of A as an equivalence class of arcs.

For matrix Lie groups (i.e., subgroups of $GL(n, \mathbf{C})$), the Maurer-Cartan form is $\mu = X^{-1}dX$, where X is an $n \times n$ matrix whose (ij)-th entry is the coordinate function x_{ij}. One verifies that $\mu(I)(I + tA) = dX(I + tA) = I + tA$ and $\mu(g)(g(I + tA)) = g^{-1}g(I + tA) = I + tA$.

Now let $G = SU(2)$. We recall that $SU(2)$ is diffeomorphic to S^3: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$, then a, b determine A and $\{|a|^2 + |b|^2 = 1\}$ is the unit sphere in $\mathbf{C}^2$.

Consider the 3-form
$$\sigma = \frac{1}{24\pi^2} Tr(\mu \wedge \mu \wedge \mu),$$
where μ is the Maurer-Cartan form.

Lemma 13.2. *$[\sigma]$ generates the integral cohomology group $H^3(SU(2); \mathbf{Z}) \simeq H^3(S^3; \mathbf{Z}) \simeq \mathbf{Z}$.*

Proof. Let us perform the calculation at $e \in G$ and rely on the left-invariance. $\mathfrak{su}(2)$ is the set of traceless skew-Hermitian matrices, and has an $\mathbf{R}$-basis:
$$\left\{ A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}.$$
We compute that:
$$\sigma(e)(A, B, C) = \frac{1}{24\pi^2}(3!) Tr(ABC) = \frac{1}{4\pi^2} Tr \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{2\pi^2}.$$
The $3!$ comes from observing that we are taking alternating sums when evaluating three tangent vectors A, B, C, and each sum is the same. Since $2\pi^2$ is the volume of the unit 3-sphere in $\mathbf{R}^4$ (HW: verify this!), $\int_{S^3} \sigma = -1$ and $[\sigma]$ generates $H^3(SU(2); \mathbf{Z})$. □

13.2. **The loop group.**

Suppose $G = SU(2)$ still. Let LG be the loop group, i.e., the group of smooth maps $S^1 \to G$.

Lemma 13.3. $H^2(LG; \mathbf{Z}) = \mathbf{Z}$.

Proof. First observe that $LG \simeq G \times \Omega G$, where ΩG is the set of *based* loops, namely smooth maps $S^1 \to G$ which map $1 \mapsto e$. (Here we are viewing $S^1 = \{|z| = 1\} \subset \mathbf{C}$ and e is the identity of G.) In fact, we can send $\gamma \in LG$ to $(\gamma(1), (\gamma(1))^{-1}\gamma)$.

Now, we have $\pi_i(\Omega G, e) \simeq \pi_{i+1}(G, e)$. (Here $e \in \Omega G$ refers to the map $S^1 \to G$ which maps to $e \in G$.) For the isomorphism, we think of a map $(S^i, pt) \to (\Omega G, e)$ as a map $(S^{i+1}, pt) \to (G, e)$.

We then have:

$$\pi_1(G) = \pi_1(S^3) = 0, \pi_2(G) = 0, \pi_3(G) = \mathbf{Z},$$

$$\pi_1(LG) = \pi_1(G) \times \pi_1(\Omega G) = \pi_1(G) \times \pi_2(G) = 0, \pi_2(LG) = \pi_2(G) \times \pi_3(G) = \mathbf{Z}.$$

By the Hurewicz isomorphism theorem, the first nontrivial π_i and H_i agree, and we have

$$H_2(LG) \simeq \pi_2(G) \simeq \mathbf{Z}.$$

□

Lemma 13.4. *A generator for $H^2(LG; \mathbf{Z})$ is given by $\omega_0 = \int_{S^1} \phi^*\sigma$, where $\phi : LG \times S^1 \to G$ is the* evaluation map $\phi(\gamma, \theta) = \gamma(\theta)$.

We need to explain the integration operation. First,

$$\phi^*(\sigma)(\gamma, \theta)\left(\xi, \eta, \frac{\partial}{\partial\theta}\right) = \sigma(\gamma(\theta))(\xi(\theta), \eta(\theta), \gamma'(\theta)),$$

where $\xi, \eta \in T_\gamma LG$. Then

$$\begin{aligned} \omega_0(\gamma)(\xi, \eta) &= \left(\int_{S^1} \phi^*(\sigma)\right)(\gamma)(\xi, \eta) \\ &= \int_0^{2\pi} \sigma(\gamma(\theta))(\xi(\theta), \eta(\theta), \gamma'(\theta))\, d\theta \\ &= \frac{1}{4\pi^2}\int_0^{2\pi} Tr(\gamma^{-1}\xi(\theta)\cdot\gamma^{-1}\eta(\theta)\cdot\gamma^{-1}\gamma'(\theta))\, d\theta. \end{aligned}$$

The composition

$$H^3(G) \xrightarrow{\phi^*} H^3(LG \times S^1) \xrightarrow{\int_{S^1}} H^2(LG)$$

is called *transgression*. It is not hard to see that the composition sends generators to generators. (Compare with the isomorphism $\pi_3(G) \simeq \pi_2(LG)$.) See Bott-Tu for more details on transgression.

Now let ω be the left-invariant 2-form on LG given by extending the Lie algebra 2-cocycle ω (with $[\omega] \in H^2(L\mathfrak{g}; \mathbf{C})$), where

$$\omega(e)(\xi, \eta) = \frac{1}{4\pi^2}\int_0^{2\pi} \langle \xi'(\theta), \eta(\theta)\rangle\, d\theta.$$

Here $\langle , \rangle$ is the Killing form for $\mathfrak{su}(2)$, which is a multiple of $(A, B) \mapsto Tr(AB)$. If we set $\xi = X \otimes t^m$ and $\eta = Y \otimes t^n$, and it $t = e^{i\theta}$, then $\frac{d}{d\theta}e^{im\theta} = ime^{im\theta}$ and

$$\omega(e)(X \otimes t^m, Y \otimes t^n) = \frac{i}{2\pi}\langle X, Y\rangle m\delta_{m+n,0}. \tag{5}$$

Observe that $\langle X, Y\rangle m\delta_{m+n,0}$ is the Lie algebra 2-cocycle. The 2-cocycle property translates into $d\omega$ being closed.

Lemma 13.5. $\omega = \omega_0 + d\beta$, *where*

$$\beta(\gamma)(\xi) = \frac{1}{8\pi^2}\int_0^{2\pi} Tr(\gamma^{-1}\gamma'(\theta)\cdot\gamma^{-1}\xi(\theta))\, d\theta.$$

Hence ω *is also a generator for* $H^2(LG;\mathbf{Z})$.

Proof. We compare

$$\omega_0(\gamma)(\xi,\eta) = \frac{1}{4\pi^2}\int_0^{2\pi} Tr(\gamma^{-1}\xi(\theta)\cdot\gamma^{-1}\eta(\theta)\cdot\gamma^{-1}\gamma'(\theta))\, d\theta \tag{6}$$

and

$$\omega(\gamma)(\xi,\eta) = \frac{1}{4\pi^2}\int_0^{2\pi} Tr((\gamma^{-1}\xi)'(\theta)\cdot\gamma^{-1}\eta(\theta))\, d\theta. \tag{7}$$

Let $\tilde{\xi}$ and $\tilde{\eta}$ be left-invariant extensions of ξ and η to all of LG. Then we can use the Cartan formula:

$$d\beta(\gamma)(\xi,\eta) = \tilde{\xi}(\beta(\tilde{\eta})) - \tilde{\eta}(\beta(\tilde{\xi})) - \beta([\tilde{\xi},\tilde{\eta}]).$$

We have $\beta(\gamma)([\tilde{\xi},\tilde{\eta}]) = \omega_0(\gamma)(\xi,\eta)$, and since $\gamma^{-1}\tilde{\xi}(\theta)$, $\gamma^{-1}\tilde{\eta}(\theta)$ are constant for all γ, we have

$$\tilde{\xi}(\beta(\tilde{\eta})) = \frac{1}{8\pi^2}\int_0^{2\pi} Tr((\gamma^{-1}\xi)'\cdot\gamma^{-1}\eta)\, d\theta,$$

$$-\tilde{\eta}(\beta(\tilde{\xi})) = -\frac{1}{8\pi^2}\int_0^{2\pi} Tr((\gamma^{-1}\eta)'\cdot\gamma^{-1}\xi)\, d\theta.$$

This proves $\omega = \omega_0 + d\beta$. □

Symplectic form: Observe that ω given by Equation 5 is degenerate on elements of $L\mathfrak{g}$ of the form $X\otimes 1$ (but nondegenerate otherwise). If we take the quotient $LG/G\simeq \Omega G$, then we quotient out the degeneracy. Hence ω is a symplectic form on ΩG. (In fact ΩG is a Kähler manifold.) For more information, consult Pressley-Segal [2].

HW: The *energy functional* $E:\Omega G\to\mathbf{R}$ is given by

$$E(\gamma) = \int_0^{2\pi}\langle\gamma^{-1}\gamma'(\theta),\gamma^{-1}\gamma'(\theta)\rangle\, d\theta.$$

Prove that the Hamiltonian vector field corresponding to E rotates the loops.

REFERENCES

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology.*
[2] A. Pressley and G. Segal, *Loop Groups*.

14. THE WESS-ZUMINO-WITTEN MODEL

Today we discuss the Wess-Zumino-Witten (WZW) model.

14.1. **Definitions.** Let Σ be a compact Riemann surface, i.e., a 1-dimensional complex manifold, without boundary (sometimes called a closed Riemann surface). Let G be the Lie group $SU(2)$. Consider $Map(\Sigma, G)$, the space of smooth maps $f : \Sigma \to G$.

We first define the *energy functional*

$$E_\Sigma(f) = -i \int_\Sigma Tr(f^{-1}\partial f \wedge f^{-1}\overline{\partial} f).$$

Interpretation: First recall that the Killing form of $G = SU(2)$ is a constant multiple of $(X, Y) \mapsto Tr(XY)$. If we use local holomorphic coordinate $z = x + iy$ for Σ, then

$$\partial f = \frac{\partial f}{\partial z} dz = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(dx + idy),$$

$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(dx - idy).$$

If $f(z) = e$ (which we may assume because $f^{-1}\partial f$ and $f^{-1}\overline{\partial} f$ are left-invariant), then:

$$-iTr(f^{-1}\partial f \wedge f^{-1}\overline{\partial} f) = -\frac{1}{2} Tr\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) dxdy.$$

Hence, $E_\Sigma(f)$ is, up to constant multiple, equal to the energy $\int_\Sigma |df|^2 dvol$ defined previously. Note that there is no metric defined for Σ. A complex structure on Σ defines a metric up to a conformal factor, i.e., $g \sim fg$, where f is a positive function on Σ. Hence $E_\Sigma(f)$ only depends on the *conformal class* of the metric, corresponding to the complex structure on Σ.

WZW action S_Σ: Let k be a nonnegative integer, called the level. Then define:

$$S_\Sigma(f) = \frac{k}{4\pi} E_\Sigma(f) - 2\pi i k \int_B \tilde{f}^* \sigma,$$

where B is a 3-manifold with $\partial B = \Sigma$, $\tilde{f} : B \to G$ is an extension of $f : \Sigma \to G$, and $\sigma = \frac{1}{24\pi^2} Tr(\mu \wedge \mu \wedge \mu)$ where μ is the Maurer-Cartan form. (Recall σ was treated in the previous lecture.)

HW: Prove that $f : \Sigma \to G$ always admits and extension $\tilde{f} : B \to G$.

Remark: The second, topological term $-2\pi i k \int_B \tilde{f}^* \sigma$ is called the Wess-Zumino term. This is apparently needed for conformal invariance. (I don't know why at this point.)

Remark: $S_\Sigma(f)$ is, strictly speaking, $S_\Sigma(\tilde{f})$. To remove the dependence on the extension $\tilde{f}$, we exponentiate it.

Lemma 14.1. $exp(-S_\Sigma(f))$ *does not depend on the choice of* $\tilde{f}: B \to G$.

Proof. Take two extensions $\tilde{f}: B \to G$ and $\tilde{f}': B' \to G$. We can glue them together to obtain $F: M = B \cup (-B') \to G$. Then

$$S_\Sigma(\tilde{f}) - S_\Sigma(\tilde{f}') = 2\pi ik \left(\int_B \tilde{f}^* \sigma - \int_{B'} (\tilde{f}')^* \sigma \right) = 2\pi ik \int_M F^* \sigma.$$

Now, $\int_M F^*\sigma$ is an integer since $[\sigma] \in H^3(G; \mathbf{Z})$. If we exponentiate, then $exp(-S_\Sigma(\tilde{f})) = exp(-S_\Sigma(\tilde{f}'))$. □

14.2. **The Polyakov-Wiegmann formula.**

Proposition 14.2 (Polyakov-Wiegmann formula)**.** *Let* Σ *be a closed Riemann surface. Given* $f, g: \Sigma \to G$, *we have:*

$$exp(-S_\Sigma(fg)) = exp(-S_\Sigma(f) - S_\Sigma(g) + \Gamma_\Sigma(f,g)),$$

where $\Gamma_\Sigma(f,g) = -\frac{ik}{2\pi}\int_\Sigma Tr(f^{-1}\overline{\partial} f \wedge \partial g g^{-1})$.

Proof. We will often use the identity

$$Tr(\omega \wedge \eta) = (-1)^{pq} Tr(\eta \wedge \omega),$$

where ω is a p-form with values in $\mathfrak{g}$ and η is a q-form with values in $\mathfrak{g}$.

We compute

$$\begin{aligned}
-\frac{k}{4\pi}E_\Sigma(f) &= \frac{ik}{4\pi}\int_\Sigma Tr((fg)^{-1}\partial(fg) \wedge (fg)^{-1}\overline{\partial}(fg)) \\
&= \frac{ik}{4\pi}\int_\Sigma Tr((g^{-1}f^{-1})(f\partial g + \partial f \cdot g) \wedge (g^{-1}f^{-1})(f\overline{\partial} g + \overline{\partial} f \cdot g)) \\
&= \frac{ik}{4\pi}\int_\Sigma Tr((g^{-1}\partial g + g^{-1}f^{-1}\partial f \cdot g) \wedge (g^{-1}\overline{\partial} g + g^{-1}f^{-1}\overline{\partial} f \cdot g)) \\
&= \frac{ik}{4\pi}\int_\Sigma Tr(g^{-1}\partial g \wedge g^{-1}\overline{\partial} g + f^{-1}\partial f \wedge f^{-1}\overline{\partial} f + \partial g g^{-1} \wedge f^{-1}\overline{\partial} f + f^{-1}\partial f \wedge \overline{\partial} g \cdot g^{-1}) \\
&= -\frac{k}{4\pi}(E_\Sigma(f) + E_\Sigma(g)) + \frac{ik}{4\pi}\int_\Sigma Tr(-f^{-1}\overline{\partial} f \wedge \partial g g^{-1} + f^{-1}\partial f \wedge \overline{\partial} g \cdot g^{-1})
\end{aligned}$$

Next,

$$\begin{aligned}
Tr(f^{-1}df \wedge dgg^{-1}) &= Tr(f^{-1}(\partial f + \overline{\partial} f) \wedge (\partial g + \overline{\partial} g)g^{-1}) \\
&= Tr(f^{-1}\partial f \wedge \overline{\partial} g g^{-1}) + Tr(f^{-1}\overline{\partial} f \wedge \partial g g^{-1}),
\end{aligned}$$

since terms of the form $\partial f \wedge \partial g$ and $\overline{\partial} f \wedge \overline{\partial} g$ are zero. Hence we conclude that:

$$-\frac{k}{4\pi}E_\Sigma(f) = \frac{ik}{4\pi}(-E_\Sigma(f) - E_\Sigma(g)) + \Gamma_\Sigma(f,g) + \frac{ik}{4\pi}\int_\Sigma Tr(f^{-1}df \wedge dgg^{-1}). \tag{8}$$

Next consider the Wess-Zumino terms. Omitting tildes for convenience, we have:

$$\begin{aligned} 2\pi ik\int_B (fg)^*\sigma &= \frac{2\pi ik}{24\pi^2}\int_B Tr((fg)^{-1}d(fg)\wedge(fg)^{-1}d(fg)\wedge(fg)^{-1}d(fg)) \\ &= \frac{ik}{12\pi}\int_B (A_1+3A_2+3A_3+A_4), \\ &= 2\pi ik\int_B (f^*\sigma+g^*\sigma)+\frac{ik}{4\pi}\int_B (A_2+A_3) \end{aligned}$$

where

$$\begin{aligned} A_1 &= Tr(f^{-1}dff^{-1}dff^{-1}df) \\ A_2 &= Tr(dgg^{-1}f^{-1}dff^{-1}df) \\ A_3 &= Tr(dgg^{-1}dgg^{-1}f^{-1}df) \\ A_4 &= Tr(g^{-1}dgg^{-1}dgg^{-1}dg) \end{aligned}$$

Now,

$$d(Tr(dgg^{-1}f^{-1}df)) = dgg^{-1}dgg^{-1}df + dgg^{-1}f^{-1}dff^{-1}df = A_2+A_3,$$

using $d(f^{-1}) = -f^{-1}dff^{-1}$. Finally,

$$\frac{ik}{4\pi}\int_B (A_2+A_3) = \frac{ik}{4\pi}\int_\Sigma Tr(dgg^{-1}f^{-1}df), \tag{9}$$

using Stokes' Theorem. Combining Equations 8 and 9 gives the result. □

14.3. **Line bundles over $LG_{\mathbf{C}}$.** Let us use the complexification $G_{\mathbf{C}}$ instead of G.

Consider $\mathbf{CP}^1 = \mathbf{C}\cup\{\infty\}$. Let $D_0 = \{|z|\le 1\}\subset \mathbf{C}$, $D_\infty = \{|z|\ge 1\}\cup\{\infty\}$, and $S^1 = \{|z| = 1\} = \partial D_0 = -\partial D_\infty$.

We define a complex line bundle $\mathcal{L}$ over $LG_{\mathbf{C}}$ as follows: Let $Map_0(D_\infty, G)$ be the set of smooth maps $f_\infty : D_\infty \to G_{\mathbf{C}}$ with $f_\infty(\infty) = e$. Then let $\mathcal{L} = Map_0(D_\infty, G_{\mathbf{C}})/\sim$, where $(f_\infty, u)\sim(g_\infty, v)$ if:

(1) $f_\infty|_{S^1} = g_\infty|_{S^1}$.
(2) If $g_\infty = f_\infty h_\infty$, then

$$v = u\cdot exp(-S_{\mathbf{CP}^1}(h)+\Gamma_{D_\infty}(f_\infty, h_\infty)).$$

Here h is an extension of h_∞ to D_0 by e. (Note that $h_\infty|_{S^1} = e$.)

The equivalence class of (f_∞, u) will be denoted $[f_\infty, u]$. The projection $\mathcal{L}\to LG_{\mathbf{C}}$ is given by $[f_\infty, u]\mapsto f_\infty|_{S^1}$.

We can view $f_0 : D_0 \to G_{\mathbf{C}}$ as an element of $\mathcal{L}$ as follows: Let f_∞ be a smooth extension of f_0, i.e., $f_\infty|_{S^1} = f_0|_{S^1}$. Then assign $f_0\mapsto [f_\infty, exp(-S_{\mathbf{CP}^1}(f))]$.

Lemma 14.3. $[f_\infty, exp(-S_{\mathbf{CP}^1}(f))]$ *does not depend on the extension* f_∞.

