1. Holonomic approximations

For more details, refer to Eliashberg-Mishachev [EM], “Introduction to the \(h\)-principle”. All manifolds we consider are smooth manifolds.

1.1. Definitions. Given a subset \(A\) of a manifold \(V\), we write \(Op(A)\) for an arbitrarily small but non-specified open neighborhood of \(A\) in \(V\).

Definition 1.1.1. Given a fiber bundle \(X \to V\), its \(r\)-jet bundle \(J^r(X) \to V\) (also written as \(X^{(r)}\)) is the bundle whose fiber over \(p \in V\) is

\[
J^r_p(X) = \{\text{sections of } X \text{ over } Op(p)\}/\sim,
\]

where \(f : U_1 \ni p \to X\) and \(g : U_2 \ni p \to X\) satisfy \(f \sim g\) if \(f\) and \(g\) have the same \(r\)th order Taylor expansion at \(p\).

Note that \(J^0(X) = X\). Let \(p_{r,r'} : J^r(X) \to J^{r'}(X)\) be the projection for \(r' \leq r\).

Example 1.1.2. \(J^1(V, \mathbb{R}) = \mathbb{R} \times T^*V\).

In local coordinates, \(J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{q d_1} \times \cdots \times \mathbb{R}^{q d_r}\), where \(d_i\) is number of partial derivatives \(D^\alpha\) of order \(i\) of a function \(\mathbb{R}^n \to \mathbb{R}\). Given a smooth map \(f : \mathbb{R}^n \to \mathbb{R}^q\) and \(x \in \mathbb{R}^n\),

\[
J^r_f(x) = (f(x), f'(x), \ldots, f^{(r)}(x)) \in J^r(\mathbb{R}^n, \mathbb{R}^q)
\]
is the \(r\)-jet of \(f\) at \(x\). Here \(f^{(i)}(D^\alpha f)\alpha\), where \(\alpha = (\alpha_1, \ldots, \alpha_n)\), \(\alpha_1 + \cdots + \alpha_n = i\).

Definition 1.1.3. A section \(F : V \to J^r(X)\) is holonomic if \(F = J^r_{p_{r,0}} F\).

1.2. Holonomic approximations.

Question 1.2.1. Can any section of \(J^r(X) \to V\) be \(C^0\)-approximated by a holonomic section?

Example 1.2.2. Consider \(F(x) = (x, x, 0)\) for \(J^1(\mathbb{R}, \mathbb{R}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}\). It’s impossible to find \(f(x)\) such that \(|f(x) - x| < \varepsilon\) and \(|f'(x)| < \varepsilon\). (A picture would help here....)

In what follows we assume we have chosen an auxiliary Riemannian metric on the relevant spaces whenever we refer to lengths.

Theorem 1.2.3 (Holonomic approximation). Let \(K \subset V\) be a polyhedron (i.e., submanifold with higher-dimensional corners) of positive codimension and \(F : Op(K) \to X^{(r)}\) a section. Then for all \(\delta, \varepsilon > 0\) small there exists a \(\delta\)-small diffeotopy (in the \(C^0\)-sense)

\[
h^\tau : V \cong V, \quad \tau \in [0, 1],
\]
(i.e., a 1-parameter family of diffeomorphisms of $V$ with $h^0 = \text{id}$) and a holonomic section

$$\tilde{F} : \text{Op}(h^1(K)) \to X^{(r)}$$

such that $\text{dist}(\tilde{F}(v), F(v)) < \varepsilon$ for all $v \in \text{Op}(h^1(K))$. [Here we are choosing $\delta > 0$ so that $\text{Op}(h^1(K)) \subset \text{Op}(K)$].

There is also a parametric/relative version:

**Theorem 1.2.4** (Parametric holonomic approximation). Let $K \subset V$ be a polyhedron of codimension $\geq 1$. Let $I^m$ be an $m$-dimensional cube and let $F_z : \text{Op}(K) \to X^{(r)}$ be a family of sections smoothly parametrized by $z \in \text{Op}(I^m)$ such that $F_z$ is holonomic for $z \in \text{Op}(\partial I^m)$. Then for all $\delta, \varepsilon > 0$ small there exists a family of $\delta$-small diffeotopies $h^\tau_z$, $\tau \in [0, 1]$, and a family of holonomic sections $\tilde{F}_z$ such that:

1. $h^\tau_z = \text{id}$ and $\tilde{F}_z = F_z$ for all $z \in \text{Op}(\partial I^m)$ and
2. $\text{dist}(\tilde{F}_z(v), F_z(v)) < \varepsilon$ for all $v \in \text{Op}(h^1_z(K))$ and $z \in I^m$. 
2. Proof of Holonomic Approximation Theorem

We explain the main idea of the proof. We stress that the basic idea of the proof is simple and pretty but the actual proof is notation-heavy and slightly unpleasant to read.

Step 1. Let us write \( J = J^r(\mathbb{R}^n, \mathbb{R}^q) \). We first reduce to proving the following relative theorem on a cube:

**Theorem 2.0.1** (Holonomic approximation on the cube). Let \( I^k = [-1, 1]^k \subset \mathbb{R}^k \subset \mathbb{R}^n \), \( k < n \), be the cube corresponding to the first \( k \) coordinates. For any section \( F : Op(I^k) \to J \) which is holonomic over \( Op(\partial I^k) \), there exists a diffeomorphism

\[
h : \mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n + \phi(x_1, \ldots, x_n)),
\]

and a holonomic section \( \tilde{F} : Op(h(I^k)) \to J \) such that

1. \( h = \text{id} \) and \( \tilde{F} = F \) on \( Op(\partial I^k) \); and
2. \( |\tilde{F} - F|_{C^0} < \varepsilon \) on \( Op(h(I^k)) \).

