NOTES FOR MATH 234: CONTACT GEOMETRY

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1. Holonomic approximations

For more details, refer to Eliashberg-Mishachev [EM], "Introduction to the h-principle". All manifolds we consider are smooth manifolds.

1.1. **Definitions.**

Given a subset A of a manifold V, we write $Op(A)$ for an arbitrarily small but non-specified open neighborhood of A in V.

Definition 1.1.1. Given a fiber bundle $X \to V$, its *r-jet bundle* $J^r(X) \to V$ (also written as $X^{(r)}$) is the bundle whose fiber over $p \in V$ is

$$J^r_p(X) = \{\text{sections of } X \text{ over } Op(p)\}/\sim,$$

where $f : U_1 \ni p \to X$ and $g : U_2 \ni p \to X$ satisfy $f \sim g$ if f and g have the same rth order Taylor expansion at p.

Note that $J^0(X) = X$. Let $p_{r,r'} : J^r(X) \to J^{r'}(X)$ be the projection for $r' \leq r$.
If $X = V \times W$, then we write $J^r(V, W) = J^r(X)$.

Example 1.1.2. $J^1(V, \mathbb{R}) = \mathbb{R} \times T^*V$.

In local coordinates, $J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qd_1} \times \cdots \times \mathbb{R}^{qd_r}$, where d_i is number of partial derivatives D^α of order i of a function $\mathbb{R}^n \to \mathbb{R}$. Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^q$ and $x \in \mathbb{R}^n$,

$$J^r_f(x) = (f(x), f'(x), \dots, f^{(r)}(x)) \in J^r(\mathbb{R}^n, \mathbb{R}^q)$$

is the r-jet of f at x. Here $f^{(i)} = (D^\alpha f)_\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_1 + \cdots + \alpha_n = i$.

Definition 1.1.3. A section $F : V \to J^r(X)$ is *holonomic* if $F = J^r_{p_{r,0}F}$.

1.2. **Holonomic approximations.**

Question 1.2.1. Can any section of $J^r(X) \to V$ be C^0-approximated by a holonomic section?

Example 1.2.2. Consider $F(x) = (x, x, 0)$ for $J^1(\mathbb{R}, \mathbb{R}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. It's impossible to find $f(x)$ such that $|f(x) - x| < \varepsilon$ and $|f'(x)| < \varepsilon$. (A picture would help here....)

In what follows we assume we have chosen an auxiliary Riemannian metric on the relevant spaces whenever we refer to lengths.

Theorem 1.2.3 (Holonomic approximation)**.** *Let $K \subset V$ be a polyhedron (i.e., submanifold with higher-dimensional corners) of positive codimension and $F : Op(K) \to X^{(r)}$ a section. Then for all $\delta, \varepsilon > 0$ small there exists a δ-small diffeotopy (in the C^0-sense)*

$$h^\tau : V \xrightarrow{\sim} V, \quad \tau \in [0, 1],$$

(i.e., a 1-parameter family of diffeomorphisms of V with $h^0 = \mathrm{id}$) and a holonomic section

$$\widetilde{F} : Op(h^1(K)) \to X^{(r)}$$

such that $\mathrm{dist}(\widetilde{F}(v), F(v)) < \varepsilon$ *for all* $v \in Op(h^1(K))$. *[Here we are choosing* $\delta > 0$ *so that* $Op(h^1(K)) \subset Op(K)$*.]*

There is also a parametric/relative version:

Theorem 1.2.4 (Parametric holonomic approximation)**.** *Let $K \subset V$ be a polyhedron of codimension ≥ 1. Let I^m be an m-dimensional cube and let $F_z : Op(K) \to X^{(r)}$ be a family of sections smoothly parametrized by $z \in Op(I^m)$ such that F_z is holonomic for $z \in Op(\partial I^m)$. Then for all $\delta, \varepsilon > 0$ small there exists a family of δ-small diffeotopies h_z^τ, $\tau \in [0,1]$, and a family of holonomic sections $\widetilde{F}_z$ such that:*

(1) $h_z^\tau = \mathrm{id}$ *and* $\widetilde{F}_z = F_z$ *for all* $z \in Op(\partial I^m)$ *and*
(2) $\mathrm{dist}(\widetilde{F}_z(v), F_z(v)) < \varepsilon$ *for all* $v \in Op(h_z^1(K))$ *and* $z \in I^m$.