Proof.

$$\begin{aligned}[f_\infty, exp(-S_{\mathbf{CP}^1}(f))] &= [g_\infty, exp(-S_{\mathbf{CP}^1}(f))exp(-S_{\mathbf{CP}^1}(h) + \Gamma_{D_\infty}(f_\infty, h_\infty))], \\ &= [g_\infty, exp(-S_{\mathbf{CP}^1}(f_0 \cup g_\infty))],\end{aligned}$$

using the Polyakov-Wiegmann formula. Here $g_\infty = f_\infty h_\infty$, and h is the extension of h_∞ to $\mathbf{CP}^1$ by setting $h(D_0) = e$. □

15. WZW, CONTINUED

We use the notation from last time. In particular, $G = SU(2)$ still.

15.1. **Line bundles.** Last time we defined the complex line bundle $\mathcal{L} = Map_0(D_\infty, G_{\mathbf{C}}) \times \mathbf{C}/\sim$, where $(f_\infty, u) \sim (g_\infty, v)$ if the following hold:

(1) $f_\infty|_{S^1} = g_\infty|_{S^1}$.
(2) If $g_\infty = f_\infty h_\infty$, then $v = u \cdot exp(-S_{\mathbf{CP}^1}(h) + \Gamma_{D_\infty}(f_\infty, h_\infty))$.

Here h is h_∞ extended to D_0 by $h(D_0) = e$.

We can also define the dual line bundle $\mathcal{L}^{-1} = Map_0(D_0, G_{\mathbf{C}}) \times \mathbf{C}/\sim$, where $(f_0, u) \sim (g_0, v)$ if:

(1) $f_0|_{S^1} = g_0|_{S^1}$.
(2) If $g_0 = f_0 h_0$, then $v = u \cdot exp(-S_{\mathbf{CP}^1}(h) + \Gamma_{D_0}(f_0, h_0))$.

Again, h is h_0 extended to D_∞ by $h(D_\infty) = e$.

Let $\gamma \in G_{\mathbf{C}}$. Then the fiber of $\mathcal{L}$ over γ is denoted $\mathcal{L}_\gamma$. Then there is a pairing $\mathcal{L}_\gamma \times \mathcal{L}_\gamma^{-1} \to \mathbf{C}$, given by

$$\langle [f_\infty, u], [g_0, v] \rangle = uv \cdot exp(S_{\mathbf{CP}^1}(f_\infty \cup g_0)). \tag{10}$$

We use the notation $f_\infty \cup g_0$ to mean the map $\mathbf{CP}^1 \to G_{\mathbf{C}}$ which restricts to f_∞ on D_∞ and g_0 on D_0. Observe that there is no minus sign in front of $S_{\mathbf{CP}^1}$.

HW: Verify that the pairing does not depend on the choice of representative of $[f_\infty, u]$ and $[g_0, v]$.

Lemma 15.1.

(1) $f_0 : D_0 \to G_{\mathbf{C}}$ *determines an element* $[f_\infty, exp(-S_{\mathbf{CP}^1}(f_0 \cup f_\infty))]$ *in* $\mathcal{L}$ *which does not depend on the choice of* f_∞.
(2) $f_\infty : D_\infty \to G_{\mathbf{C}}$ *determines an element* $[f_0, exp(-S_{\mathbf{CP}^1}(f_0 \cup f_\infty))]$ *in* $\mathcal{L}^{-1}$ *which does not depend on the choice of* f_0.

The pairing in Equation 10 can be reinterpreted as follows:

$$\langle exp(-S_{D_0}(f_0)), exp(-S_{D_\infty}(f_\infty)) \rangle = exp(-S_{\mathbf{CP}^1}(f_0 \cup f_\infty)).$$

More generally, let Σ be a compact Riemann surface with (oriented) boundary components $(\partial\Sigma)_i$, $i = 1, \dots, n$. Let $\widetilde{\Sigma}$ be a closed Riemann surface obtained from Σ by capping off $(\partial\Sigma)_i$, $i = 1, \dots, n$, by disks D_i. (We need to put complex structures on D_i.) Given $f : \Sigma \to G_{\mathbf{C}}$, extend f to $\tilde{f} : \widetilde{\Sigma} \to G_{\mathbf{C}}$ via $f_i : D_i \to G_{\mathbf{C}}$. Then define $exp(-S_\Sigma(f)) \in \otimes_{i=1}^n \mathcal{L}_{\gamma_i}$, where $\gamma_i = f|_{(\partial\Sigma)_i}$, by the following relation:

$$\langle exp(-S_\Sigma(f)), \otimes_{i=1}^n exp(-S_{D_i}(f_i)) \rangle = exp(-S_{\widetilde{\Sigma}}(\tilde{f})).$$

Observe that $exp(-S_{D_i}(f_i))$ are elements in $\mathcal{L}_{\gamma_i}$.

15.2. **Holonomy.** A connection ∇ on a rank k vector bundle $E \to M$ gives rise to *parallel transport* along arcs in M. In local coordinates $\nabla = d + A$, where A is a $k \times k$ matrix of 1-forms.

Model situation: Let $M = [0,1]$. Then $E = [0,1] \times \mathbf{R}^k$. We are looking for solutions $x(t)$ (i.e., sections of the bundle) which satisfy

$$\dot{x}(t) + A(t)x(t) = 0.$$

Here $t \in [0,1]$ and $x(t) \in \mathbf{R}^k$.

HW: Prove that there is a unique solution, given an initial condition $x(0) = x_0$.

Given a connection ∇, there is a well-defined linear map called the *holonomy map* Hol which sends E_0 to E_1: Given $x_0 \in E_0$, find a solution $x(t)$ with $x(0) = x_0$. Then $\text{Hol}(x_0) = x(1)$.

More generally, if we fix ∇, then to each arc $\gamma : [0,1] \to M$, we have the corresponding *holonomy map* $\text{Hol}_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ obtained by pulling back to the model situation on $[0,1]$.

HW: Prove that, for line bundles, $Hol_\gamma : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$ is given by multiplication by $e^{2\pi i \int_D \omega}$, if γ is a closed curve which bounds a disk D and ω is $\frac{i}{2\pi}$ times the curvature form, i.e., ω represents the first Chern class.

We now explain the following fact:

Proposition 15.2. *The line bundle $\mathcal{L}$ is the kth tensor power of the line bundle over $LG_{\mathbf{C}}$ which corresponds to $[\int_{S^1} \phi^*\sigma] \in H^2(LG;\mathbf{Z})$ given in Lemma 13.4. In other words, $c_1(\mathcal{L}) = k[\int_{S^1} \phi^*\sigma]$.*

Proof. Think of $f_\infty : D_\infty \to G_{\mathbf{C}}$ with $f_\infty(\infty) = e$ as a path γ in $LG_{\mathbf{C}}$ based at e. Here $e \in LG_{\mathbf{C}}$ maps $S^1 \to e$. More precisely, $\gamma(0) = e$ and $\gamma(t)$, $t \in [0,1]$, is $f_\infty|_{\{|z|=1/t\}}$ with the counterclockwise orientation on $\mathbf{C}$.

Given $f_\infty, g_\infty : D_\infty \to G_{\mathbf{C}}$ with $f_\infty|_{S^1} = g_\infty|_{S^1}$ and $f_\infty(\infty) = g_\infty(\infty) = e$, we have corresponding paths γ_1, γ_2 of $LG_{\mathbf{C}}$ based at e, and the holonomy around $\gamma_2^{-1}\gamma_1$ is:

$$\begin{aligned} exp(-S_{\mathbf{CP}^1}(h) + \Gamma_{D_\infty}(f_\infty, h_\infty)) &= exp(-S_{\mathbf{CP}^1}(f_0 \cup g_\infty) + S_{\mathbf{CP}^1}(f_0 \cup f_\infty)) \\ &= exp(-\frac{k}{4\pi}E_{D_\infty}(g_\infty) + \frac{k}{4\pi}E_{D_\infty}(f_\infty) + 2\pi i k \int_{B'} G^*\sigma). \end{aligned}$$

Here f_0 is an extension of f_∞ and g_∞ to D_0, and $g_\infty = f_\infty h_\infty$. If $F : B \to G_{\mathbf{C}}$ extends $f_0 \cup f_\infty$ to a 3-manifold B with boundary $\mathbf{CP}^1$ and $G : B \cup B' \to G_{\mathbf{C}}$ extends $f_0 \cup g_\infty$, then $-f_\infty \cup g_\infty$ is the restriction of G to $\partial B'$. (Here $-f_\infty$ means f_∞ with the opposite orientation.)

Moreover, we can take B' to be the 3-ball $B^3 = \{(x,y,z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 \leq 1\}$ and G to map the axis $x = y = 0$ to e. (Prove this using properties of the homotopy groups of G!) By fibering $B^3 - \{x = y = 0\}$ by circles of type $x^2 + y^2 = r^2$, $z = const$, we can view $G : B' \to G_{\mathbf{C}}$ as a map from the 2-dimensional disk to $LG_{\mathbf{C}}$. We then conclude that $exp(2\pi i k \int_{B'} G^*\sigma) = exp(2\pi i k \int_D \int_{S^1} \phi^*\sigma)$, where we are thinking of $\int_{S^1} \phi^*\sigma$ as the curvature form of a connection on the line bundle $\mathcal{L}'$ whose first Chern class generates $H^2(LG;\mathbf{Z})$. Modulo the energy terms, we would conclude that $\mathcal{L} = (\mathcal{L}')^{\otimes k}$.

One way to get rid of the energy terms is to define $\mathcal{L}_t$ as $Map_0(D_\infty, G_{\mathbf{C}}) \times \mathbf{C}/\sim$, where we replace condition (2) in the definition of $\mathcal{L}$ by:

(2') If $g_\infty = f_\infty h_\infty$, then $v = u \cdot exp(-t\frac{k}{4\pi}(E_{D_\infty}(g_\infty) - E_{D_\infty}(f_\infty)) + 2\pi ik \int_{B'} G^*\sigma)$.

$\mathcal{L}_1 = \mathcal{L}$ and $\mathcal{L}_0$ is the desired line bundle which is homotopic to $\mathcal{L}$ and does not have the energy terms. (Verify that the definition of $\mathcal{L}_t$ is consistent!) $\square$

16. More on WZW

16.1. **Group central extensions.** In the previous lectures we constructed the line bundle $\mathcal{L} \to LG_{\mathbf{C}}$ by taking $\mathcal{L} = Map_0(D_\infty, G_{\mathbf{C}}) \times \mathbf{C}/\sim$. Today we investigate the group structure on $\widehat{LG_{\mathbf{C}}}$, which is $\mathcal{L}$ minus the zero section.

Given $g_i : D \to G_{\mathbf{C}}$, $i = 1, 2$, we have $exp(-S_D(g_i)) \in \mathcal{L}_{g_i|_{S^1}}$. (When we write D we mean D_0.) Define the product:

$$exp(-S_D(g_1)) \star exp(-S_D(g_2)) = exp(-\Gamma_D(g_1, g_2))exp(-S_D(g_1 g_2)),$$

and extend linearly.

Unit: $exp(-S_D(e))$, where $e : D \to G_{\mathbf{C}}$ maps $D \mapsto e$. Then $\Gamma_D(e, g) = 0$ implies that $exp(-S_D(e)) \star exp(-S_D(g)) = exp(-S_D(g))$.

Associativity: Let us use the shorthand $g_1 * g_2$ for the above product. Then:

$$(g_1 \star g_2) \star g_3 = exp(-\Gamma_D(g_1, g_2))g_1g_2 \star g_3 = exp(-\Gamma_D(g_1, g_2) - \Gamma_D(g_1g_2, g_3))g_1g_2g_3,$$

$$g_1 \star (g_2 \star g_3) = g_1 \star exp(-\Gamma_D(g_2, g_3))g_2g_3 = exp(-\Gamma_D(g_2, g_3) - \Gamma_D(g_1, g_2g_3))g_1g_2g_3.$$

Therefore, associativity is equivalent to:

$$\Gamma_D(g_1, g_2) + \Gamma_D(g_1g_2, g_3) - \Gamma_D(g_2, g_3) - \Gamma(g_1, g_2g_3) = 0. \tag{11}$$

HW: Verify Equation 11!

Alternate definition: For $i = 1, 2$, choose h_i on D_∞ so that $h_i|_{S^1} = g_i|_{S^1}$. Then

$$exp(-S_D(g_i)) = [h_i, exp(-S_{\mathbf{CP}^1}(g_i \cup h_i))],$$

and the equation defining the product is equivalent to:

$$[h_1, exp(-S_{\mathbf{CP}^1}(g_1 \cup h_1))] \star [h_2, exp(-S_{\mathbf{CP}^1}(g_2 \cup h_2))]$$
$$= [h_1h_2, exp(-S_{\mathbf{CP}^1}(g_1g_2 \cup h_1h_2) - \Gamma_D(g_1, g_2))],$$

or

$$[h_1, 1] \star [h_2, 1] = [h_1h_2, exp(\Gamma_{D_\infty}(h_1, h_2))]. \tag{12}$$

Well-definition: We prove the well-definition using the alternate definition (although it is possible to prove directly). Let $h : D_\infty \to G_{\mathbf{C}}$ satisfy $h|_{S^1} = e$. Then

$$\begin{aligned}[h_1, 1] \star [h_2, 1] &= [h_1h_2, exp(\Gamma_{D_\infty}(h_1, h_2))] \\ &= [hh_1h_2, exp(-S_{\mathbf{CP}^1}(e \cup h) + \Gamma_{D_\infty}(h, h_1h_2) + \Gamma_{D_\infty}(h_1, h_2))].\end{aligned}$$

We also have:

$$\begin{aligned}[h_1, 1] \star [h_2, 1] &= [hh_1, exp(-S_{\mathbf{CP}^1}(e \cup h) + \Gamma_{D_\infty}(h, h_1))] \star [h_2, 1] \\ &= [hh_1h_2, exp(-S_{\mathbf{CP}^1}(e \cup h) + \Gamma_{D_\infty}(h, h_1) + \Gamma_{D_\infty}(hh_1, h_2))].\end{aligned}$$

Equation 11 shows that they are equal.

We now have the following group central extension

$$1 \to \mathbf{C}^* \xrightarrow{i} \widehat{LG_{\mathbf{C}}} \xrightarrow{\pi} LG_{\mathbf{C}} \to 1.$$

Here i maps $1 \mapsto exp(-S_D(e))$ and π maps $exp(-S_D(g)) \mapsto g|_{S^1}$. This central extension is the analog of the Lie algebra central extension

$$0 \to \mathbf{C} \to \widetilde{L\mathfrak{g}} \to L\mathfrak{g} \to 0.$$

16.2. **Left and right actions.** Suppose Σ is a Riemann surface with only one boundary component. Then $Map(\Sigma, G_{\mathbf{C}})$ acts on $\widehat{LG_{\mathbf{C}}}$ as follows: given $f : \Sigma \to G_{\mathbf{C}}$, we set:

$$l(f)exp(-S_\Sigma(g)) = exp(-S_\Sigma(f)) \star exp(-S_\Sigma(g)).$$

HW: Prove that $exp(-S_\Sigma(f)) \star exp(-S_\Sigma(g)) = exp(-S_\Sigma(fg))exp(-\Gamma_\Sigma(f,g))$. (Hint: use Equation 12.)

Similarly, we can define

$$r(f)exp(-S_\Sigma(g)) = exp(-S_\Sigma(g)) \star exp(-S_\Sigma(f)).$$

Representations of $Map(\Sigma, G_{\mathbf{C}})$ on $\Gamma(\mathcal{L})$. Let $\Gamma(\mathcal{L})$ be the space of sections of $\mathcal{L}$. The left and right actions above give rise to:

$$\rho : Map(\Sigma, G_{\mathbf{C}}) \to Aut(\Gamma(\mathcal{L})).$$

If $s \in \Gamma(\mathcal{L})$ and $\gamma \in LG_{\mathbf{C}}$, then:

$$[\rho(f)s](\gamma) = l(f)s((f|_{\partial\Sigma})^{-1}\gamma).$$

Also we have $\rho^* : Map(\Sigma, G_{\mathbf{C}}) \to Aut(\Gamma(\mathcal{L}))$ given by:

$$[\rho^*(f)s](\gamma) = r(f^*)s(\gamma(f^*|_{\partial\Sigma})^{-1}).$$

where $f^*(z) = \overline{f(z)}^T$.

Lemma 16.1. *If $h : \Sigma \to G_{\mathbf{C}}$ is holomorphic and $g : \Sigma \to G_{\mathbf{C}}$ is smooth, then*

$$l(h)exp(-S_\Sigma(g)) = exp(-S_\Sigma(hg)).$$

Similarly, if $h : \Sigma \to G_{\mathbf{C}}$ is antiholomorphic, then

$$r(h)exp(-S_\Sigma(g)) = exp(-S_\Sigma(gh)).$$

Proof. Since $\overline{\partial}h = 0$, $\Gamma_\Sigma(h,g) = C\int_\Sigma Tr(h^{-1}\overline{\partial}h \wedge \partial g g^{-1}) = 0$. (Here C is some constant.) □

16.3. **Representation of affine Lie algebras.** We now explain how $\Gamma(\mathcal{L})$ is a representation of the Lie algebra $\widetilde{L\mathfrak{g}}$.

Let $\Sigma = D$ and $X \in \mathfrak{g}$. At the infinitesimal level, define $X_{n,\varepsilon}(z) = e^{\varepsilon X z^n}$ for nonnegative integers n and $e^{\varepsilon X \overline{z}^{-n}}$ for negative integers n. Here $z \in D$ and ε is a small real number. Then define the infinitesimal action

$$X_n s = \frac{d}{d\varepsilon}|_{\varepsilon=0} \rho(X_{n,\varepsilon}) s$$

for $s \in \Gamma(\mathcal{L})$.

Proposition 16.2. *$[X_m, Y_n] = [X,Y]_{m+n} + mk\delta_{m+n,0}\langle X, Y\rangle$, as actions on $\Gamma(\mathcal{L})$.*

Therefore, $X \otimes t^m \mapsto X_m$ gives a representation of $\widetilde{L\mathfrak{g}}$ on $\Gamma(\mathcal{L})$.

Proof. Suppose $m, n \geq 0$. First note that $\Gamma_D(f, g) = 0$ if f is holomorphic by Lemma 16.1. We compute that

$$\begin{aligned}
[X_m, Y_n]s &= \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \frac{1}{\varepsilon_1 \varepsilon_2} (l(X_{m,\varepsilon_1}) l(Y_{n,\varepsilon_2}) l((-X)_{m,\varepsilon_1}) l((-Y)_{n,\varepsilon_2}) s - s) \\
&= \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \frac{1}{\varepsilon_1 \varepsilon_2} (l(e^{\varepsilon_1 X z^m} e^{\varepsilon_2 Y z^n} e^{-\varepsilon_1 X z^m} e^{-\varepsilon_2 Y z^n}) s - s) \\
&= \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \frac{1}{\varepsilon_1 \varepsilon_2} (l(1 + \varepsilon_1 \varepsilon_2 [X,Y] z^{m+n} + \dots) s - s) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (l(e^{\varepsilon [X,Y] z^{m+n}}) s - s) \\
&= [X,Y]_{m+n} s.
\end{aligned}$$

Next, suppose $m \geq 0$ and $n < 0$. We compute

$$\begin{aligned}
\Gamma_D(e^{\varepsilon_2 Y \overline{z}^{-n}}, e^{-\varepsilon_1 X z^m}) &= \frac{-ik}{2\pi} \int_D Tr((-n)\varepsilon_2 Y \overline{z}^{-n-1} d\overline{z} \wedge m\varepsilon_1 X z^{m-1} dz) \\
&= \varepsilon_1 \varepsilon_2 \frac{ik}{2\pi} (m)(-n) \langle X, Y\rangle \int_D z^{m-1} \overline{z}^{-n-1} dz d\overline{z}.
\end{aligned}$$

Now using polar coordinates $z = re^{i\theta}$ and writing $dzd\overline{z} = -2idxdy = -2irdrd\theta$, we have:

$$\begin{aligned}
\Gamma_D(e^{\varepsilon_2 Y \overline{z}^{-n}}, e^{-\varepsilon_1 X z^m}) &= \varepsilon_1 \varepsilon_2 \frac{k}{\pi} (m)(-n) \langle X, Y\rangle \int_0^{2\pi} e^{(m+n)i\theta} d\theta \int_0^1 r^{m-n-1} dr \\
&= \varepsilon_1 \varepsilon_2 \frac{k}{\pi} (m)(-n) \langle X, Y\rangle (2\pi \delta_{m+n,0}) \frac{1}{m-n} \\
&= \varepsilon_1 \varepsilon_2 km \langle X, Y\rangle \delta_{m+n,0}.
\end{aligned}$$

A similar calculation of $[X_m, Y_n]s$ as in the case $m, n \geq 0$ gives the desired result. $\square$

Recall the Feynman path integral philosophy. In order to understand

$$\int_{Map(\widetilde{\Sigma},G_{\mathbf{C}})} exp(-S_{\widetilde{\Sigma}}(f))\mathcal{D}f,$$

where $\widetilde{\Sigma}$ is a closed surface, we instead calculate:

$$s(\gamma) = \int_{Map_\gamma(\Sigma,G_{\mathbf{C}})} exp(-S_\Sigma(f))\mathcal{D}f,$$

where $\gamma \in LG_{\mathbf{C}}$, Σ has one boundary component, and Map_γ means maps that restrict to γ on $\partial\Sigma$. Since $exp(-S_\Sigma(f))$ can be interpreted as an element of $\mathcal{L}_\gamma$, so is $s(\gamma)$. We therefore obtain a section $s \in \Gamma(\mathcal{L})$. Moreover, the section is invariant under left multiplication by holomorphic maps and right multiplication by antiholomorphic maps.