HW: show that Theorem 2.0.1 implies Theorem 1.2.3.

Step 2. The starting point of the induction is to observe that given \( F : Op(I) \to J \) there exists a family \( F_y : Op(y) \to J \), \( y \in I \), of holonomic sections that agree with \( F \) at \( y \) and with \( F \) on \( Op(\partial I) \).

Step 3. We introduce some notation. Let \( \pi_{k-l} : I^k \to I^{k-l} \) be the projection to the first \( k-l \) coordinates with \( 0 \leq l < k \). The fibers are \( y \times I^l \) where \( y \in I^{k-l} \). Given \( K \subset \mathbb{R}^n \), let \( N_\delta(K) \) be its cubical \( \delta \)-neighborhood in \( \mathbb{R}^n \), i.e.,

\[
N_\delta(K) = \bigcup_{(x_1, \ldots, x_n) \in K} [x_1 - \delta, x_1 + \delta] \times \cdots \times [x_n - \delta, x_n + \delta].
\]

We then set

\[
U_\delta(y) = N_\delta(y \times I^l), \quad V_\delta(y) = N_\delta(y \times \partial I^l),
\]

\[
A_\delta(y) = (U_\delta(y) - V_\delta(y)) \cap (y \times \mathbb{R}^{n-(k-l)}),
\]

where \( 0 < \theta < 1 \) and the neighborhoods are all taken in \( \mathbb{R}^n \). A picture would be good here (cf. Figures 3.2 and 3.3 of [EM]).

We state the following inductive lemma:

**Lemma 2.0.2** (Inductive lemma). With \( I^k \subset \mathbb{R}^n \) and \( F \) as in Theorem 2.0.1, suppose \( F \) is fiberwise holonomic with respect to \( \pi_{l-1} : I^k \to I^{k-1} \) for \( l = k - 1 \), i.e., given \( \delta > 0 \) small, for each \( y \in I = I^1 \) there exist a cubical \( \delta \)-neighborhood \( U_\delta(y) \) of \( y \times I^l \) and a family \( F_y : U_\delta(y) \to J \) of holonomic sections such that \( F_y = F \) on \( (y \times I^1) \cup V_\delta(y) \) and \( y \in Op(\partial I) \). Then for \( \varepsilon > 0 \) there exists a large integer \( N > 0 \) and a holonomic section

\[
\tilde{F} : \Omega \to J \quad \text{over} \quad \Omega = Op(\bigcup_{i=-N+1}^N A_\delta(c_i) \cup I^k) \cup \bigcup_{i=-N+1}^N A_\delta(c_i),
\]

where \( c_i = \frac{2i-1}{2N} \), \( i = -N + 1, \ldots, N \), and

1. \( \tilde{F} = F \) on \( \Omega \cap Op(\partial I^k) \);
2. \( |\tilde{F} - F|_{C^0} < \varepsilon \) on \( \Omega \).

Draw picture of \( \Omega \) for \( n = 2, k = 1, l = 0 \) (cf. Figure 3.4 of [EM]).

**Proof.** For sufficiently large \( N > 0 \), the holonomic family \( F_y : U_\delta(y) \to J \), \( y \in I \), exists for \( \delta = 1/N \). Define \( F^\tau_{c_i} : U_{\delta/2}(c_i) \cap \{y > c_i\} \to J \), \( \tau \in [0, 1] \), such that
\begin{itemize}
  \item $F_0^0 = F_{c_i}$;
  \item $F_{c_i}^\tau = F_{c_i}$ on $V_{\delta/2}(c_i) \cap \{y > c_i\}$ for all $\tau \in [0, 1]$;
  \item $F_{c_{i+1}}^1 = F_{c_i}$ on $N_{\delta}(I_k) \cap U_{\delta/2}(c_i) \cap \{y > c_i\}$; and
  \item all the $F_{c_i}^\tau$ are $C^0$-close to each other.
\end{itemize}

Then let $\tilde{F} = F_{c_i}$ on $U_{\delta/2}(c_i) \cap \{y \leq c_i\} - A_{\delta/2}(c_i)$ and $\tilde{F} = F_{c_i}^1$ on $U_{\delta/2}(c_i) \cap \{y > c_i\}$.

\textbf{Step 4.} A corollary of Lemma 2.0.2 is the following:

\textbf{Corollary 2.0.3.} With the assumptions of Lemma 2.0.2, there exists a diffeomorphism $h : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n + \phi(x_1, \ldots, x_n))$, and a holonomic section $\tilde{F} : Op(h(I^k)) \to J$ such that

1. $h = \text{id}$ and $\tilde{F} = F$ on $Op(\partial I^k)$; and
2. $|\tilde{F} - F|_{C^0} < \varepsilon$ on $Op(h(I^k))$.

\textbf{Proof.} There exists the desired diffeomorphism $h$ such that $h = \text{id}$ on $Op(\partial I^k)$ and such that $h(I^k) \subset \Omega$ (see Figure 3.6 of [EM]). Then $\tilde{F}$ given by Lemma 2.0.2 and restricted to $Op(h(I^k))$ satisfies Conditions (1) and (2).

Given $F : Op(I) \to J$ and the family $F_y : Op(y) \to J$, $y \in I$, of holonomic sections that agree with $F$ at $y$ and with $F$ on $Op(\partial I)$, Corollary 2.0.3 implies the existence of a $C^0$-close holonomic section $\tilde{F} : Op(h(I)) \to J$.

Applying the argument parametrically, given $F : Op(I^2) \to J$, we obtain a family $F_y : Op(h_1(y \times I)) \to J$ of holonomic sections and then $\tilde{F} : Op(h_2(I^2)) \to J$ holonomic, and so on.