2. Proof of Holonomic Approximation Theorem

We explain the main idea of the proof. We stress that the basic idea of the proof is simple and pretty but the actual proof is notation-heavy and slightly unpleasant to read.

Step 1. Let us write $J = J^r(\mathbb{R}^n, \mathbb{R}^q)$. We first reduce to proving the following relative theorem on a cube:

Theorem 2.0.1 (Holonomic approximation on the cube)**.** *Let $I^k = [-1,1]^k \subset \mathbb{R}^k \subset \mathbb{R}^n$, $k < n$, be the cube corresponding to the first k coordinates. For any section $F : Op(I^k) \to J$ which is holonomic over $Op(\partial I^k)$, there exists a diffeomorphism*

$$h : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n, \qquad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, x_n + \phi(x_1, \dots, x_n)),$$

and a holonomic section $\widetilde{F} : Op(h(I^k)) \to J$ such that

1. *$h = \mathrm{id}$ and $\widetilde{F} = F$ on $Op(\partial I^k)$; and*
2. *$|\widetilde{F} - F|_{C^0} < \varepsilon$ on $Op(h(I^k))$.*

HW: show that Theorem 2.0.1 implies Theorem 1.2.3.

Step 2. The starting point of the induction is to observe that given $F : Op(I) \to J$ there exists a family $F_y : Op(y) \to J$, $y \in I$, of holonomic sections that agree with F at y and with F on $Op(\partial I)$.

Step 3. We introduce some notation. Let $\pi_{k-l} : I^k \to I^{k-l}$ be the projection to the first $k-l$ coordinates with $0 \leq l < k$. The fibers are $y \times I^l$ where $y \in I^{k-l}$. Given $K \subset \mathbb{R}^n$, let $N_\delta(K)$ be its cubical δ-neighborhood in $\mathbb{R}^n$, i.e.,

$$N_\delta(K) = \cup_{(x_1,\dots,x_n)\in K} [x_1 - \delta, x_1 + \delta] \times \cdots \times [x_n - \delta, x_n + \delta].$$

We then set

$$U_\delta(y) = N_\delta(y \times I^l), \quad V_\delta(y) = N_\delta(y \times \partial I^l),$$
$$A_\delta(y) = (U_{\theta\delta}(y) - V_\delta(y)) \cap (y \times \mathbb{R}^{n-(k-l)}),$$

where $0 < \theta < 1$ and the neighborhoods are all taken in $\mathbb{R}^n$. A picture would be good here (cf. Figures 3.2 and 3.3 of [EM]).

We state the following inductive lemma:

Lemma 2.0.2 (Inductive lemma)**.** *With $I^k \subset \mathbb{R}^n$ and F as in Theorem 2.0.1, suppose F is* fiberwise holonomic with respect to *$\pi_{k-l} : I^k \to I^{k-l}$ for $l = k-1$, i.e., given $\delta > 0$ small, for each $y \in I = I^1$ there exist a cubical δ-neighborhood $U_\delta(y)$ of $y \times I^l$ and a family $F_y : U_\delta(y) \to J$ of holonomic sections such that $F_y = F$ on $(y \times I^l) \cup V_\delta(y)$ and on $y \in Op(\partial I)$. Then for $\varepsilon > 0$ there exists a large integer $N > 0$ and a holonomic section*

$$\widetilde{F} : \Omega \to J \quad \text{over} \quad \Omega = Op(\cup_{i=-N+1}^{N} A_\delta(c_i) \cup I^k) - \cup_{i=-N+1}^{N} A_\delta(c_i),$$

where $c_i = \frac{2i-1}{2N}$, $i = -N+1, \dots, N$, and

1. *$\widetilde{F} = F$ on $\Omega \cap Op(\partial I^k)$;*
2. *$|\widetilde{F} - F|_{C^0} < \varepsilon$ on Ω.*

Draw picture of Ω for $n = 2$, $k = 1$, $l = 0$ (cf. Figure 3.4 of [EM]).

Proof. For sufficiently large $N > 0$, the holonomic family $F_y : U_\delta(y) \to J$, $y \in I$, exists for $\delta = 1/N$. Define $F^\tau_{c_i} : U_{\delta/2}(c_i) \cap \{y > c_i\} \to J$, $\tau \in [0,1]$, such that

- $F^0_{c_i} = F_{c_i}$,
- $F^\tau_{c_i} = F_{c_i}$ on $V_{\delta/2}(c_i) \cap \{y > c_i\}$ for all $\tau \in [0,1]$;
- $F^1_{c_i} = F_{c_{i+1}}$ on $N_\delta(I^k) \cap U_{\delta/2}(c_i) \cap \{y > c_i\}$; and
- all the $F^\tau_{c_i}$ are C^0-close to each other.