17. Conformal blocks

Let Σ be a closed Riemann surface and $p_1, \dots, p_n$ be n distinct points on Σ. Let $\mathcal{M}_{p_1,\dots,p_n}$ be the space of meromorphic functions on Σ with poles of arbitrary order at most at $p_1, \dots, p_n$. (Note that our setting is slightly different from last time — last time we had a Riemann surface with boundary and this time we have a Riemann surface with punctures.)

Define $\mathfrak{g}(p_1, \dots, p_n) = \mathfrak{g} \otimes_{\mathbf{C}} \mathcal{M}_{p_1,\dots,p_n}$. This is a Lie algebra with bracket $[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$. Note that $\mathfrak{g}(p_1, \dots, p_n)$ is supposed to correspond to the infinitesimal version of the Lie group $Map(\text{surface with boundary}, G_{\mathbf{C}})$.

For each $i = 1, \dots, n$ there is a linear map $\phi_i : \mathfrak{g}(p_1, \dots, p_n) \to \widetilde{L\mathfrak{g}}$ (not a Lie algebra homomorphism), defined as follows: Choose local holomorphic coordinates t_i about p_i. Given $X \otimes f \in \mathfrak{g}(p_1, \dots, p_n)$, write f as a Laurent series $f(t_i)$ in t_i and map $X \otimes f \mapsto X \otimes f(t_i)$. ϕ_i is the composition of this map with the natural inclusion $\mathfrak{g} \otimes \mathbf{C}((t_i)) \to (\mathfrak{g} \otimes \mathbf{C}((t_i))) \oplus \mathbf{C}c = \widetilde{L\mathfrak{g}}$.

Fix a level k and let H_{λ_i} be an integral highest weight representation of $\widetilde{L\mathfrak{g}}$ with highest weight λ_i (i.e., if v is the highest weight vector, then $Hv = \lambda_i v$). We define the diagonal action Δ of $\mathfrak{g}(p_1, \dots, p_n)$ on $H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n}$ as follows:

$$\Delta(X \otimes f)(\xi_1 \otimes \dots \otimes \xi_n) = \sum_{i=1}^{n} \xi_1 \otimes \dots \otimes \phi_i(X \otimes f)\xi_i \otimes \xi_n.$$

Lemma 17.1. $\delta : \mathfrak{g}(p_1, \dots, p_n) \to End(H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n})$ *is a representation of* $\mathfrak{g}(p_1, \dots, p_n)$.

Proof. We show that $\Delta([X \otimes f, Y \otimes g]) = \Delta([X, Y] \otimes fg)$ is equal to $[\Delta(X \otimes f), \Delta(Y \otimes g)]$.

$$\begin{aligned} \Delta(X \otimes f)\Delta(Y \otimes g)(\xi_1 \otimes \dots \otimes \xi_n) &= \sum_{i \neq j} \xi_1 \otimes \dots \otimes \phi_i(X \otimes f)\xi_i \otimes \phi_j(Y \otimes g)\xi_j \otimes \dots \otimes \xi_n \\ &+ \sum_i \xi_1 \otimes \dots \otimes \phi_i(X \otimes f)\phi_j(Y \otimes g)\xi_i \otimes \dots \otimes \xi_n. \end{aligned}$$

Taking commutators, what remains is:

$$[\Delta(X \otimes f), \Delta(Y \otimes g)](\xi_1 \otimes \dots \otimes \xi_n) = \sum_i \xi_1 \otimes \dots \otimes [X \otimes f(t_i), Y \otimes g(t_i)]\xi_i \otimes \dots \otimes \xi_n.$$

Now,

$$[X \otimes f(t_i), Y \otimes g(t_i)]\xi_i = ([X, Y] \otimes f(t_i)g(t_i) + \langle X, Y \rangle Res_{p_i}(df \cdot g)k)\xi_i,$$

where Res_{p_i} is the residue at p_i. The lemma follows once we show that the sum of the residues of $df \cdot g$ is zero. □

Residues: A *meromorphic 1-form* ω is a 1-form which can be written locally as as Fdz, where z is the holomorphic coordinate and F is a meromorphic function. The residue of a meromorphic 1-form at a point a is: $\frac{1}{2\pi i}\int_\gamma \omega$, where γ is a sufficiently small closed curve which encircles a

counterclockwise exactly once. The residue is a purely topological quantity, since the integral only depends on the homology class of γ.

Lemma 17.2. *The sum of residues of a meromorphic 1-form ω is zero on a closed Riemann surface Σ.*

Proof. A meromorphic 1-form ω is closed: If $\omega = F dz$ locally, then $d\omega = (\partial F + \overline{\partial} F) dz = \partial F \wedge dz = \frac{\partial f}{\partial z} dz \wedge dz = 0$. Now if $D \subset \Sigma$ is a disk that contains all the poles, then the sum of the residues is $\frac{1}{2\pi i}\int_{\partial D}\omega$. However, it is also $\frac{1}{2\pi i}\int_{-\partial(\Sigma - D)}\omega$, which is zero by Stokes' theorem (since there are no poles in $\Sigma - D$). □

Conformal blocks: The space of *conformal blocks* $\mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$ is the space of multilinear maps

$$\Psi : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \to \mathbf{C}$$

which are invariant under the diagonal action Δ of $\mathfrak{g}(p_1, \dots, p_n)$. By *invariance* we mean

$$\sum_{i=1}^{n} \Psi(\xi_1 \otimes \cdots \otimes (X \otimes f(t_i))\xi_i \otimes \cdots \otimes \xi_n) = 0$$

for all $X \otimes f$. (This is also often called "coinvariance".) We can also write

$$\mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n) = Hom_{\mathfrak{g}(p_1, \dots, p_n)}(\otimes_{i=1}^{n} H_{\lambda_i}, \mathbf{C}),$$

where the action of $\mathfrak{g}(p_1, \dots, p_n)$ on $\mathbf{C}$ is the trivial action.

The definition of the space of conformal blocks is consistent with our discussion of $\Gamma(\mathcal{L})$ from last time (although not exactly on the nose). Each pole p_i corresponds to a boundary component of a Riemann surface and a line bundle $\mathcal{L}$ is attached to it. Invariance under left multiplication by holomorphic maps $f : \Sigma \to G_{\mathbf{C}}$ (here Σ has boundary) corresponds to $\mathfrak{g}(p_1, \dots, p_n)$-invariance.

The space of conformal blocks seems like a daunting infinite-dimensional object. The next lemma shows that, $\mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$ is *finite-dimensional*! Recall first the weight space diagram of H_λ. $H_\lambda = \oplus_{d=0}^{\infty} H_\lambda(d)$, where $H_\lambda(0) = V_\lambda$ is the top row of the weight space diagram (with conformal weight Δ_λ, defined before), $H_\lambda(1)$ is the next row down, etc.

Lemma 17.3. *Suppose $\Sigma = \mathbf{CP}^1$. Then the map $\mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n) \to Hom_{\mathfrak{g}}(\otimes_{i=1}^{n} V_{\lambda_i}, \mathbf{C})$ is injective.*

Proof. Suppose $\Psi = 0$ on $\otimes_i V_{\lambda_i}$. We show that $\Psi = 0$ on all of $\otimes_i H_{\lambda_i}$.

Let z be the usual coordinate on $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$. Consider $f(z) = (z - z_i)^r$ on $\mathbf{CP}^1$, where z_i is the ith pole and $r < 0$. If we expand about the other poles and write $t_i = z - z_i$, then we have $f(t_i) = \sum_{m=0}^{\infty} a_m^{(j)} t_j^m$, and

$$\Psi(\xi_1, \dots, (X \otimes t_i^r)\xi_i, \dots, \xi_n) = -\sum_{j \neq i}\sum_{m \geq 0} a_m^{(j)} \Psi(\xi_1, \dots, (X \otimes t_j^m)\xi_j, \dots, \xi_n).$$

Note that $r < 0$ but the m's are ≥ 0.

We argue inductively, starting with all $\xi_i \in V_{\lambda_i} = H_{\lambda_i}(0)$. Then $(X \otimes t_j^m)\xi_j$ for $m > 0$ are zero since they raise ξ_j further up (and there's nothing above it). Also, $\Psi = 0$ on $\otimes_i V_{\lambda_i}$, so the RHS of the equation vanishes. Hence so does the LHS. Now apply induction. $\square$

18. More on conformal blocks

Recall from last time that the space of conformal blocks

$$\mathcal{H}(p_1, \ldots, p_n; \lambda_1, \ldots, \lambda_n) = Hom_{\mathfrak{g}(p_1,\ldots,p_n)}(H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n}, \mathbf{C})$$

injects into $Hom_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, \mathbf{C})$. Today we determine its image.

18.1. Quantum Clebsch-Gordan rule.

Theorem 18.1. $\dim_{\mathbf{C}} \mathcal{H}(p_1, p_2, p_3; \lambda_1, \lambda_2, \lambda_3) = 1$ *if* $(*)$ *holds and* $= 0$ *otherwise.*

Here, $(*)$ is:

(1) $\lambda_1 + \lambda_2 + \lambda_3 \in 2\mathbf{Z}$;
(2) $\lambda_1 + \lambda_2 \geq \lambda_3$, $\lambda_2 + \lambda_3 \geq \lambda_1$, $\lambda_1 + \lambda_3 \geq \lambda_2$;
(3) $\lambda_1 + \lambda_2 + \lambda_3 \leq 2k$.

Conditions (1) and (2) were called the Clebsch-Gordan rule. These were the conditions for $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. (3) is the quantum condition, and (1), (2), and (3) together are called the *quantum Clebsch-Gordan rule*.

Step 1:

Lemma 18.2. *If* $\Psi \in \mathcal{H}(p_1, p_2, p_3; \lambda_1, \lambda_2, \lambda_3)$, *then*

$$\Psi(v_1, E^{m_2}\xi_2, E^{m_3}\xi_3) = 0,$$

if $v_1 \in V_{\lambda_1}$ *of highest weight,* $\xi_j \in V_{\lambda_j}$, $j \neq 1$, *and* $m_2 + m_3 = d_1 = k - \lambda_1 + 1$.

Proof. We use the $\mathfrak{g}(p_1, p_2, p_3)$-invariance with $E \otimes f$, where $f(z) = (z - z_1)^{-1}$. Let us use the standard complex coordinate $z \in \mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$. Also write p_i as z_i, and let $t_i = z - z_i$ be the local coordinate centered about z_i.

HW: Prove that the power series expansion of $f(z) = (z - z_1)^{-1}$ about z_j is $\sum_0^\infty (-1)^n (z_j - z_1)^{-n-1}(z - z_j)^n$. (Hint: use the binomial theorem.)

The invariance gives:

$$\begin{aligned}
\Psi((E \otimes t_1^{-1})v_1, \xi_2, \xi_3) &= -\sum_{m \geq 0} (-1)^m (z_2 - z_1)^{-m-1} \Psi(v_1, (E \otimes t_2^m)\xi_2, \xi_3) \\
&\quad - \sum_{m \geq 0} (-1)^m (z_3 - z_1)^{-m-1} \Psi(v_1, \xi_2, (E \otimes t_2^m)\xi_3) \\
&= -(z_2 - z_1)^{-1} \Psi(v_1, E\xi_2, \xi_3) - (z_3 - z_1)^{-1} \Psi(v_1, \xi_2, E\xi_3).
\end{aligned}$$

Note that $(X \otimes t^m)(\xi) = 0$ if $\xi \in V_\lambda$ and $m > 0$, since the ξ are on the top row of the weight space diagram and t^m raises it.

By repeated application of the invariance, we have:
(13)

$$\Psi((E\otimes t_1^{-1})^{d_1}v_1,\xi_2,\xi_3)=(-1)^{d_1}\sum_{m_2+m_3=d_1}\frac{d_1!}{m_2!m_3!}(z_2-z_1)^{-m_2}(z_3-z_1)^{-m_3}\Psi(v,E^{m_2}\xi_2,E^{m_3}\xi_3)=0.$$

Here Kohno immediately concludes the lemma by arguing that z_1,z_2,z_3 are arbitrary. Since Ψ depends on the particular choice of z_1,z_2,z_3, the argument seems incomplete. Instead, we use the following trick: Using the fact that Ψ is $\mathfrak{g}$-invariant, it follows that:

$$\Psi(E(v\otimes\xi_2\otimes E^{d_1-1}\xi_3))=0.$$

We therefore have:

$$\begin{aligned}\Psi(v,E\xi_2,E^{d_1-1}\xi_3)+\Psi(v,\xi_2,E^{d_1}\xi_3) &= 0\\ &\vdots\\ \Psi(v,E^{d_1}\xi_2,\xi_3)+\Psi(v,E^{d_1-1}\xi_2,E\xi_3) &= 0\end{aligned}$$

HW: Verify that these d_1 equations and Equation 13 are linearly independent.

The proof of the lemma is complete. $\square$

Remark: Equation 13 is the only relation that only involves elements in $V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}$, besides those that come from the $\mathfrak{g}$-invariance. We should think of Equation 13 as the *extra conditions* that need to be satisfied in addition to the $\mathfrak{g}$-invariance.

Step 2. Suppose $\lambda_1+\lambda_2+\lambda_3\leq 2k$. Write $Hv_1=\lambda_1v_1$, $H\xi_2=(-\lambda_2+2n_2)\xi_2$, and $H\xi_3=(-\lambda_3+2n_3)\xi_3$, where $n_2,n_3\geq 0$. Then $v_1\otimes E^{m_2}\xi_2\otimes E^{m_3}\xi_3$ is an eigenvector of H with eigenvalue

$$\begin{aligned}\lambda &= \lambda_1+2(m_2+m_3)-\lambda_2-\lambda_3+2(n_2+n_3)\\ &= (2k-\lambda_1-\lambda_2-\lambda_3)+2+2(n_2+n_3)>0.\end{aligned}$$

Hence,

$$\Psi(H(v_1\otimes E^{m_2}\xi_2\otimes E^{m_3}\xi_3))=\lambda\cdot\Psi(v_1\otimes E^{m_2}\xi_2\otimes E^{m_3}\xi_3)=0,$$

and $\lambda\neq 0$ implies:

$$\Psi(v,E^{m_2}\xi_2,E^{m_3}\xi_3)=0.$$

In other words, the only extra conditions — those of Equation 13 — are automatically satisfied by $\mathfrak{g}$-invariance. Therefore, $dim_{\mathbf{C}}\mathcal{H}(p_1,p_2,p_3;\lambda_1,\lambda_2,\lambda_3)=1$.

Step 3. Suppose $\lambda_1+\lambda_2+\lambda_3>2k$. Observe that if $\eta_i\in V_{\lambda_i}$ are eigenvectors of H and $H(\eta_1\otimes\eta_3\otimes\eta_3)\neq 0$, then $\Psi(\eta_1,\eta_2,\eta_3)=0$.

We inductively show that $\Psi=0$ on all of $V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}$.

Take v_1, ξ_2, ξ_3 so that $Hv_1 = \lambda_1 v_1$, $H\xi_2 = -\lambda_2\xi_2$, and $H\xi_3 = (\lambda_2 - \lambda_1)\xi_3$. We claim that $\Psi(v_1, \xi_2, \xi_3) = 0$. Indeed,

$$\lambda_2 - \lambda_1 \geq -\lambda_3 + 2d_1 = -\lambda_3 + 2(k - \lambda_1 + 1).$$

Therefore, we can write $\xi_3 = E^{d_1}\xi$ for some $\xi \in V_{\lambda_3}$, and

$$\Psi(v_1, \xi_2, \xi_3) = \Psi(v_1, \xi_2, E^{d_1}\xi) = 0$$

by Equation 13. Moreover, Equation 13 tells us that

$$\Psi(v_1, E^{m_2}\xi_2, E^{m_3}\xi) = 0$$

whenever $m_2 + m_3 = d_1$. Hence $\Psi(v_1, \xi_2, \xi_3) = 0$ if v_1 is the highest weight vector and the eigenvalues of H add up to zero.

We continue by taking

$$\Psi(F(v_1 \otimes \xi_2 \otimes E\xi_3)) = \Psi(Fv_1, xi_2, E\xi_3) + \Psi(v_1, F\xi_2, E\xi_3) + \Psi(v_1, \xi_2, FE\xi_3).$$

The last two terms on the right-hand side are zero by the previous paragraph, so

$$\Psi(Fv_1 \otimes \xi_2 \otimes E\xi_3).$$

Now continue in like manner....

18.2. **The general case.** We will now describe a basis for the space of conformal blocks of the form

$$\mathcal{H}(p_1, \ldots, p_n, p_{n+1}; \lambda_1, \ldots, \lambda_n, \lambda_{n+1}^*) = Hom_{\mathfrak{g}(p_1,\ldots,p_{n+1})}(H_{\lambda_1} \otimes H_{\lambda_n}, H_{\lambda_{n+1}}).$$

As before, this injects into $Hom_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, V_{\lambda_{n+1}})$.

Fix a directed graph Γ with n incoming edges and one outgoing edge. The incoming edges are labeled $\lambda_1, \ldots, \lambda_n$ and the outgoing one is labeled λ_{n+1}. The interior vertices are all trivalent, with two incoming edges and one outgoing.

Then a basis for $Hom_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, V_{\lambda_{n+1}})$ is given by all labelings of Γ (subject to the above conditions) so that the Clebsch-Gordan rule is satisfied at every trivalent vertex.

Theorem 18.3. *A basis for the space of conformal blocks is given by all labelings of* Γ *so that the* quantum *Clebsch-Gordan rule is satisfied at every trivalent vertex.*

Remark: I do not know how to prove this theorem. If someone could point me to a proof, I'd appreciate it!

If we take a different directed graph Γ' (also satisfying the above conditions), then we obtain a different basis for either $Hom_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, V_{\lambda_{n+1}})$ or $\mathcal{H}(p_1, \ldots, p_n, p_{n+1}; \lambda_1, \ldots, \lambda_n, \lambda_{n+1}^*)$. In the classical case, when $n = 3$, the transformation between the two bases corresponding to the two possible Γ is encoded in the *classical* $6j$*-symbol*. The quantum version is the *quantum* $6j$*-symbol*.