Then let $\widetilde{F} = F_{c_i}$ on $U_{\delta/2}(c_i) \cap \{y \leq c_i\} - A_{\delta/2}(c_i)$ and $\widetilde{F} = F^1_{c_i}$ on $U_{\delta/2}(c_i) \cap \{y > c_i\}$. □

Step 4. A corollary of Lemma 2.0.2 is the following:

Corollary 2.0.3. *With the assumptions of Lemma 2.0.2, there exists a diffeomorphism*

$$h : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n, \qquad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, x_n + \phi(x_1, \dots, x_n)),$$

and a holonomic section $\widetilde{F} : Op(h(I^k)) \to J$ *such that*

(1) $h = \mathrm{id}$ *and* $\widetilde{F} = F$ *on* $Op(\partial I^k)$*; and*
(2) $|\widetilde{F} - F|_{C^0} < \varepsilon$ *on* $Op(h(I^k))$.

Proof. There exists the desired diffeomorphism h such that $h = \mathrm{id}$ on $Op(\partial I^k)$ and such that $h(I^k) \subset \Omega$ (see Figure 3.6 of [EM]). Then $\widetilde{F}$ given by Lemma 2.0.2 and restricted to $Op(h(I^k))$ satisfies Conditions (1) and (2). □

Given $F : Op(I) \to J$ and the family $F_y : Op(y) \to J$, $y \in I$, of holonomic sections that agree with F at y and with F on $Op(\partial I)$, Corollary 2.0.3 implies the existence of a C^0-close holonomic section $\widetilde{F} : Op(h(I)) \to J$.

Applying the argument parametrically, given $F : Op(I^2) \to J$, we obtain a family $F_y : Op(h_1(y \times I)) \to J$ of holonomic sections and then $\widetilde{F} : Op(h_2(I^2)) \to J$ holonomic, and so on.

3. Tangential homotopies

3.1. **Tangential homotopies.** Let $\pi : Gr_n(W) \to W$ be the Grassmannian n-plane bundle tangent to a q-manifold W, where $n < q$, i.e., $\pi^{-1}(y) = Gr_n(T_yW)$.

Given an n-manifold V and a fiberwise injective bundle map $F : TV \to TW$ covering $f : V \to W$, let $GF : V \to Gr_nW$ be the map $x \mapsto (f(x), F(T_xV))$. In particular, if $f : V \to W$ is an immersion, there exists a corresponding map $Gdf : V \to Gr_nW$.

Assume $V \subset W$ is an embedded submanifold and $f_0 : V \to W$ the inclusion map.

Definition 3.1.1. A *tangential homotopy* of $f_0 : V \to W$ is a homotopy $G_t : V \to Gr_nW$, $t \in [0, 1]$, such that $G_0 = Gdf_0$ and $\pi \circ G_t = f_0$.

Theorem 3.1.2 (Approximate integration of tangential homotopies)**.** *Let $K \subset V$ be a polyhedron of positive codimension and $G_t : V \to Gr_n(W)$ a tangential homotopy. Then there exists an approximation of G_t near K by an isotopy of embeddings, i.e., for all $\delta, \varepsilon > 0$ small there exists a δ-small diffeotopy*

$$h^\tau : V \xrightarrow{\sim} V, \quad \tau \in [0, 1],$$

and an isotopy

$$\widetilde{f}_t : Op(h^1(K)) \to W, \quad \widetilde{f}_0 = f_0|_{Op(h^1(K))},$$

such that $Gd\widetilde{f}_t : Op(h^1(K)) \to Gr_n(W)$ is ε-close to the tangential homotopy $G_t|_{Op(h^1(K))}$.

Proof. Note that the theorem is basically the same statement as the holonomic approximation theorem with jet spaces replaced by Gr_nW. Hence it's natural to reduce it to Theorem 1.2.3.

Assume G_t, $t \in [0, 1]$, is small, i.e., the angle between $G_{t_1}(w)$ and $G_{t_2}(w)$ is less than $\frac{\pi}{4}$ for all $w \in Op(f_0(K))$ and $t_1, t_2 \in [0, 1]$. Let X be a tubular neighborhood of V in W, which we view as a normal bundle $X \to V$. Then the space $X^{(1)}$ of 1-jets of sections $V \to X$ can be viewed as the space of n-planes that are tangent to X and transverse to the fibers of $X \to V$. Hence the tangential homotopy $G_t : V \to Gr_nW$ can be viewed as a homotopy of sections $F_t : V \to X^{(1)}$. We can now apply Theorem 1.2.3.

If G_t, $t \in [0, 1]$, is not small, then we subdivide into small intervals on which G_t is small and successively apply Theorem 1.2.3. Note that at each iteration of Theorem 1.2.3, the "wiggles" are one order of magnitude small than the previous ones. □

3.2. **Directed embeddings.**

Definition 3.2.1. Let $A \subset Gr_n(W)$ be a subset. An immersion $f : V \to W$ is A-*directed* if $Gdf(V) \subset A$.

Theorem 3.2.2 (A-directed embeddings for open manifolds)**.** *Let V be an open manifold, $A \subset Gr_n(W)$ an open subset, and $f_0 : V \to W$ an embedding whose lift $G_0 = Gdf_0$ is tangentially homotopic to $G_1 : V \to Gr_n(W)$ with $G_1(V) \subset A$. Then f_0 can be isotoped to an A-directed embedding $f_1 : V \to W$.*

Proof. We use the following well-known fact (give proof for HW) for open manifolds V:

Fact: Given an open manifold V, there exists a polyhedron $K \subset V$ of codimension at least 1 such that for an arbitrarily small neighborhood $N(K)$ there is an isotopy $\phi_t : V \xrightarrow{\sim} V$, $t \in [0, 1]$, $\phi_0 = \mathrm{id}$, which takes V to $N(K)$ while fixing K pointwise.

The theorem follows immediately from (i) the Fact, (ii) Theorem 3.1.2, and (iii) the fact that A is open. □

We can say slightly more:

Corollary 3.2.3. *In Theorem 3.2.2, suppose that the tangential homotopy* G_t *is induced by a homotopy* $F_t : TV \to TW$, $t \in [0,1]$, *of injective bundle maps covering* f_0 *with* $F_0 = df_0$. *Then we may assume that* F_1 *is homotopic to* df_1 *through a homotopy* $\widetilde{F}_t : TV \to TW$, $t \in [0,1]$, *which covers* f_t *and such that* $G\widetilde{F}_t(V) \subset A$ *for all* t.

4. Differential relations and h-principles

4.1. Definitions.

Definition 4.1.1. A *differential relation* $\mathcal{R}$ is a subset of $X^{(r)}$. A *formal solution to* $\mathcal{R}$ is a section $V \to \mathcal{R}$ and a *genuine solution to* $\mathcal{R}$ is a holonomic section $V \to \mathcal{R}$.

Example 4.1.2.

(1) $\mathcal{R}_{\mathrm{imm}} \subset J^1(V, W)$ consists of (v, w, ϕ) such that $(v, w) \in V \times W$ and $\phi : T_v V \to T_w W$ is injective. Note that $\mathcal{R}_{\mathrm{imm}}$ is an open relation (i.e., is an open subset).

(2) Let (V, ω_V) and (W, ω_W) be symplectic manifolds. $\mathcal{R}_{\mathrm{isosymp}}$ (for "isosymplectic") is the subset of $\mathcal{R}_{\mathrm{imm}}$ consisting of (v, w, ϕ) such that $\phi^* \omega_W = \omega_V$ at v.

(3) Let (V, ξ_V) and (W, ξ_W) be contact manifolds. $\mathcal{R}_{\mathrm{isocont}}$ (for "isocontact") is the subset of $\mathcal{R}_{\mathrm{imm}}$ consisting of (v, w, ϕ) such that $\phi^{-1}(\xi_W) = \xi_V$ at v and $\phi : \xi_V(v) \to \xi_W(w)$ is conformally symplectic with respect to the conformally symplectic structures on ξ_V and ξ_W.

Definition 4.1.3 (Types of h-principles)**.**

(1) The *h-principle holds for* $\mathcal{R}$ if a formal solution of $\mathcal{R}$ is homotopic in $\operatorname{Sec}\mathcal{R}$ (the sections of $\mathcal{R}$) to a genuine solution of $\mathcal{R}$. We write $\operatorname{Hol}\mathcal{R}$ for the set of genuine solutions of $\mathcal{R}$.