19. Bundle of conformal blocks

Today we try to understand what happens to the conformal blocks as we vary $p_1, \dots, p_n \in \mathbf{CP}^1$. Let us fix the level k and highest weights $\lambda_1, \dots, \lambda_n$.

Let $\text{Conf}_n(\mathbf{CP}^1) = \{(p_1, \dots, p_n) \mid p_i \in \mathbf{CP}^1, p_i \neq p_j \text{ if } i \neq j\}$ be the configuration space of n distinct ordered points in $\mathbf{CP}^1$.

19.1. **Conformal block bundle.**

The goal is to try to put a vector bundle structure on

$$\mathcal{E}_{\lambda_1,\dots,\lambda_n} \stackrel{def}{=} \bigcup_{(p_1,\dots,p_n)\in \text{Conf}_n(\mathbf{CP}^1)} \mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n).$$

Once the vector bundle property is verified, it will be called the *conformal block bundle*. (We won't quite succeed — see the remarks at the end of the lecture.)

First we spend some time discussing the family version of the constructions done in the last two lectures.

1. Consider the projection $\pi : (\mathbf{CP}^1)^{n+1} \to (\mathbf{CP}^1)^n$ onto the first n factors. In other words, coordinates $(z_1, \dots, z_n, z_{n+1})$ map down to $(z_1, \dots, z_n)$. (We will equivalently write $p_1, \dots, p_n$ or $z_1, \dots, z_n$.) At this point, $(\mathbf{CP}^1)^{n+1}$ is a bundle of $\mathbf{CP}^1$'s over the base. Next restrict to $\text{Conf}_n(\mathbf{CP}^1) \subset (\mathbf{CP}^1)^n$. Then we have $\pi^{-1}(\text{Conf}_n(\mathbf{CP}^1)) \to \text{Conf}_n(\mathbf{CP}^1)$. Define the divisors $D_j = \{z_{n+1} = z_j\}$. The D_j are disjoint on $\pi^{-1}(\text{Conf}_n(\mathbf{CP}^1))$. On $\pi^{-1}(z_1, \dots, z_n)$ the divisors restrict to $z_1, \dots, z_n$.

2. Let $U \subset \text{Conf}_n(\mathbf{CP}^1)$ be an open set. Then define $\mathcal{M}_{D_1,\dots,D_n}$ as the set of meromorphic functions on $\pi^{-1}(U)$ with poles of any order at most along $D_1, \dots, D_n$. We then consider $\mathfrak{g} \otimes_{\mathbf{C}} \mathcal{M}_{D_1,\dots,D_n}(U)$. An element $f \in \mathfrak{g} \otimes_{\mathbf{C}} \mathcal{M}_{D_1,\dots,D_n}(U)$ can be written locally along D_j as:

$$f_{D_j}(t_j) = \sum_{m=-N}^{\infty} a_m(z_1, \dots, z_m) t_j^m.$$

Here $t_j = z_{n+1} - z_j$. We will also write $\tau_j(f)$ for $f_{D_j}(t_j)$.

3. $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$ can naturally be viewed as a subset of $E = \text{Conf}_n(\mathbf{CP}^1) \times Hom_{\mathbf{C}}(\otimes H_{\lambda_j}, \mathbf{C})$, which is an infinite-dimensional vector bundle over $\text{Conf}_n(\mathbf{CP}^1)$. Define $\mathcal{E}_{\lambda_1,\dots,\lambda_n}(U)$ as the space of smooth sections $\Psi : U \to E$ such that $\Psi(p_1, \dots, p_n)$ is $\mathfrak{g} \otimes \mathcal{M}_{D_1,\dots,D_n}(U)$-invariant for all $(p_1, \dots, p_n) \in U$. In other words:

$$\sum_{j=1}^{n} \Psi(p_1, \dots, p_n)(\xi_1, \dots, \tau_j(f)\xi_j, \dots, \xi_n) = 0$$

for all $(p_1, \dots, p_n) \in U$.

At this point, we do not know whether $\mathcal{E}_{\lambda_1,\dots,\lambda_n}(U)$ has any nontrivial elements.

19.2. **Main proposition.** Given $\Psi : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \to \mathbf{C}$ and $X \in U(\widetilde{L\mathfrak{g}})$, define the multilinear map

$$X^{(j)}\Psi : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \to \mathbf{C},$$

$$(\xi_1, \ldots, \xi_n) \mapsto \Psi(\xi_1, \ldots, X\xi_j, \ldots, \xi_n).$$

In other words, we insert X in the jth spot.

Also we recall the *Sugawara operator*

$$L_{-1} = \frac{1}{2(k+2)} \sum_{\mu} \sum_{j \in \mathbf{Z}} I_\mu \otimes t^{j-1} \cdot I_\mu \otimes t^{-j},$$

where $\{I_\mu\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form.

Proposition 19.1. *If Ψ is a smooth section of $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}$ over U, then so is $\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)}\Psi$.*

Proof. Given $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1,\ldots,D_n}(U)$, we show that

$$\sum_i \left(\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)}\Psi \right)(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n) = 0.$$

First observe that if $\tau_j(f) = \sum a_m(z_1, \ldots, z_n) t_j^m$, where $t_j = z_{n+1} - z_j$, then

$$\tau_j(f_{z_j}) = \sum \frac{\partial a_m}{\partial z_j} t_j^m - a_m m t_j^{m-1}.$$

If $\partial_j = \frac{\partial}{\partial z_j}$ with respect to the variables $(z_1, \ldots, z_n, t_j)$ (which we view as independent), then we write

$$\begin{aligned} \tau_j(f_{z_j}) &= \partial_j \tau_j(f) - \frac{\partial}{\partial t_j}\tau_j(f), \\ \tau_i(f_{z_j}) &= \partial_j \tau_i(f). \end{aligned}$$

Since the Sugawara operator L_{-1} satisfies $[L_{-1}, X \otimes t^n] = -nX \otimes t^{n-1}$, it follows that:

$$[L_{-1}, \tau_j(f)] = -\frac{\partial}{\partial t_j}\tau_j(f).$$

Also observe that:

$$\frac{\partial}{\partial z_j}(\Psi(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n)) = \frac{\partial \Psi}{\partial z_j}(\xi_1, \ldots, \tau_i(f)\xi_i, \ldots, \xi_n) + \Psi(\xi_1, \ldots, \partial_j \tau_i(f)\xi_i, \ldots, \xi_n),$$

by the product rule.

Now,

$$\begin{aligned}\sum_i \left(\frac{\partial\Psi}{\partial z_j} - L_{-1}^{(j)}\Psi\right)(\xi_1,\dots,\tau_i(f)\xi_i,\dots,\xi_n) &= \sum_i \frac{\partial}{\partial z_j}(\Psi(\xi_1,\dots,\tau_i(f)\xi_i,\dots,\xi_n)) \\ &\quad -\sum_i \Psi(\xi_1,\dots,\partial_j\tau_i(f)\xi_i,\dots,\xi_n) \\ &\quad -\sum_{i\neq j}\Psi(\xi_1,\dots,\tau_i(f)\xi_i,\dots,L_{-1}\xi_j,\dots,\xi_n) \\ &\quad -\Psi(\xi_1,\dots,L_{-1}\tau_j(f)\xi_j,\dots,\xi_n)\end{aligned}$$

The second term on the RHS equals

$$-\left(\sum_{i\neq j}\Psi(\xi_1,\dots,\tau_i(f_{z_j})\xi_i,\dots,\xi_n)\right) - \Psi(\xi_1,\dots,\partial_j\tau_j(f)\xi_j,\dots,\xi_n).$$

The fourth term on the RHS can be written as

$$-\Psi(\xi_1,\dots,(\tau_j(f)L_{-1} + \tau_j(f_{z_j}) - \partial_j\tau_j(f))\xi_j,\dots,\xi_n).$$

Finally, $\sum_i \left(\frac{\partial\Psi}{\partial z_j} - L_{-1}^{(j)}\Psi\right)(\xi_1,\dots,\tau_i(f)\xi_i,\dots,\xi_n)$ is the sum of the following three terms, each of which is zero by invariance.

$$\frac{\partial}{\partial z_j}\left(\sum_i \Psi(\xi_1,\dots,\tau_i(f)\xi_i,\dots,\xi_n)\right),$$

$$-\sum_i \Psi(\xi_1,\dots,\tau_i(f_{z_j})\xi_i,\dots,\xi_n),$$

$$-\sum_i \Psi(f(\xi_1\otimes\cdots\otimes L_{-1}\xi_j\otimes\cdots\otimes\xi_n)).$$

□

19.3. **Flatness.** A connection ∇ on a vector bundle $E \xrightarrow{\pi} M$ is *flat* if it has zero curvature, i.e., if A is the connection 1-form, then $dA + A\wedge A = 0$. By the Frobenius integrability theorem, through any point in E there is a local section $s : U \to \pi^{-1}(U)$ which passes through it and satisfies $\nabla s = 0$. (Let us call these sections *covariant constant* sections.)

In our case, let $\omega = \sum_{j=1}^n L_{-1}^{(j)} dz_j$ be the connection 1-form on the bundle E defined previously. Then

$$\nabla\Psi = d\Psi - \sum_{j=1}^n L_{-1}^{(j)}\Psi dz_j,$$

and we have a covariant derivative

$$\nabla_{\frac{\partial}{\partial z_j}} : \mathcal{E}_{\lambda_1,\dots,\lambda_n}(U) \to \mathcal{E}_{\lambda_1,\dots,\lambda_n}(U),$$

$$\Psi \mapsto \frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)}\Psi.$$

(The restriction to $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}(U)$ follows from the previous proposition.)

Lemma 19.2. *ω is a flat connection.*

Proof. Since L_{-1} does not depend on $z_1, \ldots, z_n$, $d\omega = 0$. Also, $\omega \wedge \omega = \sum_{i,j}[L_{-1}^{(i)}, L_{-1}^{(j)}]dz_i \wedge dz_j = 0$ since $[L_{-1}^{(i)}, L_{-1}^{(j)}] = 0$ for all i, j. We then have $d\omega + \omega \wedge \omega = 0$. □

Remark: Lemma 19.2 implies the existence of local covariant constant sections that pass through any point on E. Together with Proposition 19.1 one would like to conclude that through each $s(0) \in \mathcal{E}_{\lambda_1,\ldots,\lambda_n}$ there is a covariant constant section s which is a section of $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}$. (Hence this would show that $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}$ is a vector bundle.) To first order near $s(0)$ that is true, but I do not know how to do this to higher orders.

Instead, one can use the following strategy to show that $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}$ is a vector bundle. Restrict to $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, which is finite-dimensional. The $\mathfrak{g}$-invariance does not depend on $z_1, \ldots, z_n$ and conditions analogous to Equation 13 gives extra linear conditions which depend holomorphically on $z_1, \ldots, z_n$. The dimension of $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}(p_1, \ldots, p_n)$ (the fiber over $(p_1, \ldots, p_n)$) is an upper semi-continuous function of $(p_1, \ldots, p_n)$. If we accept that the dimension of each $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}(p_1, \ldots, p_n)$ is the same (from the previous lecture), it follows that $\mathcal{E}_{\lambda_1,\ldots,\lambda_n}$ is a vector bundle.

20. The KZ equation

20.1. **The KZ equation.** KZ = Knizhhnik-Zamolodchikov. Last time we constructed the conformal block bundle $\mathcal{E}_{\lambda_1,\dots,\lambda_n} \rightarrow \mathrm{Conf}_n(\mathbf{C})$ and a flat connection on it. Horizontal (=covariant constant) sections Ψ of $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$ satisfied:

$$\frac{\partial \Psi}{\partial z_i} = L_{-1}^{(i)}\Psi.$$

Let us now restrict this equation to $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$.

Theorem 20.1. *Let Ψ be a horizontal section of $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$. Then Ψ restricted to $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$ (call it Ψ_0) satisfies:*

$$\frac{\partial \Psi_0}{\partial z_i} = \frac{1}{k+2}\sum_{j,j\neq i}\frac{\Omega^{(ij)}\Psi_0}{z_i-z_j}. \tag{14}$$

Equation 14 is called the *KZ equation*.

Here $\Omega=\sum_\mu I_\mu\otimes I_\mu$, where $\{I_\mu\}$ is an orthonormal basis for $\mathfrak{g}=\mathfrak{sl}(2,\mathbf{C})$ with respect to the Killing form. Ω is not quite a Casimir (we'll have more to say about their relation next time). Also, for $i\neq j$, we set

$$\Omega^{(ij)}\Psi_0(\xi_1,\dots,\xi_n)=\sum_\mu \Psi_0(\xi_1,\dots,I_\mu\xi_i,\dots,I_\mu\xi_j,\dots,\xi_n),$$

where $\xi_i\in V_{\lambda_i}$, $i=1,\dots,n$. If $i=j$, then we set

$$\Omega^{(ii)}\Psi_0(\xi_1,\dots,\xi_n)=\sum_\mu \Psi_0(\xi_1,\dots,I_\mu I_\mu\xi_i,\dots,\xi_n).$$

Proof. Recall that

$$L_{-1}\xi=\frac{1}{2(k+2)}\sum_{\mu,j} I_\mu\otimes t^{j-1}\cdot I_\mu\otimes t^{-j}\xi,$$

and if $\xi\in V_\lambda$, then the only nonzero terms in the sum are $(I_\mu\cdot I_\mu\otimes t^{-1})\xi$ and $(I_\mu\otimes t^{-1}\cdot I_\mu)\xi$, which are equal. Hence,

$$\begin{aligned}
(L_{-1}^{(i)}\Psi)_0(\xi,\dots,\xi_n) &= \frac{1}{k+2}\sum_\mu \Psi_0(\xi_1,\dots,(I_\mu\otimes t^{-1}\cdot I_\mu)\xi_i,\dots,\xi_n)\\
&= \frac{1}{k+2}\sum_\mu\sum_{j,j\neq i}(z_i-z_j)^{-1}\Psi_0(\xi_1,\dots,I_\mu\xi_i,\dots,I_\mu\xi_j,\dots,\xi_n)\\
&= \frac{1}{k+2}\sum_{j,j\neq i}\frac{\Omega^{(ij)}\Psi_0}{z_i-z_j}(\xi_1,\dots,\xi_n).
\end{aligned}$$

Here we are using the fact that

$$((X\otimes t^{-1})^{(i)}\Psi)_0(\xi_1,\dots,\xi_n)=\sum_{j,j\neq i}(z_i-z_j)^{-1}\Psi(\xi_1,\dots,X\xi_j,\dots,\xi_n)$$

by the $\mathfrak{g}(p_1, \dots, p_n)$-invariance. □

20.2. **Conformal invariance.**

Lemma 20.2. *If Ψ is a horizontal section of $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$, then its restriction Ψ_0 to $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ satisfies:*

(1) $\sum_{i=1}^n \frac{\partial}{\partial z_i}\Psi_0 = 0$;
(2) $\sum_{i=1}^n \left(z_i \frac{\partial}{\partial z_i} + \Delta_{\lambda_i}\right)\Psi_0 = 0$;
(3) $\sum_{i=1}^n \left(z_i^2 \frac{\partial}{\partial z_i} + 2z_i\Delta_{\lambda_i}\right)\Psi_0 = 0$.

Here $\Delta_\lambda = \frac{j(j+1)}{k+2}$ is the conformal weight, *where $j = \frac{\lambda}{2}$.*

Proof.

(1) $\sum_i \frac{\partial \Psi_0}{\partial z_i} = \frac{1}{k+2}\sum_{i\neq j} \frac{\Omega^{(ij)}\Psi_0}{z_i - z_j} = 0$ since $\Omega^{(ij)} = \Omega^{(ji)}$. The term $\frac{\Omega^{(ij)}\Psi_0}{z_i-z_j}$ is canceled by $\frac{\Omega^{(ji)}\Psi_0}{z_j - z_i}$.

(2) Since Ψ_0 is $\mathfrak{g}$-invariant,

$$\sum_{j=1}^n \Omega^{(ij)}\Psi_0(\xi_1, \dots, \xi_n) = \sum_{j,\mu} \Psi_0(\xi_1, \dots, I_\mu \xi_i, \dots, I_\mu \xi_j, \dots, \xi_n) = 0.$$

Here j is summed from 1 to n in the RHS. If $j = i$, then we have $\Psi_0(\xi_1, \dots, I_\mu I_\mu \xi_i, \dots, \xi_n)$. Hence,

$$\begin{aligned} \sum_{j, j\neq i} \Omega^{(ij)}\Psi_0(\xi_1, \dots, \xi_n) &= -\Omega^{(ii)}\Psi_0(\xi_1, \dots, \xi_n) \\ &= -\Psi_0(\xi_1, \dots, C\xi_i, \dots, \xi_n) \\ &= -2j(j+1)\Psi_0(\xi_1, \dots, \xi_i, \dots, \xi_n) \end{aligned}$$

$$\sum_{j,j\neq i} \Omega^{(ij)}\Psi_0(\xi_1, \dots, \xi_n) = -2(k+2)\Delta_{\lambda_i}\Psi_0(\xi_1, \dots, \xi_n). \tag{15}$$

Here $C = \sum_\mu I_\mu \cdot I_\mu$ is the Casimir and has eigenvalue $2j(j+1)$ on all of V_λ. Also the j in the third line is $\frac{\Delta_{\lambda_i}}{2}$. Summing over all the i's we can write:

$$\sum_{i<j} \Omega^{(ij)}\Psi_0 = -(k+2)\sum_j \Delta_j \Psi_0.$$

Finally, since $\frac{z_i}{z_i - z_j} + \frac{z_j}{z_j - z_i} = 1$,

$$\sum_i z_i \frac{\partial \Psi_0}{\partial z_i} = \frac{1}{k+2}\sum_{i\neq j} \frac{z_i}{z_i - z_j}\Omega^{(ij)}\Psi_0 = \frac{1}{k+2}\sum_{i<j}\Omega^{(ij)}\Psi_0 = -\sum_j \Delta_j \Psi_0.$$

(3) Since $\frac{z_i^2}{z_i - z_j} + \frac{z_j^2}{z_j - z_i} = z_i + z_j$,

$$\begin{aligned}\sum_i z_i^2 \frac{\partial \Psi_0}{\partial z_i} &= \frac{1}{k+2}\sum_{i<j}(z_i + z_j)\Omega^{(ij)}\Psi_0 \\ &= \frac{1}{k+2}\sum_i z_i \sum_{j, j\neq i}\Omega^{(ij)}\Psi_0 \\ &= \frac{1}{k+2}\sum_i z_i(-2(k+2)\Delta_{\lambda_i})\Psi_0 \\ &= -2\sum_i z_i \Delta_{\lambda_i}\Psi_0.\end{aligned}$$

Here we are using Equation 15 to go from the second line to the third. □

Theorem 20.3 (Conformal Invariance)**.** *Let Ψ be a horizontal section of $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$. Under the fractional linear transformation $w = \frac{az+b}{cz+d}$ with $ad - bc = 1$, applied to each $z_1, \dots, z_n$, we have:*

$$\Psi_0(w_1, \dots, w_n) = \prod_j (cz_j + d)^{2\Delta_{\lambda_j}} \Psi_0(z_1, \dots, z_n).$$

Proof. We verify the equation on generators of $PSL(2, \mathbf{C})$.

1. Consider the translation $f_s(z) = z + \alpha s$, where $s \in \mathbf{R}$ and $\alpha \in \mathbf{C}$. Then

$$\frac{d}{ds}\Psi_0(f_s(z_1), \dots, f_s(z_n)) = \sum_i \frac{\partial \Psi_0}{\partial z_i}\frac{d}{ds}(z_i + \alpha s) = \alpha \sum_i \frac{\partial \Psi_0}{\partial z_i} = 0,$$

where the last equality follows from Lemma 20.2. Hence Ψ_0 is invariant under translation.