(2) The *parametric h-principle holds for* $\mathcal{R}$ if the inclusion $\operatorname{Hol}\mathcal{R} \to \operatorname{Sec}\mathcal{R}$ is a weak homotopy equivalence. In other words, for every $\phi_0 : (D^k, S^{k-1}) \to (\operatorname{Sec}\mathcal{R}, \operatorname{Hol}(\mathcal{R}))$, $k = 0, 1, \dots$, there is a homotopy $\phi_t : (D^k, S^{k-1}) \to (\operatorname{Sec}\mathcal{R}, \operatorname{Hol}\mathcal{R})$, $t \in [0,1]$, such that $\phi_1(D^k) \subset \operatorname{Hol}\mathcal{R}$.

(3) The *local h-principle holds for $\mathcal{R}$ near $A \subset V$* if the h-principle holds for V replaced by $Op(A)$.

(4) The *C^0-dense h-principle holds for* $\mathcal{R}$ if the usual h-principle holds for $\mathcal{R}$ and, for any formal solution $F_0 : V \to \mathcal{R}$ and an arbitrarily small neighborhood $N(f_0(V))$ of the underlying section $f_0 : V \to X$, the underlying section f_t of the homotopy F_t from F_0 to F_1 genuine can be chosen such that $f_t(V) \subset N(f_0(V))$, $t \in [0,1]$.

4.2. Basic version of h-principle.

Let $p : X \to V$ be a fiber bundle and let $\operatorname{Diff}_V(X)$ be the group of diffeomorphisms of X that send fibers to fibers. Each $h_X \in \operatorname{Diff}_V(X)$ covers $h_V \in \operatorname{Diff}(V)$ and there is a homomorphism $\pi : \operatorname{Diff}_V(X) \to \operatorname{Diff}(V)$ sending $h_X \mapsto h_V$.

The fiber bundle $p : X \to V$ is *natural* if there is a homomorphism $j : \operatorname{Diff}(V) \to \operatorname{Diff}_V(X)$ going the other way such that $\pi \circ j = \mathrm{id}$. This also induces $j : \operatorname{Diff}(V) \to \operatorname{Diff}_V(X^{(r)})$. [For example, when $X = V \times W$, then we can take $j(h_V)(v, w) = (h_V(v), w)$.] Given $h \in \operatorname{Diff}(V)$, we write $h_* = j(h)$ for any $X^{(r)}$.

We say $\mathcal{R}$ is $\operatorname{Diff}(V)$-*invariant* if $h_*(\mathcal{R}) = \mathcal{R}$ for all $h \in \operatorname{Diff}(V)$.

Theorem 4.2.1 (Local h-principle for $\operatorname{Diff}(V)$-invariant $\mathcal{R}$)**.** *Let $X \to V$ be a natural fiber bundle and $\mathcal{R} \subset X^{(r)}$ be an open $\operatorname{Diff}(V)$-invariant differential relation. Then all forms of the local h-principle hold near any polyhedron $K \subset V$ of positive codimension.*

Proof. We will explain the non-parametric case, i.e., show that given $F \in \operatorname{Sec}\mathcal{R}|_{Op(K)}$ there exists $G \in \operatorname{Hol}\mathcal{R}|_{Op(K)}$ homotopic to F in $\operatorname{Sec}\mathcal{R}|_{Op(K)}$. By the holonomic approximation theorem and the openness of $\mathcal{R}$, there is a C^0-small diffeotopy $h^\tau : V \xrightarrow{\sim} V$, $\tau \in [0,1]$, and a section $\widetilde{F}^1 \in \operatorname{Hol}\mathcal{R}|_{Op(h^1(K))}$ that is C^0-close to $\widetilde{F}^0 := F|_{Op(h^1(K))}$. We linearly interpolate between $\widetilde{F}^0$ and $\widetilde{F}^1$ to obtain $\widetilde{F}^t$, $t \in [0,1]$, which lies in $\operatorname{Sec}\mathcal{R}|_{Op(h^1(K))}$.

The homotopy from F to $G = (h^1)^{-1}_* \widetilde{F}^1$ in $\operatorname{Sec}\mathcal{R}|_{Op(K)}$ is the concatenation of the following:

- $(h^\tau)_*^{-1}F$, $\tau \in [0,1]$, from F to $(h^1)_*^{-1}F$, restricted to $Op(K)$;
- $(h^1)_*^{-1}\widetilde{F}^t$, $t \in [0,1]$, from $(h^1)_*^{-1}\widetilde{F}^0$ to $G = (h^1)_*^{-1}\widetilde{F}^1$, restricted to $Op(K)$.

The $\mathrm{Diff}(V)$-invariance of $\mathcal{R}$ implies that G is holonomic over $Op(K)$. □

Theorem 4.2.1 in turn implies:

Theorem 4.2.2 (Gromov)**.** *If V is open and $X \to V$ is natural, then an open $\mathrm{Diff}(V)$-invariant $\mathcal{R}$ satisfies the parametric h-principle.*

4.3. **Smale-Hirsch h-principle.** We now explain the *microextension trick*, which can upgrade to the case where V is closed manifold.

Theorem 4.3.1 (Hirsch)**.** *The C^0-dense h-principle holds for immersions of an n-manifold V into a q-manifold W, $n < q$.*

Proof. We treat the non-parametric case. The differential relation we use is $\mathcal{R}_{\mathrm{imm}} \subset J^1(V,W)$. Let F be a formal solution to $\mathcal{R}$ which covers $f : V \to W$. Let $N \to V$ be the "formal normal bundle" to TV, viewed as a subbundle of f^*TW under F. We can then lift $F : TV \to TW$ to $\widetilde{F} : TN \to TW$. Since TN is an open manifold which is effectively of dimension $< q$, we can apply Theorem 4.2.1 to obtain the theorem. □

4.4. **Strengthening of Theorems 4.2.1 and 4.2.2.** Let $\mathcal{R} \subset X^{(r)}$ be a differential relation. There are two ways of generalizing Theorems 4.2.1 and 4.2.2:

(A) Replace $\mathcal{R}$ open by $\mathcal{R}$ locally integrable and microflexible.
(B) Replace $\mathrm{Diff}(V)$ by a capacious subgroup G.

Theorem 4.4.1. *Theorem 4.2.1 holds with (A) and (B).*

Definition 4.4.2. $\mathcal{R}$ is *locally integrable* if for any $v \in V$ and any section $F : \{v\} \to \mathcal{R}$ there exist a holonomic extension $\widetilde{F} : Op(\{v\}) \to \mathcal{R}$, i.e., $\widetilde{F}(v) = F(v)$.

The following definition is my interpretation of microflexibility in [EM, Section 13.3] (which probably has typos and does not make sense as stated). Let $I = [-1,1]$ as before and view

$$I^n = I^{k-l} \times I^l \times I^{n-k} = I^k \times I^{n-k},$$

where $l < k < n$. Consider the *standard triple*

$$(I^n, I^{k-l} \times \partial I^l \times I^{n-k}, I^k \times \{0\}).$$

A picture would be helpful here. A triple $(A,B,C) \subset V$, where $\dim V = n$, is a *θ-triple* if it is diffeomorphic to a standard triple. [Here $B, C \subset A$ but $C \not\subset B$.]

Definition 4.4.3. $\mathcal{R}$ is *k-microflexible* if for any sufficiently small θ-triple and

(1) holonomic section $F^0 : Op(A) \to \mathcal{R}$ and
(2) homotopy $F^\tau : Op(C) \to \mathcal{R}$, $\tau \in [0,1]$, of holonomic sections that extend $F^0|_{Op(C)}$ and are constant on $Op(B)$,

there exist $\sigma \in (0,1]$ and a homotopy $F^\tau : Op(A) \to \mathcal{R}$, $\tau \in [0,\sigma]$, of holonomic sections that are constant over $Op(B)$ and extend F^τ on $Op(C)$. $\mathcal{R}$ is *microflexible* if it is k-microflexible for all $k = 0, \dots, n-1$ for $\dim V = n$.

Example 4.4.4.

(1) Open differential relations are microflexible.
(2) $\mathcal{R}_{\rm isocont}$ is microflexible.
(3) $\mathcal{R}_{\rm isosymp}$ is microflexible for $k \neq 1$.

We have the following strengthening of Theorem 1.2.3 where we replace $X^{(r)}$ by locally integrable and microflexible $\mathcal{R}$. The proof is still the same.