2. Consider the dilatation $f_s(z) = e^{\alpha s} z$. (In this case, we rewrite $f_s(z) = \frac{e^{\alpha s/2} z}{e^{-\alpha s/2}}$. Hence $cz + d = e^{-\alpha s/2}$.) Then

$$\frac{d}{ds}\Psi_0(f_s(z_1), \dots, f_s(z_n)) = \sum_i \frac{\partial \Psi_0}{\partial z_i}\frac{d}{ds}(e^{\alpha s} z_i) = \sum_i \frac{\partial \Psi_0}{\partial z_i} z_i(\alpha e^{\alpha s}) = -\alpha e^{\alpha s}\sum_i \Delta_{\lambda_i}\Psi_0,$$

by Lemma 20.2. Hence

$$\Psi_0(e^{\alpha s} z_1, \dots, e^{\alpha s} z_n) = e^{\alpha s(-\sum_i \Delta_{\lambda_i})}\Psi_0(z_1, \dots, z_n) = (e^{-\alpha s/2})^{(\sum_i 2\Delta_{\lambda_i})}\Psi_0(z_1, \dots, z_n).$$

3. If $f_s(z) = \frac{z}{-sz+1}$, then

$$\left.\frac{d}{ds}\right|_{s=0}\Psi_0(f_s(z_1), \dots, f_s(z_n)) = \sum_i \frac{\partial \Psi_0}{\partial z_i} z_i^2((-sz_i + 1)^{-2})|_{s=0} = -(\sum_i 2 z_i \Delta_{\lambda_i})\Psi_0(z_1, \dots, z_n).$$

This is the infinitesimal version of the equation:

$$\Psi_0(f_s(z_1),\dots,f_s(z_n)) = \prod_j (-sz_j+1)^{2\Delta_{\lambda_j}}\Psi_0(z_1,\dots,z_n).$$

Since the above three fractional linear transformations infinitesimally generate $PSL(2,\mathbf{C})$, the theorem follows. (The first two fractional linear transformations are much more satisfactory, since we do not need to specialize at $s=0$.) □

What is truly conformally invariant is:

$$\Psi_0(dz_1)^{\Delta_{\lambda_1}}\dots(dz_n)^{\Delta_{\lambda_n}},$$

where dz^a are weighted differentials with weight $a\in\mathbf{R}$.

HW: Check that if $w=\frac{az+b}{cz+d}$, then $dw=\frac{1}{(cz+d)^2}dz$.

20.3. **The KZ equation in general.** Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and V_i, $i=1,\dots,n$, be finite-dimensional irreducible representations of $\mathfrak{g}$. Given a map

$$\Phi:\mathrm{Conf}_n(\mathbf{C})\to Hom_{\mathbf{C}}(V_1\otimes\cdots\otimes V_n,\mathbf{C}),$$

we have the KZ equation:

$$\frac{\partial\Phi}{\partial z_i}=\frac{1}{\kappa}\sum_{j,j\neq i}\frac{\Omega^{(ij)}\Phi}{z_i-z_j},$$

where κ is some complex parameter.

Let us write $\omega_{ij}=d\log(z_i-z_j)=\frac{dz_i-dz_j}{z_i-z_j}$ and $\omega=\frac{1}{\kappa}\sum_{i<j}\Omega^{(ij)}\omega_{ij}$. Then the KZ equation is

$$d\Phi=\omega\Phi.$$

As before, we have:

Theorem 20.4. *The KZ connection is flat connection on the trivial* $Hom_{\mathbf{C}}(V_1\otimes\cdots\otimes V_{\lambda_n},\mathbf{C})$*-bundle over Conf*$_n(\mathbf{C})$.

21. Braid groups and the monodromy representation

21.1. **Braid groups.** Let $\text{Conf}_n(\mathbf{C})$ be the configuration space of n ordered distinct points on $\mathbf{C}$, i.e., $\{(z_1, \dots, z_n) \mid z_i \in \mathbf{C}, z_i \neq z_j \text{ for } i \neq j\}$.

The *pure braid group* P_n is $\pi_1(\text{Conf}_n(\mathbf{C}))$. Think of the motion of n distinct points $z_1, \dots, z_n$ which begin and end at the same location. If we take $\mathbf{C} \times [0,1]$, where $t \in [0,1]$ represents the time direction, then we can represent an element of P_n by n strands starting at $\mathbf{C} \times \{0\}$ and ending at $\mathbf{C} \times \{1\}$. (Picture needed here!)

There is also a configuration space of n *unordered* distinct points. That is given by $\text{Conf}_n(\mathbf{C})/S_n$, where S_n is the symmetric group on n elements. The *braid group* B_n is $\pi_1(\text{Conf}_n(\mathbf{C})/S_n)$.

Fact: B_n has generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i, \ |i-j| > 1; \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{aligned}$$

Here σ_i switches the ith and $i+1$st stands in a particular way – if we think of strands as going from bottom to top in $\mathbf{C} \times [0,1]$, then the strand from z_i to z_{i+1} is in front of the strand from z_{i+1} to z_i. (Picture needed here!)

HW: Verify that $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ by drawing a picture of the two braids. (Note they are equivalent by a type III Reidemeister move.)

HW: Verify that we have an exact sequence:

$$0 \to P_n \to B_n \to S_n \to 0.$$

21.2. **Monodromy representation of the KZ equation.** Recall that given $\Phi : \text{Conf}_n(\mathbf{C}) \to Hom_{\mathbf{C}}(V_1 \otimes V_n, \mathbf{C})$, the KZ equation is the differential equation:

$$d\Phi = \omega\Phi,$$

and ω is a flat connection, i.e., $d\omega + \omega \wedge \omega = 0$.

Holonomy: The holonomy of a flat connection on a vector bundle $E \to M$ only depends on the homotopy class of paths relative to the endpoints (i.e., fixing the endpoints). If γ is a path from a to b in M, then the holonomy

$$Hol_\gamma : E_a \xrightarrow{\sim} E_b$$

only depends on the homotopy class $[\gamma]$.

Hence, the flat connection ω on the trivial bundle

$$\text{Conf}_n(\mathbf{C}) \times Hom_{\mathbf{C}}(V_1 \otimes \cdots \otimes V_n, \mathbf{C}) \to \text{Conf}_n(\mathbf{C})$$

(or the conformal block bundle $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$) gives rise to a representation:

$$\rho : P_n = \pi_1(\text{Conf}_n(\mathbf{C})) \to GL(V_1^* \otimes \cdots \otimes V_n^*).$$

For the braid group B_n, take $V_1 = \cdots = V_n = V$. Then S_n acts diagonally on $\mathrm{Conf}_n(\mathbf{C}) \times (V^*)^{\otimes n}$, where $\sigma \in S_n$ permutes $(z_1, \ldots, z_n)$ and $\sigma \cdot \phi(v_1, \ldots, v_n) = \phi(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$. The quotient is a vector bundle over $\mathrm{Conf}_n(\mathbf{C})/S_n$. The corresponding representation is:

$$\rho : B_n \to GL((V^*)^{\otimes n}).$$

21.3. **Comultiplication.** The *comultiplication* for $U(\mathfrak{g})$ is an *algebra homomorphism*

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}),$$

where $\Delta(X) = X \otimes 1 + 1 \otimes X$ if $X \in \mathfrak{g}$. Since Δ is an algebra homomorphism, we extend the definition by writing $\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y)$.

The topologist's way of expressing a comultiplication is to draw a pair-of-pants with one boundary component on the left and two boundary components on the right. The left-hand boundary corresponds to X and the right-hand ones correspond to $X \otimes 1 + 1 \otimes X$. Then *coassociativity* $(\Delta \otimes 1)\Delta = \Delta(\Delta \otimes 1)$ can be interpreted as in the following diagram.

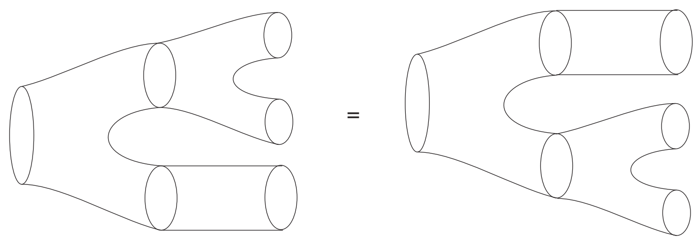

FIGURE 1. Coassociativity.

We denote $(\Delta \otimes 1)\Delta$ by $\Delta^{(3)}$. For our purposes, Δ is important because the tensor product representation $V_1 \otimes V_2$ is a representation of $\mathfrak{g}$ via the diagonal action Δ which sends X to $X \otimes 1 + 1 \otimes X$. The same holds true for $\Delta^{(3)}$ and the higher $\Delta^{(n)}$, defined analogously.

Calculation: $\Omega = \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C)$, where C is the Casimir $\sum_\mu I_\mu \cdot I_\mu$. Indeed, omitting summations,

$$\begin{aligned} \Delta(C) &= \Delta(I_\mu) \cdot \Delta(I_\mu) = (I_\mu \otimes 1 + 1 \otimes I_\mu) \cdot (I_\mu \otimes 1 + 1 \otimes I_\mu) \\ &= I_\mu \cdot I_\mu \otimes 1 + 1 \otimes I_\mu \cdot I_\mu + 2 I_\mu \otimes I_\mu \\ &= C \otimes 1 + 1 \otimes C + 2\Omega. \end{aligned}$$

Similarly, one calculates that

$$\begin{aligned} \Delta^{(3)}(C) &= (I_\mu \otimes 1 \otimes 1 + 1 \otimes I_\mu \otimes 1 + 1 \otimes 1 \otimes I_\mu)(\text{same}) \\ &= C \otimes 1 \otimes 1 + 1 \otimes C \otimes 1 + 1 \otimes 1 \otimes C \\ &\quad +2(\Omega^{(12)} + \Omega^{13} + \Omega^{(23)}). \end{aligned}$$

21.4. **Computation of the holonomy with respect to a preferred basis.** Let us use $Hom_{\mathbf{C}}(V_1 \otimes \cdots \otimes V_n, V_{n+1})$ instead. Suppose $n = 3$ and $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. Use the basis of conformal blocks which corresponds to the tree $((12)3)$, which means 1 and 2 are merged together first, and the resulting outgoing edge with 3. The edge obtained by merging 1 and 2 is labeled λ. Let Φ be a basis element corresponding to λ.

Setting $\zeta_1 = z_2 - z_1$ and $\zeta_2 = z_3 - z_1$, we write

$$\begin{aligned} \omega &= \frac{1}{\kappa}\left(\Omega^{(12)}\frac{dz_1 - dz_2}{z_1 - z_2} + \Omega^{(13)}\frac{dz_1 - dz_3}{z_1 - z_3} + \Omega^{(23)}\frac{dz_2 - dz_3}{z_2 - z_3}\right) \\ &= \frac{1}{\kappa}\left(\Omega^{(12)}\frac{d\zeta_1}{\zeta_1} + \Omega^{(13)}\frac{d\zeta_2}{\zeta_2} + \Omega^{(23)}\frac{d(\zeta_2 - \zeta_1)}{\zeta_2 - \zeta_1}\right). \end{aligned}$$

Effectively we are setting $z_1 = 0$.

We calculate the holonomy/monodromy as ζ_1 circles once about 0. Here we assume that $|\zeta_2| \gg |\zeta_1|$ and ζ_2 is fixed. Then the second and third terms of the sum do not contribute, and the monodromy is given by:

$$\text{res}_{\zeta_1=0}\omega = \frac{1}{\kappa}\Omega^{(12)}.$$

(Note that the $\Omega^{(ij)}$ do not depend on the z_i or ζ_i.)

Computation of $\frac{1}{\kappa}\Omega^{(12)}\Phi$. Φ is the composition

$$V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \to V_\lambda \otimes V_{\lambda_3} \to V_{\lambda_4}.$$

Here the first map is given by taking the direct sum of irreducible factors $V_{\lambda_1} \otimes V_{\lambda_2}$ and projecting down to the factor V_λ, and the second map is the projection to the factor V_{λ_4}. Hence we are interested in $\Phi((\sum v_1 \otimes v_2) \otimes v_3)$, where $\sum v_1 \otimes v_2$ is in V_λ. We easily verify that:

$$(C \otimes 1 \otimes 1)(v_1 \otimes v_2 \otimes v_3) = 2(k+1)\Delta_{\lambda_1}(v_1 \otimes v_2 \otimes v_3),$$
$$(1 \otimes C \otimes 1)(v_1 \otimes v_2 \otimes v_3) = 2(k+1)\Delta_{\lambda_2}(v_1 \otimes v_2 \otimes v_3).$$

Using our interpretation of Δ as giving the tensor product representation, we have:

$$(\Delta(C) \otimes 1)((\sum v_1 \otimes v_2) \otimes v_3) = 2(k+1)\Delta_\lambda((\sum v_1 \otimes v_2) \otimes v_3).$$

Via the formula $\Omega = \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C)$, we compute that:

$$(\text{res}_{\zeta_1=0}\omega)\Phi = (\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2})\Phi.$$

Next we calculate the monodromy as ζ_1 circles once about 0 and ζ_2 circles once about 0 and ζ_1. (They both happen at the same time, and $|\zeta_2| \gg |\zeta_1|$ again!) Hence we are adding up the residues to get $\frac{1}{\kappa}(\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)})$.

HW: Verify that $\frac{1}{\kappa}(\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)})\Phi = (\Delta_{\lambda_4} - \Delta_{\lambda_1} - \Delta_{\lambda_2} - \Delta_{\lambda_3})\Phi$. (Hint: use the formula for $\Delta^{(3)}(C)$ above.)

22. Link invariants (preliminaries)

22.1. **Monodromy representation.** Let us summarize our findings about the monodromy representation from last time.

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. Fix a level k and let $\lambda_1, \ldots, \lambda_n$ be highest weights at $0 < z_1 < z_2 < \cdots < z_n$. The n-tuple $(z_1, \ldots, z_n)$ will be our basepoint in the configuration space $\mathrm{Conf}_n(\mathbf{C})$. We will also take weights $\lambda_0 = 0$ at $z_0 = 0$ and $\lambda_{n+1} = 0$ at $z_{n+1} = \infty$. (These additional points correspond to the trivial $\mathfrak{g}$-representation $\mathbf{C}$.) Let $V_{\lambda_1,\ldots,\lambda_n}$ be the space of conformal blocks $\mathcal{H}(z_0, \ldots, z_{n+1}; \lambda_0, \ldots, \lambda_n, \lambda^*_{n+1})$. The KZ equation gives rise to a representation:

$$\rho : P_n \to GL(V_{\lambda_1,\ldots,\lambda_n}).$$

Take a basis for $V_{\lambda_1,\ldots,\lambda_n}$ corresponding to a tree Γ of form $((((12)3)4)\ldots n)$. By this we mean the edges 1 and 2 come together first, then the outgoing edge is merged with edge 3, and the outgoing edge is merged with edge 4, etc. Last time we computed that $\rho(\sigma_1^2)$ is diagonal with respect to the basis corresponding to Γ and that:

$$\rho(\sigma_1^2)(v_\mu) = e^{2\pi i(\Delta_\mu - \Delta_{\lambda_1} - \Delta_{\lambda_2})} v_\mu.$$

Here, v_μ is any basis element whose labeling of the third edge at a vertex where λ_1 and λ_2 come together is μ. Also, we use the standard generators $\sigma_1, \ldots, \sigma_n$ of B_n and view $P_n \subset B_n$. Then one full twist of strands 1 and 2 is σ_1^2.

For the braid group B_n, we let $\lambda = \lambda_1 = \cdots = \lambda_n$. Then ρ is a representation:

$$\rho : B_n \to GL(V_{\lambda,\ldots,\lambda}).$$

With respect to the Γ-basis, we have:

$$\rho(\sigma_1)(v_\mu) = e^{\pi i(\Delta_\mu - \Delta_{\lambda_1} - \Delta_{\lambda_2})} v_\mu.$$

Notice the loss of the factor 2 in front of $2\pi i$.

Example: Suppose $\lambda = 1$, i.e., all the representations V_1 are the standard 2-dimensional representation of $\mathfrak{sl}(2, \mathbf{C})$. Then a basis of $V_{1,\ldots,1}$ is given by ordered $n+1$-tuples $(\mu_0, \ldots, \mu_{n+1})$, where:

(1) $\mu_0 = \mu_{n+1} = 0$;
(2) $|\mu_i - \mu_{i+1}| = 1$ for $0 \leq i \leq n$;
(3) $0 \leq \mu_i \leq k$ for $1 \leq i \leq n$.

The μ_i are labelings of the edges of graphs of type $((((01)2)3)\ldots n+1)$. (2) holds because we are always fusing μ_i with $\lambda_{i+1} = 1$. (3) is the quantum Clebsch-Gordan condition.

Remark: I have been told that this has something to do with the *Catalan number*, if we disregard the quantum condition.

22.2. **Iwahori-Hecke algebra representations.** The *Iwahori-Hecke algebra* is an associative algebra over **C** with unit, generated by $\sigma_1, \dots, \sigma_{n-1}$ and relations:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| > 1$;
(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$;
(3) $(\sigma_i - q^{1/2})(\sigma_i - q^{-1/2}) = 0$,

where q is some nonzero complex number. Conditions (1) and (2) are just the braid relations, and condition (3) tells us that a representation of the Iwahori-Hecke algebra is a representation of the braid group, where each σ_i has at most two eigenvalues. See Jones' fundamental paper [1].

Again let $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$ and $\lambda = 1$ (corresponding to the 2-dimensional irrep). We define $q^{1/2} = e^{\pi i/k+2}$ and $\eta = e^{\pi i/2(k+2)}$.

Theorem 22.1. *Given the monodromy representation $\rho : B_n \to GL(V_{1,\dots,1})$ of the KZ equation, the modified representation $\tilde{\rho} : B_n \to GL(V_{1,\dots,1})$, $\tilde{\rho}(\sigma_i) = \eta\rho(\sigma_i)$ is an Iwahori-Hecke algebra representation.*

Remark: Note that the modification $\tilde{\rho}$ is also a representation of B_n. This is the multiplicative version of the fact that the *linking number* $lk : B_n \to \mathbf{Z}$ given by $\sigma_i = 1$ for all i is a homomorphism. This follows from observing that the braid relations always preserve *word length*.

Proof. Without loss of generality, we show the theorem for σ_1. With respect to $\Gamma = (((01)2)\dots n+1)$, μ_1 is either 0 or 2. Hence $\rho(\sigma_1)$ has eigenvalues $e^{\pi i(\Delta_0 - \Delta_1 - \Delta_1)} = e^{-\frac{3}{2}\frac{\pi i}{k+2}}$ or $e^{\pi i(\Delta_2 - \Delta_1 - \Delta_1)} = e^{\frac{1}{2}\frac{\pi i}{k+2}}$. Multiplying with η gives $q^{\pm 1/2}$. □

Remark: In Kohno's book, the Iwahori-Hecke algebra is supposed to satisfy $(\sigma_i - q^{1/2})(\sigma_i + q^{-1/2}) = 0$. I do not see where the plus sign comes from, and would appreciate any suggestions!

22.3. **Knots and links.** A *link* is an embedding $f : S^1 \sqcup \dots \sqcup S^1 \hookrightarrow S^3$ (or the image of the embedding). Here $\sqcup$ represents disjoint union. The image will be denoted $L = L_1 \sqcup \dots \sqcup L_m$. We will also blur the distinction between knots and *isotopy classes of knots*. Also, we will switch back and forth between S^3 and $\mathbf{R}^3$ with impunity. We will usually project the knot from $\mathbf{R}^3$ to $\mathbf{R}^2$ so that the projection has only transverse intersections, and denote the knot by the crossing data.