Theorem 4.4.5 (Holonomic $\mathcal{R}$-approximation)**.** *Let $\mathcal{R} \subset X^{(r)}$ be a locally integrable and microflexible differential relation. Let $K \subset V$ be a polyhedron of positive codimension and $F : Op(K) \to \mathcal{R}$ a section. Then for all $\delta, \varepsilon > 0$ small there exists a δ-small diffeotopy (in the C^0-sense) $h^\tau : V \xrightarrow{\sim} V$, $\tau \in [0,1]$, and a holonomic section*

$$\widetilde{F} : Op(h^1(K)) \to \mathcal{R}$$

such that $\operatorname{dist}(\widetilde{F}(v), F(v)) < \varepsilon$ *for all* $v \in Op(h^1(K))$.

Definition 4.4.6. Let G be a Lie subgroup of the group of compactly supported diffeomorphisms of V and $\mathfrak{g}$ be its Lie algebra of vector fields. G and $\mathfrak{g}$ are *capacious* if:

(1) for any $v \in \mathfrak{g}$, any compact subset $A \subset V$, and its neighborhood $U \supset A$, there exists a vector field $\tilde{v} \in \mathfrak{g}$ such that $\tilde{v} = v$ on A and $\operatorname{Supp}(\tilde{v}) \subset U$.
(2) for any $x \in V$ and any tangent hyperplane $\xi \subset T_xV$, there is a vector field $v \in \mathfrak{g}$ transverse to ξ.

Moreover, (1) and (2) are required to hold parametrically for any compact space of parameters.

Roughly speaking, a capacious group contains enough diffeomorphisms with small support.

Example 4.4.7. The identity component of the group of compactly supported contactomorphisms of (M^{2n+1}, ξ) contact and the group of compactly supported Hamiltonian diffeomorphisms of (M^{2n}, ω) symplectic are capacious. This is because the corresponding vector fields are (roughly) in bijection with the space of functions on M.

5. Examples of h-principles

5.1. **Contact structures on open manifolds.** We assume all contact structures are cooriented. Let V be a manifold of odd dimension. Let $\mathcal{S}$ be the space of almost contact structures on V, i.e., a pair (ξ,ω) consisting of a hyperplane distribution ξ on V and a conformal class ω of symplectic structures on ξ. Let $\mathbb{S}$ be the space of contact structures on V.

Theorem 5.1.1 (Gromov). *If V is open, then the inclusion $\mathbb{S}\to\mathcal{S}$ is a weak homotopy equivalence.*

Proof. The proof is a consequence of a much more general result.

Step 1. Using holonomic approximation, we obtain:

Lemma 5.1.2. *Given a polyhedron $K\subset V$ of positive codimension and a pair (α,ω) consisting of a $(p-1)$-form and a p-form on V, there is a C^0-small diffeotopy $h^\tau: V\stackrel{\sim}{\to} V$ and a $(p-1)$-form $\widetilde{\alpha}$ on V such that (α,ω) is C^0-close to $(\widetilde{\alpha},d\widetilde{\alpha})$ on $\widetilde{K}=h^1(K)$.*

Briefly, there is a bundle map $D:(\Lambda^{p-1}V)^{(1)}\to\Lambda^pV$ given by the symbol of the exterior derivative d. In local coordinates, suppose $\sum_I f_I dx_I$ is a $(p-1)$-form and let $(a_{I,j})$, where $j=1,\dots,n$ and $I=(i_1<i_2<\dots<i_{p-1})$, be the fiber coordinates on $(\Lambda^{p-1}V)^{(1)}$ corresponding to $\frac{\partial f_I}{\partial x_j}$. Then D takes $(a_{I,j})$ to $(\sum_{I'=I\cup\{j\}}\pm a_{I,j})$.

We can also view $(\Lambda^{p-1}V)^{(1)}$ as a bundle over $\Lambda^{p-1}V\oplus\Lambda^pV$ with affine fibers (by the definition of D). Hence any section $\omega:V\to\Lambda^pV$ can be lifted to $F_\omega:V\to(\Lambda^{p-1}V)^{(1)}$ so that $\omega=D\circ F_\omega$ and $p_{1,0}\circ F_\omega=\alpha$.

We then apply holonomic approximation to $X=\Lambda^{p-1}(V)$, $X^{(1)}=(\Lambda^{p-1}(V))^{(1)}$, and F_ω.