A *Seifert surface* Σ of an oriented knot L is an embedded oriented surface $\Sigma \subset S^3$ such that $\partial\Sigma = L$ (and the boundary orientation of Σ agrees with the orientation of L). The Seifert surface is not unique.

There is an algorithm, called the *Seifert algorithm* for finding a Seifert surface. We locally resolve each crossing so that the orientations (arrows) are well-defined after each resolution. See Figure 2. The result is a union of circles, and the Seifert surface is obtained from the disjoint union of disks that bound these circles, by banding in a manner dictated by the crossing data.

The *linking number* $lk(L_1, L_2)$ between two links L_1 and L_2 is the (oriented) intersection number of L_2 with the Seifert surface for L_1.

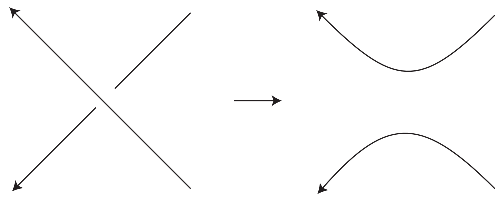

FIGURE 2. Resolving crossings.

HW: Prove that $lk(L_1, L_2) = lk(L_2, L_1)$.

We can assign a sign to each crossing in the knot projection. See Figure 3.

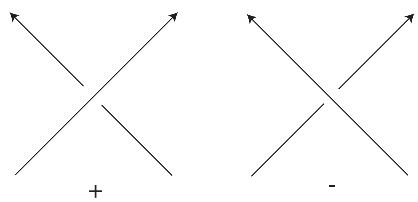

FIGURE 3. Crossing signs.

HW: Prove that $lk(L_1, L_2)$ is the number of positive crossings of L_1 and L_2 minus the number of negative crossings of L_1 and L_2.

Framings: A *framing* of L is a homotopy class of trivializations of the normal bundle νL of L (in S^3 or $\mathbf{R}^3$). In other words, it is a homotopy class of nowhere zero sections of νL. For example, a Seifert surface Σ of L induces a framing $T\Sigma \cap \nu L$ along L, which is usually called the 0-framing. In general, take a pushoff L' of L in the direction dictated by the section of νL. Then the framing is given by $lk(L, L')$.

REFERENCES

[1] V. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335–388.

23. LINK INVARIANTS

Today's goal is to define an invariant of oriented, framed links.

Blackboard framing: Given a planar diagram of a link L, the *blackboard framing* is given as the normal to the tangent space TL of the link inside $\mathbf{R}^2$. (We need to exercise some care at the crossings....) Figure 4 gives the blackboard framing of the right-handed trefoil.

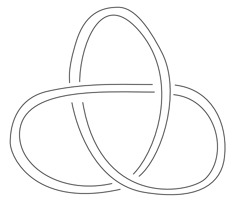

FIGURE 4. Blackboard framing.

Decomposition into elementary tangles: Suppose the link is in planar form, with coordinates x, t for the plane. After perturbing the link L if necessary, we decompose the link into slices $t_i \leq t \leq t_{i+1}$, where $i = 0, \dots, s-1$, so that on each slice there is only one of the following (1) crossing, (2) maximum (with respect to t), or (3) minimum. (Such a slice is called an *elementary tangle*.) See Figure 5 for an elementary tangle with a crossing (there is another with the overcrossing/undercrossing switched).

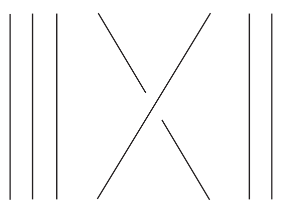

FIGURE 5. An elementary tangle with a crossing.

Write L as a disjoint union $L_1 \cup \dots \cup L_m$ of knots L_j. We will keep track of the order of the L_j, and such a link is often called a *colored link*. Fix a level k. Label each L_j by the highest weight μ_j.

We will use the TQFT philosophy to assign a vector space $V(t_i)$ to each level $t = t_i$ and a morphism $Z_i : V(t_i) \to V(t_{i+1})$ to each $[t_i, t_{i+1}]$. We set $V(t_0) = \mathbf{C}$ and $V(t_s) = \mathbf{C}$. Once the $V(t_i)$ and Z_i are defined, we define:

$$Z(L; \mu_1, \dots, \mu_m) = Z_{s-1} \circ Z_{s-2} \circ \dots \circ Z_1 \circ Z_0(1).$$

Definition of $V(t_i)$: If $t = t_i$ intersects L at n points (n is even), then we assign μ_j if the intersection is in $L_j \cap \{t = t_j\}$ and L_j is oriented downwards; we assign μ_j^* if L_j is oriented upwards. (By μ_j^* we mean the dual to V_{μ_j}.) Then we set $V(t_i) = V_{\lambda_1,\ldots,\lambda_n}$, where each λ_l, $l = 1, \ldots, n$, is one of the μ_j, μ_j^*, $j = 1, \ldots, m$. We will also write $V(t_i) = V_{0,\lambda_1,\ldots,\lambda_n,0}$. In this way, we see that $V(t_0) = V(t_s) = V_{0,0}$, and its basis can be denoted by a tree with one edge which is labeled 0.

Definition of Z_i for a crossing: We will give $Z_i : V(t_i) \to V(t_{i+1})$ if the elementary tangle is a crossing given in Figure 5. Write σ_a for the corresponding braid in B_n, where σ_a switches the ath and $(a+1)$st strands. It is convenient to define Z_i with respect to the basis corresponding to some tree $\Gamma = (\ldots(a, a+1)\ldots)$. In other words, we have an edge λ_a and λ_{a+1} come together at a trivalent vertex and an edge ν emanating from the vertex. Let v_ν be any basis element corresponding to such a labeling. Then

$$Z_i(v_\nu) = e^{\pi i(\Delta_\nu - \Delta_{\lambda_a} - \Delta_{\lambda_{a+1}})} P_{a,a+1} v_\nu,$$

where $P_{a,a+1}$ permutes the ath and $(a+1)$st factors. If the overcrossing and undercrossing are reversed, then the multiplicative factor changes to $e^{-\pi i(\Delta_\nu - \Delta_{\lambda_a} - \Delta_{\lambda_{a+1}})}$. See Figure 6.

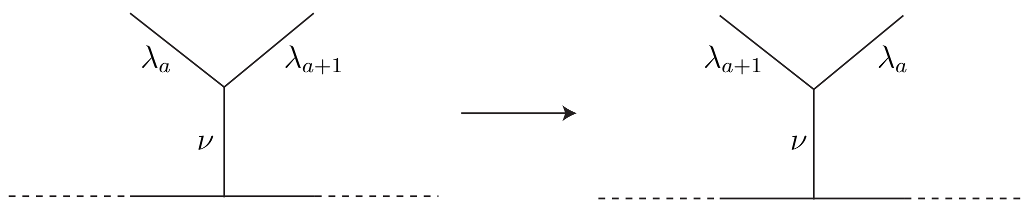

FIGURE 6. Z_i for a crossing.

Definition of Z_i for a minimum: Suppose $L \cap \{t = t_i\}$ has n intersection points. Then

$$Z_i : V_{0,\lambda_1,\ldots,\lambda_n,0} \to V_{0,\lambda_1,\ldots,\lambda_a,\mu_j,\mu_j^*,\lambda_{a+1},\ldots,\lambda_n,0}$$

is given as follows. (Here the minimum lies on L_j, corresponding to μ_j.) First embed $V_{0,\lambda_1,\ldots,\lambda_n,0}$ into $V_{0,\lambda_1,\ldots,\lambda_a,0,\lambda_{a+1},\ldots,\lambda_n,0}$ by adding an edge with label 0 onto some Γ for $V_{0,\ldots,\lambda_1,\ldots,\lambda_n,0}$. Then add two edges labeled μ_j and μ_j^* onto the univalent vertex of the edge labeled 0. This corresponds to the embedding $V_{0,\lambda_1,\ldots,\lambda_a,0,\lambda_{a+1},\ldots,\lambda_n,0}$ into $V_{0,\lambda_1,\ldots,\lambda_a,\mu_j,\mu_j^*,\lambda_{a+1},\ldots,\lambda_n,0}$. See Figure 7.

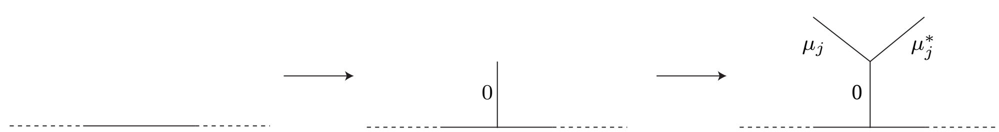

FIGURE 7. Z_i for a maximum.

Definition of Z_i for a maximum:

$$Z_i : V_{0,\lambda_1,\ldots,\lambda_a,\mu_j,\mu_j^*,\lambda_{a+1},\ldots,\lambda_n,0} \to V_{0,\lambda_1,\ldots,\lambda_n,0}$$

is given by starting with a basis corresponding to Γ as in the right-hand side of Figure 7, but the labelings are μ_j, μ_j^* and ν instead of 0. Then all the basis elements with $\nu \neq 0$ are mapped to zero and the basis elements with $\nu = 0$ are mapped to itself. (In other words, we have a projection to a subspace consisting of $\nu = 0$.) Now we can (naturally) map to the basis in the center of Figure 7, and hence to the basis on the left.

Lemma 23.1. *$Z(L; \mu_1, \ldots, \mu_m)$ is invariant under the moves given below.*

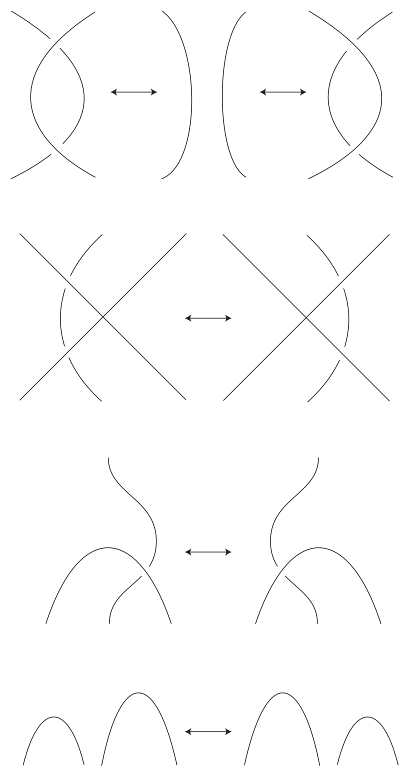

FIGURE 8

The top three moves are the "horizontal moves". The invariance of Z under the top two moves follows from the flatness of the KZ connection, since the moves represent homotopic paths in the configuration space $\mathrm{Conf}_n(\mathbf{C})$ (relative to the endpoints). I have not been able to figure out the proof for the third move. The fourth move represents moving maxima and minima above one another, and is straightforward to verify.

It turns out that $Z(L; \mu_1, \ldots, \mu_m)$ is not yet an invariant of the oriented framed colored link. Under the move below which increases the number of maxima and minima by one each, $Z(L)$ satisfies the following:

Fact: $Z(L'; \mu_1, \ldots, \mu_m) = Z(K_0; \mu_j) Z(L; \mu_1, \ldots, \mu_m)$, where K_0 is an unknot with two maxima and two minima and no crossings. Also, μ_j is the labeling on the component L_j of the link L which has its number of maxima/minima increased. (This is not too hard to verify.)

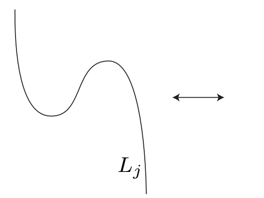

FIGURE 9

Write $d(\mu) = Z(K_0; \mu)^{-1}$. Also let $\max_j$ be the number of maxima in L_j. Then define:

$$J(L; \mu_1, \ldots, \mu_m) = d(\mu_1)^{\max_1} \ldots d(\mu_m)^{\max_m} Z(L; \mu_1, \ldots, \mu_m).$$

Theorem 23.2. *$J(L; \mu_1, \ldots, \mu_m)$ is an invariant of the colored oriented framed link.*

Proof. Two colored oriented framed links L and L' are isotopic iff L' is obtained from L' by a sequence of moves already considered, namely: local horizontal moves, moves that shift maxima/minima above one another, and cancellations of critical points. □

24. The Jones polynomial

24.1. **Link invariants.** Last time we defined $Z(L;\mu_1,\dots,\mu_m)$, where $L = L_1\cup\dots\cup L_m$ is an oriented colored link, k is the level, and μ_j is the highest weight for L_j. Z was obtained by decomposing a link diagram into elementary tangles and defining maps $Z_i : V(t_i)\to V(t_{i+1})$.

To get an invariant of an oriented colored link, we need two modifications:

1. Z is not invariant under the modification given in Figure 9. Instead, $Z(L';\mu_1,\dots,\mu_m) = Z(K_0;\mu_j)Z(L;\mu_1,\dots,\mu_m)$, where K_0 is an unknot with two maxima and two minima and no crossings, and μ_j is the labeling of the component L_j involved. Hence, we modify:

$$J(L;\mu_1,\dots,\mu_m) = d(\mu_1)^{\text{max}_1}\dots d(\mu_m)^{\text{max}_m} Z(L;\mu_1,\dots,\mu_m).$$

Here $d(\mu_j) = Z(K_0;\mu_j)^{-1}$.

2. Now J is an oriented framed link invariant, but not an oriented link invariant.

Fact: If L' is the link obtained from L by increasing the framing of L_j by 1, then

$$J(L';\mu_1,\dots,\mu_m) = e^{2\pi i\Delta_{\mu_j}} J(L;\mu_1,\dots,\mu_m).$$

(I do not know how to prove this.)

For example, the (blackboard) framing can be increased or decreased as follows: Take a vertical line in one of the elementary tangles $[t_i,t_{i+1}]$ and add an extra loop (do a Reidemeister I move).

To keep track of the number of Reidemeister I moves performed, we use the *writhe* $w(L)$, which is the number of positive crossings minus the number of negative crossings. If we modify

$$J(L;\mu_1,\dots,\mu_m)\mapsto e^{-2\pi i(\sum_j \Delta_{\mu_j} w(L_j))} J(L;\mu_1,\dots,\mu_m),$$

then the new polynomial is an invariant of the oriented colored link.

24.2. **The Jones polynomial.** To define the Jones polynomial, we specialize to $\mu_1=\dots=\mu_m=1$. Write $J(L;1,\dots,1) = J_L$. Let L_+ be a (planar diagram of) a link, and let p be a positive crossing of L_+. Let L_- be obtained from L_+ by replacing the positive crossing p by a negative crossing (without changing L_+ away from p), and L_0 be obtained from L_+ by resolving the crossing p (in a way which preserves the orientation).

Lemma 24.1 (Skein relation). *If $q = e^{\frac{2\pi i}{k+2}}$, then*

$$q^{1/4}J_{L_+} - q^{-1/4}J_{L_-} = (q^{1/2}-q^{-1/2})J_{L_0}.$$

We also write $\kappa = k+2$.

Proof. I seem to get

$$q^{1/4}J_{L_+} + q^{-1/4}J_{L_-} = (q^{1/2}+q^{-1/2})J_{L_0}$$

instead....

At any rate, consider an elementary tangle which contains the crossing p. Consider the basis corresponding to the tree $(\dots(a, a+1)\dots)$, where the crossing involves the ath and $(a+1)$st strands. The eigenvalues of $\rho(\sigma_a)$ are $e^{\pi i(\Delta_0 - 2\Delta_1)} = q^{-3/4}$ or $e^{\pi i(\Delta_2 - 2\Delta_1)} = q^{1/4}$. Hence $Z_i(L) : V(t_i) \to V(t_{i+1})$ are given by:

$$Z_i(L_+) = \text{diag}(q^{-3/4}, \dots, q^{-3/4}, q^{1/4}, \dots, q^{1/4}),$$
$$Z_i(L_-) = \text{diag}(q^{3/4}, \dots, q^{3/4}, q^{-1/4}, \dots, q^{-1/4}),$$
$$Z_i(L_0) = id.$$

Here diag means the diagonal matrix with the given entries. We then have:

$$q^{1/4} Z_i(L_+) + q^{-1/4} Z_i(L_-) = (q^{1/2} + q^{-1/2}) Z_i(L_0).$$

All the other elementary tangles are the same for L_+, L_-, and L_0, implying the lemma. □

Jones polynomial normalization: Let $P_L = d(1)^{-1} e^{-2\pi i \Delta_1 w(L)} J_L$. Then the skein relation becomes

$$q P_{L_+} - q^{-1} P_{L_-} = (q^{1/2} - q^{-1/2}) P_{L_0}.$$

Note that $d(1)^{-1}$ is a constant which is thrown in to make P_L of the unknot equal to 1. Finally, if we set $t^{1/2} = -q^{-1/2}$, then we obtain the original *Jones polynomial* V_L which satisfies the skein relation

$$t^{-1} V_{L_+} - t V_{L_-} = (t^{1/2} - t^{-1/2}) V_{L_0}.$$

FS: The way we defined the Jones polynomial is not the most straightforward way. There is a straightforward combinatorial way due to Kauffman, using Kauffman brackets.

FS: There is a homology theory, called *Khovanov homology*, whose (graded) Euler characteristic gives the coefficients of the Jones polynomial.

24.3. **Calculations.** The skein relation, together with the normlization $V(\text{unknot}) = 1$, completely determines the Jones polynomial for all knots and links. The proof is by induction on the number of crossings. To give plausibility to this assertion, we compute the Jones polynomial for several knots and links, with an increasing number of crossings.

1. Let L_0 be the union of two unknots K_1, K_2, where the K_i are contained in disjoint 3-balls. Applying the skein relation to the links in Figure 10, we obtain

$$V(L_0) = \frac{t^{-1} - t}{t^{1/2} - t^{-1/2}} = -(t^{1/2} + t^{-1/2}).$$

2. If L is the Hopf link given on the LHS of Figure 11, then $V(L) = -t^{5/2} - t^{1/2}$. If L is the Hopf link given on the RHS, then $V(L) = -t^{-5/2} - t^{-1/2}$.

3. If L is the right-handed trefoil, then $V(L) = -t^4 + t^3 + t$. If L is the left-handed trefoil, then $V(L) = -t^{-4} + t^{-3} + t^{-1}$.

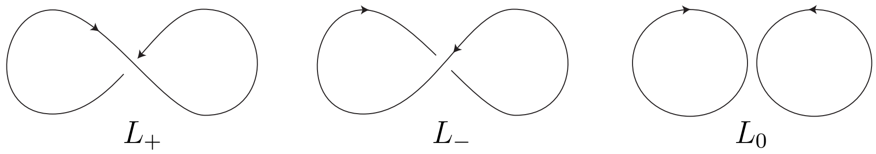

FIGURE 10

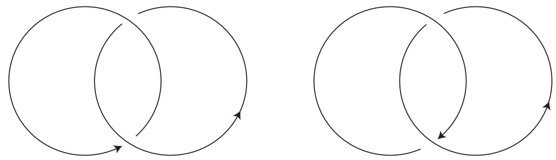

FIGURE 11

Properties of the Jones polynomial:

A. If L_1 and L_2 are contained in disjoint 3-balls, then

$$J(L_1 \cup L_2) = J(L_1)J(L_2).$$

This implies that $P(L_1 \cup L_2) = d(1)P(L_1)P(L_2)$ and $V(L_1 \cup L_2) = d(1)V(L_1)V(L_2)$.

B. If $\overline{L}$ is the mirror image of L (the mirror image is obtained by reflecting the planar diagram across a line in the plane), then $J(\overline{L}) = \overline{J(L)}$, where the latter refers to the complex conjugate of $J(L)$. In terms of t, $J_{\overline{L}}(t) = J_L(t^{-1})$. This property explains why the mirror images in 2 and 3 above differ by substituting t by t^{-1}.

25. Towards the Witten invariant of 3-manifolds

25.1. **Invariants of embedded trivalent graphs.** Although it is not explicitly stated in Kohno's book, it appears that there are invariants of embeddings of closed (finite) trivalent graphs. (Trivalent means all the vertices have three edges.)

Given an elementary tangle as in Figure 12, $Z_i : V(t_i) \to V(t_{i+1})$ is defined as in Figure 7, where the labels $0, \mu_j, \mu_j^*$ are changed to ν, μ_1, μ_2, respectively.

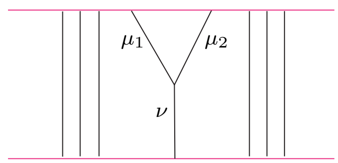

Figure 12

Similarly, if we have an elementary tangle given by the upside-down version of Figure 12, the Z_i is given by the projection on the right-hand diagram of Figure 7 (with labels ν, μ_1, μ_2) onto the middle diagram of Figure 7 (with label ν instead of 0).

Let $N_{\lambda\mu\nu}$ be the dimension of the space of conformal blocks $\mathcal{H}(p_1, p_2, p_2; \lambda, \mu, \nu)$, which is either 0 or 1, depending on whether the quantum Clebsch-Gordan rule is satisfied or not.

Lemma 25.1. $Z_i(A) = \sum_\nu N_{\mu_1\mu_2\nu} Z_i(B)$*, where* A *and* B *are elementary tangles given in Figure 13.*

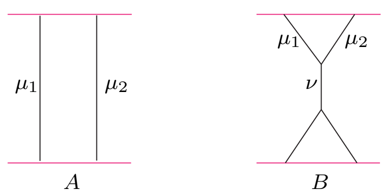

Figure 13

Proof. This is straightforward to verify, by taking a basis corresponding to the right-hand diagram of Figure 7. ☐

Remark: We must check that J (defined similarly as before) of a trivalent graph is independent of the slicing into elementary tangles and Reidemeister II and III moves. I have some trouble showing this.

25.2. $J(L; \mu_1, \dots, \mu_m)$**.** The goal of this section is to give a recursive formula which expresses $J(L; \mu_1, \dots, \mu_m)$ in terms of $J(L; 1, \dots, 1)$, i.e., the Jones polynomial.

Surgery convention: Given a knot K, take its tubular neighborhood $N(K)$. Take an oriented identification of $-\partial N(K)$ with $T^2 \simeq \mathbf{R}^2/\mathbf{Z}^2$ so that the meridian of $N(K)$ (i.e., the nontrivial curve on $\partial N(K)$ that bounds a disk in $N(K)$) corresponds to $\pm(0, 1)$ and the longitude (given as the boundary of the Seifert surface of K) corresponds to $\pm(1, 0)$. We observe that the longitude is independent of the choice of Seifert surface, provided K is a knot. (More precisely, we are choosing the longitude to be the intersection of the Seifert surface and $\partial N(K)$.) With respect to the identification with $\mathbf{R}^2/\mathbf{Z}^2$, the meridian has slope ∞ and the longitude has slope 0.

A (p, q)-cable of K is a knot on $\partial N(K)$ which represents a (q, p)-curve with respect to the above coordinates on T^2. Alternatively, it winds p times around the meridian and q times around the longitude. For example, an $(n, 1)$-cable is a pushoff of K in the n-framing direction.

Computation of $d(\lambda)$**:** Let us apply this to the $(0, 1)$-cable of the unknot. See Figure 14.

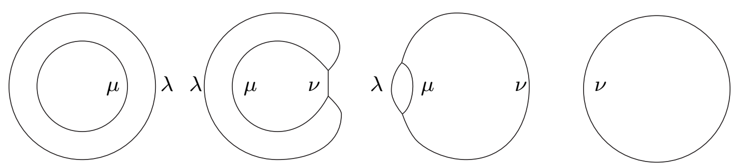

FIGURE 14

J of the leftmost diagram L equals the sum, over all ν, of $N_{\lambda\mu\nu}$ times J of any of the other three diagrams. Rephrasing in terms of Z, we have:

$$d(\lambda)d(\mu)Z(L; \mu, \lambda) = \sum_\nu N_{\lambda\mu\nu} d(\nu) Z(\text{unknot}; \nu).$$

Since $Z(L; \mu, \lambda) = Z(\text{unknot}; \nu) = 1$, it follows that:

$$d(\lambda)d(\mu) = \sum_\nu N_{\lambda\mu\nu} d(\nu). \tag{16}$$

Lemma 25.2. *$d(1) = \frac{q - q^{-1}}{q^{1/2} - q^{-1/2}}$, where $q = e^{\frac{2\pi i}{k+2}}$.*

We will often call $[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$ the *quantum integer* n. Hence $d(1) = [2]$.

Proof. Apply the skein relation of P to Figure 10. Then we have

$$q \cdot 1 - q^{-1} \cdot 1 = (q^{1/2} - q^{-1/2}) d(1) \cdot 1,$$

since $P(L) = d(1)^{-1} e^{-2\pi i \Delta_1 w(L)} J(L)$ and $J(L_1 \cup L_2) = J(L_1)J(L_2)$ by the previous lecture. □

Equation 16, together with Lemma 25.2, can be used to recursively compute $d(\lambda)$. One checks that $d(0) = 1$. By Equation 16, using the quantum Clebsch-Gordan rule we can write

$$d(1)d(1) = d(0) + d(2),$$

provided $k \geq 2$. This allows us to compute that $d(2) = [3]$. In general, we obtain (with more work):

Lemma 25.3. $d(\lambda) = [\lambda + 1]$.

Reduction to $J(L)$: Using the same technique of fusing together strands, we obtain the following:

Theorem 25.4. *If K_1 is the cable of K_0 with respect to the blackboard framing, then*

$$J(K_0, K_1; \lambda, \mu) = \sum_{\nu} N_{\lambda\mu\nu} J(K_0, \nu).$$

If we take $\lambda, \mu = 1$, then we can find $J(K_0; 2)$, etc., using the above formula.

25.3. **Dehn surgery.** Let K be a knot in S^3. The $\frac{p}{q}$-surgery on K is the closed (= compact without boundary) 3-manifold obtained by first removing $N(K) \simeq (S^1 \times D^2)$ from S^3 and gluing it back so that the new meridian has slope $\frac{p}{q}$ with respect to the coordinates introduced earlier. Our notation for the resulting 3-manifold is $S^3_{p/q}(K)$. If $\frac{p}{q} \in \mathbf{Z}$, then the surgery is called an *integer surgery*. If K is a framed knot, then the framing gives rise to an integer surgery coefficient $\frac{p}{q} = n$, and we often write $S^3(K)$.

Similarly, given a link $L = L_1 \cup \cdots \cup L_m$, each L_j is a knot and has a framing coming from its Seifert surface. We write $S^3_{(p_1/q_1,\ldots,p_m/q_m)}(L_1, \ldots, L_m)$ for the result of $\frac{p_j}{q_j}$-surgery along L_j, done simultaneously on all L_j.

Theorem 25.5. *Any oriented, closed 3-manifold is obtained by integer surgery on a link in S^3. (The coefficients may be different for different components of the link L.)*

Theorem 25.6 (Kirby, improved by Fenn-Rourke)**.** *Let L and L' be framed links in S^3. Then $S^3(L)$, $S^3(L')$ are diffeomorphic iff L' is obtained from L by applying the following moves, called* blowing up/blowing down.

In Figure 15, from the left to the right we are *blowing down* and from the right to the left we are *blowing up*. Let L_0 be the unknot with Dehn surgery coefficient ± 1 which links with strand L_j with surgery coefficient n_j. Then blowing down entails removing L_0 and replacing it with ∓ 1 full twists, and further changing the framing of L_j from n_j to $n_j \mp (lk(L_0, L_j))^2$.

References

[1] R. Gompf and A. Stipsicz, *4-Manifolds and Kirby Calculus.*

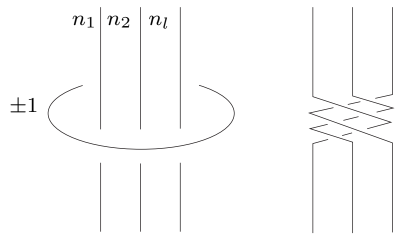

FIGURE 15. Blowing up/blowing down

26. THE WITTEN INVARIANT OF 3-MANIFOLDS

Today we define the *Witten invariant* of a closed oriented 3-manifold M. It is obtained by taking a suitable linear combination of $J(L;\lambda_1,\dots,\lambda_m)$, where $L = L_1 \cup L_m$ is a framed link such that $M \simeq S^3(L)$.

26.1. Some preparation.

Modularity: Let

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}}\sin\left(\pi\frac{(\lambda+1)(\mu+1)}{k+2}\right),$$

$$\Delta_\lambda = \frac{\lambda(\lambda+2)}{4(k+2)}.$$

Also let $c = \frac{3k}{k+2}$ and $C = e^{-\pi i c/4}$. Then consider the $(k+1)\times(k+1)$-matrices $S = (S_{\lambda\mu})$, where λ,μ range from 0 to k, and $T = \text{diag}(e^{2\pi i(\Delta_0 - c/24)},\dots,e^{2\pi i(\Delta_k - c/24)})$.

Proposition 26.1. *There is a representation $PSL(2,\mathbf{Z}) \to GL(k+1,\mathbf{C})$ which sends $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto S$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T$.*

In particular, $S^2 = (ST)^3 = I$ and hence $TSTST = S$. This gives us the following equation:

$$C\sum_{\mu=0}^{k} S_{\lambda\mu}S_{\mu\nu}e^{2\pi i(\Delta_\lambda+\Delta_\mu+\Delta_\nu)} = S_{\lambda\nu}. \tag{17}$$

FS: Apparently the representation arises from the modular transformation properties of characters of representations of affine Lie algebras.

Signature: Let $L = L_1 \cup \dots \cup L_m$ be a framed link. Note that we need the link to be framed in order to define $lk(L_i, L_i)$. The linking number is symmetric, i.e., $lk(L_i, L_j) = lk(L_j, L_i)$. Given the symmetric $m \times m$ matrix $A = (lk(L_i, L_j))$ (called the *Seifert matrix*), we define the *signature* of L to be

$$\sigma(L) = \#(\text{positive eigenvalues}) - \#(\text{negative eigenvalues}).$$

26.2. **The Witten invariant.** Let M be a closed oriented 3-manifold. If $M = S^3(L)$, then set

$$Z_k(M) = S_{00} C^{\sigma(L)} \sum_{\lambda} S_{0\lambda_1} \dots S_{0\lambda_m} J(L; \lambda_1, \dots, \lambda_m). \tag{18}$$

Here the sum is over all $\lambda : \{1, \dots, m\} \to P_+(k) = \{0, \dots, k\}$, and $\lambda_i = \lambda(i)$.

Theorem 26.2. *$Z_k(M)$ is a topological invariant, i.e., does not depend on the choice of L such that $M \simeq S^3(L)$.*

A formulation similar to Equation 18 was first given by Reshetikhin-Turaev.

We need to show that the RHS expression in Equation 18 does not change under blowing up and blowing down. Let us write $L = L_0 \cup L_1 \cup \dots \cup L_m$, where L_0 is an unknot with ± 1 framing.

Case 1. Suppose L_0 and $L_1 \cup \dots \cup L_m$ are contained in disjoint 3-balls, i.e., L_0 is not linked to the rest.

HW: Show directly that $S^3(L_0 \cup \dots \cup L_m) \simeq S^3(L_1 \cup \dots \cup L_m)$.

Recall that by Property A in Section 24.3,

$$J(L; \lambda_0, \dots, \lambda_m) = J(L_1 \cup \dots \cup L_m; \lambda_1, \dots, \lambda_m) \cdot J(L_0; \lambda_0),$$

since L_0 and the $L_1 \cup \dots \cup L_m$ are contained in disjoint 3-balls. Suppose L_0 is the $+1$-framed unknot. Then

$$J(L_0; \lambda_0) = e^{2\pi i \Delta_{\lambda_0}} J(\text{0-framed unknot}; \lambda_0) = e^{2\pi i \Delta_{\lambda_0}} d(\lambda_0) = e^{2\pi i \Delta_{\lambda_0}} \frac{S_{0\lambda_0}}{S_{00}}.$$

HW: Verify that $d(\lambda) = \frac{S_{0\lambda}}{S_{00}}$.

Next we compare signatures. Let $A = (lk(L_i, L_j))$ where $i, j = 1, \dots, m$. Then the Seifert matrix for $L_0 \cup \dots \cup L_m$ is $\text{diag}(1, A)$. Hence

$$\sigma(L_0 \cup \dots \cup L_m) = \sigma(L_1 \cup \dots \cup L_m) + 1.$$

Hence,

$$\begin{aligned} & S_{00} C^{\sigma(L_0 \cup \dots \cup L_m)} \sum_{\lambda} S_{0\lambda_0} \dots S_{0\lambda_m} J(L; \lambda_0, \dots, \lambda_m) \\ = \quad & S_{00} C^{\sigma(L_1 \cup \dots \cup L_m)+1} \sum_{\lambda_1, \dots, \lambda_m} S_{0\lambda_1} \dots S_{0\lambda_m} J(L; \lambda_1, \dots, \lambda_m) \sum_{\lambda_0} S_{0\lambda_0} e^{2\pi i \Delta_{\lambda_0}} \frac{S_{0\lambda_0}}{S_{00}}. \end{aligned}$$

Hence, to show the invariance we need:

$$C \sum_{\lambda_0} S_{0\lambda_0} \frac{S_{0\lambda_0}}{S_{00}} e^{2\pi i \Delta_{\lambda_0}} = 1,$$

which follows from Equation 17 by setting $\lambda = 0$, $\nu = 0$, $\mu = \lambda_0$.

Case 2. Suppose the Seifert surface of L_0 (a disk) has nontrivial intersection with only L_1 and they intersect in only one point.

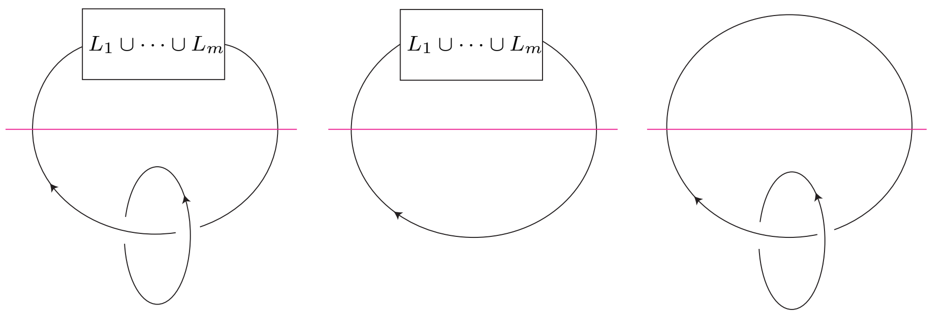

FIGURE 16. Chopping and rearranging

As shown in Figure 16, we divide the link $L_0 \cup \cdots \cup L_m$ on the left-hand side into two tangles, and complete the two tangles by attaching half of an unknot (with 0 framing) to each.

We have

$$J(L_0 \cup \cdots \cup L_m; \lambda_0, \dots, \lambda_m) = \frac{J(L_1 \cup \cdots \cup L_m; \lambda_1, \dots, \lambda_m) \cdot J(H'; \lambda_0, \lambda_1)}{J(\text{0-framed unknot}; \lambda_1)}.$$

Here H' is the Hopf link with framings $+1$ and 0. (Prove the above relation!)

Fact: If H is the Hopf link with framings $0, 0$, then $J(H; \lambda, \mu) = \frac{S_{\lambda\mu}}{S_{00}}$.

This can be verified by viewing the Hopf link as a cable of the unknot. Hence,

$$J(H'; \lambda_0, \lambda_1) = e^{2\pi i \Delta_{\lambda_0}} J(H; \lambda_0, \lambda_1) = e^{2\pi i \Delta_{\lambda_0}} \frac{S_{\lambda_0\lambda_1}}{S_{00}}.$$

Since

$$J(\text{0-framed unknot}; \lambda_1) = \frac{S_{0\lambda_1}}{S_{00}},$$

it follows that:

$$J(L_0 \cup \cdots \cup L_m; \lambda_0, \dots, \lambda_m) = J(L_1 \cup \cdots \cup L_m; \lambda_1, \dots, \lambda_m) \left(e^{2\pi i \Delta_{\lambda_0}} \frac{S_{\lambda_0\lambda_1}}{S_{00}} \right) \left(\frac{S_{00}}{S_{0\lambda_1}} \right).$$

Also observe that, after blowing down, the new link is $L_1' \cup L_2 \cup \cdots \cup L_m$, where if L_0 has framing $+1$ and L_1 has framing n_1, then L_1' is L_1 with framing $n_1 - (lk(L_0, L_1))^2 = n_1 - 1$. Hence

$$J(L_1' \cup \cdots \cup L_m; \lambda_1, \ldots, \lambda_m) = e^{-2\pi i \Delta_{\lambda_1}} J(L_1 \cup \cdots \cup L_m; \lambda_1, \ldots, \lambda_m).$$

Next we consider Seifert matrices. Let $A = (a_{ij})$ be the Seifert matrix for $L_1 \cup \cdots \cup L_m$. Then the Seifert matrices for $L_1' \cup \cdots \cup L_m$ and $L_0 \cup \cdots \cup L_m$ are (respectively):

$$A' = \begin{pmatrix} a_{11} - 1 & a_{12} & \ldots \\ a_{21} & a_{22} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}, A'' = \begin{pmatrix} 1 & 1 & 0 & \ldots \\ 1 & a_{11} & a_{12} & \ldots \\ 0 & a_{21} & a_{22} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

HW: Verify that $\sigma(A'') = \sigma(A') + 1$. (Hint: A change of coordinates for A'' is helpful.)

Finally, the following equality (a special case of Equation 17) gives the result.

$$C \sum_{\lambda_0} S_{0\lambda_0} S_{\lambda_0 \lambda_1} e^{2\pi i \Delta_{\lambda_0}} = S_{0\lambda_1} e^{-2\pi i \Delta_{\lambda_1}}.$$

Case 3. Suppose there the Seifert surface of L_0 has multiple intersections with $L_1 \cup \cdots \cup L_m$. In that case, we fuse together two strands. Refer to Figure 17. Then $J(\text{LHS})$ is equal to $\sum_\nu N_{\nu\lambda_1\lambda_2} J(\text{RHS})$, where ν is the label of the new edge which intersects the Seifert surface of L_0, and λ_1, λ_2 are the labels of the strands that are fused together. By induction, we reduce to Case 2.

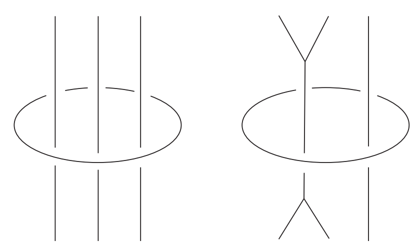

FIGURE 17. Fusing together strands

26.3. **Properties of $Z_k(M)$.**

(1) $Z_k(S^3) = S_{00}$, since S^3 is given by the empty knot.
(2) Since $S^1 \times S^3$ is given by 0-surgery on the unknot in S^3,

$$Z_k(S^1 \times S^3) = S_{00} \sum_\mu S_{0\mu} \frac{S_{0\mu}}{S_{00}} = \sum_\mu (S_{0\mu})^2 = 1.$$

(The last equality follows from $S^2 = I$, where S is the matrix given earlier.)

(3) If $M_1 \# M_2$ is the *connected sum* of M_1 and M_2, then

$$Z_k(M_1 \# M_2) = \frac{1}{S_{00}} Z_k(M_1) \cdot Z_k(M_2).$$

Here $M_1 \# M_2 = (M_1 - B^3) \cup (M_2 - B^3)$ where ∂B^3 from M_1 is identified with ∂B^3 from M_2 via a diffeomorphism.

(4) If $-M$ is M with reversed orientation, then

$$Z_k(-M) = \overline{Z_k(M)}.$$

27. Representations of mapping class groups

27.1. **Mapping class groups.** Let Σ be a closed oriented surface of genus g. Denote by $\text{Diff}^+(\Sigma)$ the group of orientation-preserving diffeomorphisms of Σ. We now put an equivalence relation on $\text{Diff}^+(\Sigma)$. Two $h, h' \in \text{Diff}^+(\Sigma)$ are equivalent ($h \sim h'$) iff there exists a smooth map $H : \Sigma \times [0,1] \to \Sigma$ such that $H_t(x) = H(t,x)$, $H_t : \Sigma \to \Sigma$ is a diffeomorphism, and $H_0 = h$, $H_1 = h'$. We say that h and h' are *isotopic*.

Define the *mapping class group* to be $Map(\Sigma) = \text{Diff}^+(\Sigma)/\sim$. We usually blur the distinction between diffeomorphisms and equivalence classes of diffeomorphisms.

Fundamental Example: Let γ be a homotopically nontrivial simple closed curve on Σ. A *positive Dehn twist* is an element of $\text{Diff}^+(\Sigma)$ which is the identity outside an annular neighborhood $N(\gamma)$ of γ. On $N(\gamma)$, cut $N(\gamma)$ along γ and reglue after doing one full twist along one of the (cut-open) copies of γ, as given in Figure 18. We denote the positive Dehn twist along γ by R_γ. (If you are looking towards γ on Σ, then an arc will be sent "to the right" after the positive Dehn twist.)

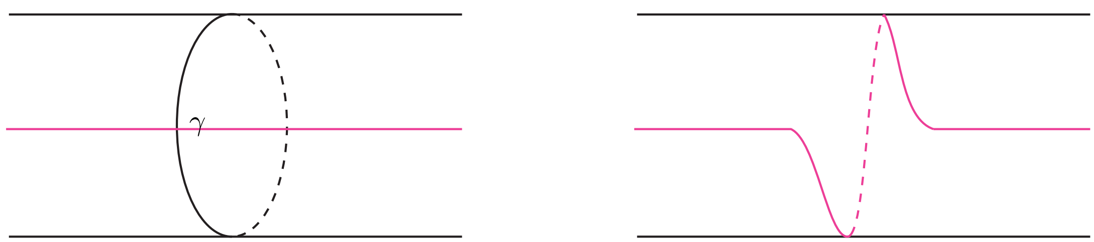

FIGURE 18. A positive Dehn twist

Theorem 27.1 (Lickorish-Humphreys). *$Map(\Sigma)$ is generated by (positive) Dehn twists about α_i, β_i, δ given in Figure 19.*

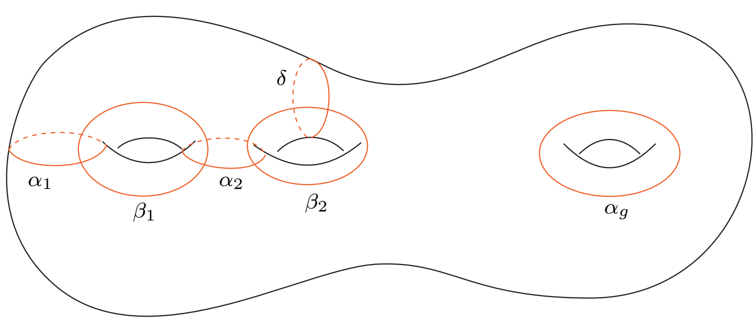

FIGURE 19. Generators of the mapping class group

27.2. **Heegaard splittings.** Any closed oriented 3-manifold M admits a decomposition $M = H_1 \cup H_2$, where H_i are handlebodies. A *handlebody* H of genus g is a compact 3-manifold with boundary which is bounded by a closed surface Σ of genus g inside $\mathbf{R}^3$. Equivalently, H can be given as a tubular neighborhood (inside $\mathbf{R}^3$) of a bouquet of g circles.

If $H_1 = H$, $H_2 = -H$ (by this we mean take two copies of H with opposite orientations), then we can identify $\partial H_1 = \partial H$ with $-\partial H_2 = \partial H$ via the diffeomorphism $h : \partial H \xrightarrow{\sim} \partial H$. The diffeomorphism type of M only depends on $[h] \in Map(\Sigma)$, where $\Sigma = \partial H$.

HW: Show that if $h = id$, then M is the connected sum of g copies of $S^1 \times S^2$.

$h = id$ corresponds to the following link diagram:

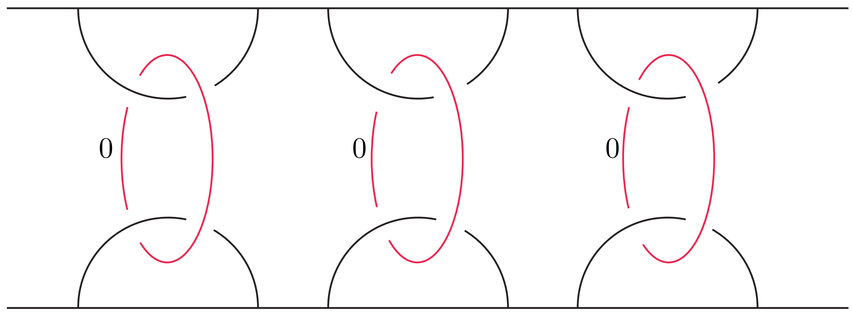

FIGURE 20. 3-manifold corresponding to $id \in Map(\Sigma)$

The top and bottom graphs are Γ' and Γ, respectively. H_1 is the thickening of Γ and H_2 is the thickening of Γ'. We are surgering along three unknots with framing 0 which lie in disjoint 3-balls. One can check that the complement of $\Gamma \cup \Gamma'$ in S^3, after surgery, is $\Sigma \times [0, 1]$, where Σ is a closed surface of genus g.

If $h = R_{\alpha_1}$, R_{α_2}, or R_{β_1}, then we have the link diagram:

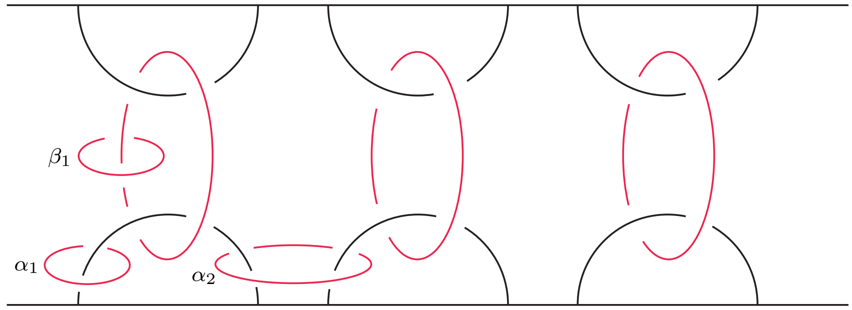

FIGURE 21. 3-manifolds corresponding to $R_{\alpha_1}, R_{\alpha_2}, R_{\beta_1} \in Map(\Sigma)$.

Each link diagram is obtained from the link in Figure 20 by adding a (-1)-framed unknot, labeled α_1, α_2, α_3, respectively, in Figure 21.

HW: What about R_δ?

27.3. **The TQFT representations of** $Map(\Sigma)$**.**

The vector space: Define the complex vector space V_Σ to be the vector space generated by labelings of the uni/trivalent graph Γ given below. The edges on the left and on the right have labelings 0, and at each trivalent vertex the level k quantum Clebsch-Gordan rule must be satisfied.

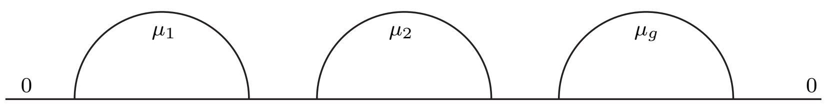

FIGURE 22. The graph Γ.

We can alternatively write V_Σ as a direct sum, over $\mu_1, \mu_1^*, \dots, \mu_g, \mu_g^* \in P_+(k) = \{0, \dots, k\}$, of the space of conformal blocks

$$V_{\mu_1\mu_1^*\dots\mu_g\mu_g^*} \overset{def}{=} \mathcal{H}(0, p_1, \dots, p_{2g}, \infty; 0, \mu_1, \mu_1^*, \dots, \mu_g, \mu_g^*, 0).$$

(We can think of cutting Γ along the edges with labelings $\mu_1, \dots, \mu_g$ so it becomes a tree.)

Fact: $\dim V_\Sigma = \sum_{\lambda \in P_+(k)} \left(\frac{1}{S_{0\lambda}}\right)^{2g-2}$.

The morphisms: If h is one of the generators R_{α_i}, R_{β_i}, R_δ of $Map(\Sigma)$, then consider the corresponding tangle $T(h)$ given in Figure 21. If Γ has labels $\mu_1, \dots, \mu_g$ and Γ' has labels $\nu_1, \dots, \nu_g$ (these are the labels on the semicircular edges), then consider

$$J(T(h); \lambda)_{\mu\nu} : V_{\mu_1\mu_1^*\dots\mu_g\mu_g^*} \to V_{\nu_1\nu_1^*\dots\nu_g\nu_g^*}.$$

Then define

$$\rho(h) : V_\Sigma \to V_\Sigma$$

by $\rho(h) = \otimes_{\mu\nu}\rho(h)_{\mu\nu}$, where

$$\rho(h)_{\mu\nu} = \sqrt{S_{0\mu_1}\dots S_{0\mu_g}}\sqrt{S_{0\nu_1}\dots S_{0\nu_g}}C^{\sigma(L(h))}\sum_\lambda S_{0\lambda_1}\dots S_{0\lambda_m}J(T(h);\lambda)_{\mu\nu},$$

where λ is the set of all labelings (with values in $P_+(k)$) of the link components.

Theorem 27.2. $\rho : Map(\Sigma) \to GL(V_\Sigma)$ *is a projective representation.*

For a *projective representation* ρ, $\rho(h_1h_2) \neq \rho(h_1)\rho(h_2)$ but instead

$$\rho(h_1h_2) = \xi(h_1, h_2)\rho(h_1)\rho(h_2),$$

where $\xi(h_1, h_2) = C^{\sigma(h_1h_2)-\sigma(h_1)-\sigma(h_2)}$.

Let us consider what happens when you stack two tangles $T(h_1)$ and $T(h_2)$ ($T(h_1)$ below $T(h_2)$). It is easy to see that ρ is a projective representation. (HW: verify this!) What seems a little strange, however, is that if we stack two "elementary" tangles corresponding to one of the Dehn twist generators, then the lower semicircles from $T(h_1)$ and the upper semicircles of $T(h_2)$ glue to give 0-framed unknots. All of a sudden new unknots appears into the surgery picture! Upon further inspection, this turns out to be quite natural:

Fact: $T(h_1 h_2)$ represents $M = H \cup_{h_1 h_2} (-H)$. (Hint: there is a way to cancel certain chains of 0-framed unknots.)

We end this lecture with an important fact:

Fact: Suppose v_λ is an element of V_Σ corresponding to a labeling of Γ in Figure 22. Let e be an edge of Γ and $C(e)$ be a curve which is the meridian corresponding to e for the thickening of Γ. Then:

$$\rho(R_{C(e)})(v_\lambda) = e^{2\pi i \Delta_{\lambda(e)}} v_\lambda.$$

28. CHERN-SIMONS THEORY

In this lecture, we briefly describe the original way Witten's invariant for 3-manifolds was defined (using path integrals).

28.1. **Connections on principal bundles.**

Let M be a compact oriented 3-manifold. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For simplicity we set $G = SU(2)$. Some of the discussion below will be valid for arbitrary G, and others will depend on $G = SU(2)$.

Let P be a principal G-bundle over M. A *principal* G-bundle admits a *right* G-action and a local trivialization $\pi^{-1}(U) \xrightarrow{\sim} U \times G$, where the identification commutes with the G-action. Here $\pi : P \to M$ is the projection. One can show that if $G = SU(2)$, then P is trivial, i.e., $P \simeq M \times G$.

Next, a connection ω is a 1-form on P with values in $\mathfrak{g}$ which satisfies the following:

(1) $\omega(p)((i_p)_*\xi) = \xi$, if $p \in P$ and $\xi \in \mathfrak{g}$;
(2) $R_g^*\omega = Ad(g^{-1})\omega$.

Here $R_g : P \to P$ is right multiplication by g, i.e., $p \mapsto pg$, and $i_p : G \to P$ is the inclusion $p \mapsto pg$. If condition (1) reminds you of the Maurer-Cartan form, you are right: if $P = M \times G$, then, with respect to the second projection $\pi_G : M \times G \to G$, the Maurer-Cartan form μ on G gets pulled back to a connection 1-form $\pi_G^*(\mu)$ on P.

It is possible to push ω down to M by taking the difference with a fixed connection, say $\omega_0 = \pi_G^*(\mu)$. Then $A = \omega - \omega_0 \in \Omega^1(M; \mathfrak{g})$. We write $\mathcal{A}_M = \Omega^1(M; \mathfrak{g})$ for the space of G-connections on $P = M \times G$.

Now, we define the *gauge group* $\mathcal{G}_M$ to be the set of maps $g : M \to G$. Here g acts on P by right multiplication. If $g \in \mathcal{G}_M$, then

$$g^*A = g^{-1}Ag + g^{-1}dg. \tag{19}$$

(Observe that this is similar to the gauge change for affine connections, discussed earlier.)

FS: For more information on principal G-bundles and their connections, refer to my second semester differential geometry (Math 535b) notes, available from my website.

28.2. **The Chern-Simons functional.**

The *Chern-Simons functional* is a function:

$$CS : \mathcal{A}_M \to \mathbf{R},$$

$$A \mapsto \frac{1}{8\pi^2}\int_M Tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A).$$

FS: We briefly mention Chern-Weil theory. Given a connection A on P, one can construct characteristic classes out of the curvature $F_A = dA + A \wedge A$ by taking $\omega_k = Tr(F_A^k)$. It can be shown that $d\omega_k = 0$ and $[\omega_k] \in H^{2k}(M; \mathbf{R})$ does not depend on the choice of connection A. The integrand of the Chern-Simons functional is supposed to be a primitive of $Tr(F_A \wedge F_A)$.

Lemma 28.1. *Suppose $\partial M = \emptyset$. Then the critical points of CS are flat connections, i.e., connections A such that $F_A = 0$.*

Proof. At a *critical point* $A \in \mathcal{A}$ all directional derivatives

$$\lim_{t\to 0} \frac{CS(A+ta) - CS(A)}{t} = 0,$$

where $a \in T_A\mathcal{A} = \Omega^1(M;\mathfrak{g})$. (Note that what we are trying to do is an analog, in infinite dimensions, of exploring the topology of the space by looking at the critical points of a Morse function.)

Using the invariance of the trace under cyclic permutation $Tr(a_1a_2a_3) = Tr(a_3a_1a_2)$, where a_i are $n \times n$ matrices, and ignoring terms which are quadratic or higher in t, we compute that:

$$\begin{aligned} \lim_{t\to 0} \frac{CS(A+ta) - CS(A)}{t} &= \frac{1}{8\pi^2}\int_M Tr(a\wedge dA + A\wedge da + 2a\wedge A\wedge A) \\ &= \frac{1}{8\pi^2}\int_M 2\cdot Tr(a\wedge(dA + A\wedge A)). \end{aligned}$$

Here, $d(Tr(A\wedge a)) = Tr(dA\wedge a) - Tr(A\wedge da)$ and $\int_M d(Tr(A\wedge a)) = 0$ (since M is closed), so $\int_M Tr(A\wedge da) = \int_M Tr(a\wedge dA)$.

Since a was arbitrary, F_A must equal zero at a critical point A. □

One can similarly compute:

Lemma 28.2. *Suppose ∂M is not necessarily empty. Then*

$$CS(g^*A) = CS(A) + \frac{1}{8\pi^2}\int_{\partial M} Tr(A\wedge dgg^{-1}) - \int_M g^*\sigma.$$

Here σ is the 3-form on $G = SU(2)$ given by $\frac{1}{24\pi^2}Tr(\mu\wedge\mu\wedge\mu)$, and μ is the Maurer-Cartan form on G. (Recall $[\sigma] \in H^3(SU(2);\mathbf{Z})$ is the generator.)

HW: Prove the lemma!

Observe that the last term is the Wess-Zumino term.

If $\partial M = \emptyset$, then the boundary term drops out, and

$$CS(g^*A) = CS(A) - \int_M g^*\sigma.$$

Notice $\int_M g^*\sigma \in \mathbf{Z}$ since it is the pullback of an integral class of G. Hence CS is a function

$$CS : \mathcal{A}_M/\mathcal{G}_M \to \mathbf{R}/\mathbf{Z}.$$

It also makes sense to write $e^{2\pi i CS([A])}$, where $[A] \in \mathcal{A}_M/\mathcal{G}_M$.

28.3. **The path integral.** Just as for the WZW model, consider the Feynman path integral:

$$Z_k(M) = \int_{\mathcal{A}_M/\mathcal{G}_M} e^{2\pi ikCS(A)} d\mu.$$

(Here k is the level.) Cut a closed oriented 3-manifold M along an oriented surface Σ so that $M = M_1 \cup M_2$, $\partial M_1 = \Sigma$, $\partial M_2 = -\Sigma$.

We construct a line bundle over $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$ in much the same way as before. Extend $a \in \mathcal{A}_\Sigma$ to $A \in \mathcal{A}_{M_1}$ and consider $e^{2\pi iCS(A)}$. Also, given $g \in \mathcal{G}_\Sigma$, extend to $\tilde{g} \in \mathcal{G}_{M_1}$. If we define

$$c(a,g) = e^{2\pi i(CS(\tilde{g}^*A) - CS(A))} = e^{2\pi i[\frac{1}{8\pi^2}\int_\Sigma a\wedge dgg^{-1} - \int_{M_1} \tilde{g}^*\sigma]},$$

the $c(a,g)$ does not depend on the extensions A and $\tilde{g}$ to M_1. Now take the trivial bundle $\mathcal{A}_\Sigma \times \mathbf{C} \to \mathcal{A}_\Sigma$ and quotient out by the equivalence relation: $(a,1) \sim (g^*a, c(a,g))$.

Now, $\mathcal{A}_\sigma/\mathcal{G}_\Sigma$ is still infinite, so we further restrict to the space of flat connections on Σ. The quotient of the space of flat connections on Σ by $\mathcal{G}_\Sigma$ will be denoted $\mathcal{M}_\Sigma$, and will be called the *moduli space* of flat connections on Σ. It is equivalent to $Hom(\pi(\Sigma), G)/G$, and can be given the structure of a complex manifold. Also, the restriction of the above line bundle to $\mathcal{M}_\Sigma$ will be written $\mathcal{L}_\Sigma$, and is a holomorphic line bundle. The k-fold tensor power of $\mathcal{L}_\Sigma$ will be written $\mathcal{L}_\Sigma^{\otimes k}$.

The space of holomorphic sections $\Gamma(\mathcal{L}_\Sigma^{\otimes k})$ is called the *quantum Hilbert space* of level k.

FS: The quantum Hilbert space of level k is isomorphic to the space of conformal blocks of level k. (See Beauville-Laslo, *Conformal blocks and theta functions*.)

If $L = L_1 \cup \cdots \cup L_m$ is a link in a 3-manifold M, then assign a representation V_j of G to each component L_j. Let $W_{L_j,R_j}(A)$ be the trace of the holonomy of A around L_j. Then Witten's invariant is given by:

$$Z_k(M; L_1, \ldots, L_m) = \int e^{2\pi ikCS(A)} \prod_{j=1}^m W_{L_j,R_j}(A)\, d\mu.$$