Step 2. Using Lemma 5.1.2 we obtain (as usual):

Lemma 5.1.3. *Let V be open and let $\mathcal{R}\subset\Lambda^{p-1}V\oplus\Lambda^pV$ be an open $\mathrm{Diff}(V)$-invariant subset. Let $\mathrm{Ex}\,\mathcal{R}\subset\mathrm{Sec}\,\mathcal{R}$ be the subspace of pairs (α,ω) such that $d\alpha=\omega$. Then the inclusion $\mathrm{Ex}\,\mathcal{R}\to\mathrm{Sec}\,\mathcal{R}$ is a homotopy equivalence.*

Step 3. Now apply Lemma 5.1.3 to the case where $\dim V=2n+1$, $\mathcal{R}\subset\Lambda^1V\oplus\Lambda^2V$ subject to $\alpha\wedge\omega^n\neq 0$. □

There is an analogous theorem for symplectic structures: Let $\mathcal{S}_{\mathrm{symp}}$ be the space of almost symplectic structures on V and let $\mathbb{S}^a_{\mathrm{symp}}$ be the space of symplectic structures on V in a fixed cohomology class $a\in H^2(V)$.

Theorem 5.1.4 (Gromov). *If V is open, then the inclusion $\mathbb{S}^a_{\mathrm{symp}}\to\mathcal{S}_{\mathrm{symp}}$ is a weak homotopy equivalence.*

5.2. **Contact and isocontact embeddings.** The proof of the following theorem uses the discussion of A-directed embeddings.

Theorem 5.2.1. *Let (V,ξ_V) and (W,ξ_W) be contact manifolds of dimension n and q, respectively. Suppose $n<q$ and V is open. If the differential $F_0=df_0$ of an embedding $f_0:(V,\xi_V)\to(W,\xi_W)$ is homotopic through injective bundle maps $F_t:TV\to TW$ covering f_0 to an isocontact (by definition injective) bundle map $T_1:TV\to TW$. Then there is an isotopy $f_t:V\to W$ such that f_1 is isocontact and df_1 is homotopic to F_1 through isocontact bundle maps.*

Proof. We consider the subset $A_{\text{cont}} \subset Gr_n W$ of n-planes P on which $\xi_W \cap P$ is $(n-1)$-dimensional and the (conformal) symplectic structure $CS(\xi_W)$ restricts to a (conformal) symplectic structure. Since A_{cont} is open, we can apply Theorem 3.2.2: View F_t as a tangential homotopy $G_t : V \to Gr_n W$ covering f_0. Then G_1 is A_{cont}-directed and Theorem 3.2.2 and Corollary 3.2.3 imply that f_0 can be isotoped to an A_{cont}-directed embedding $f_1 : V \to W$ and df_1 and F_1 are homotopic through injective bundle maps $\widetilde{F}_t : TV \to TW$ that cover f_t, $t \in [0,1]$ and are A_{cont}-directed.

Note that the above is already sufficient to show the isotopy to a *contact embedding*, i.e., an embedding where the contact structure of the domain is unspecified and the image of the map is a contact submanifold of (W, ξ_W).

It remains to "match up" ξ_V and $f_1^*\xi_W$: Let ξ_t, $t \in [0,1]$, be a family of contact structures connecting $f_1^*\xi_W$ to ξ_V. This exists by Theorem 5.1.1, since ξ_V and $f_1^*\xi_W$ are homotopic. Then one needs to show that if $g_0 : (V, \xi_0) \to (W, \xi_W)$ is an isocontact embedding, then there is a contact isotopy $g_t : V \to W$ such that $g_1 : (V, \xi_1) \to (W, \xi_W)$ is an isocontact embedding. The proof is a contact topology result (not really an h-principle result). It is nontrivial and is given in [EM, 12.3.2-12.3.5]. $\square$

5.3. **Legendrian and isocontact immersions.**

By the holonomic $\mathcal{R}$-approximation theorem (Theorem 4.4.5):

Theorem 5.3.1. *If (V, ξ_V) and (W, ξ_W) are contact and $A \subset V$ is a polyhedron of positive codimension, then all forms of the local h-principle hold for isocontact immersions $(Op(A), \xi_V|_{Op(A)}) \to (W, \xi_W)$.*

The following follow from Theorem 5.3.1, together with the microextension trick which reduces to the case of *equidimensional* isocontact immersions where the domain is open:

Corollary 5.3.2. *If $\dim V < \dim W$, then all forms of the h-principle hold for isocontact immersions $(V, \xi_V) \to (W, \xi_W)$.*

Corollary 5.3.3. *All forms of the h-principle hold for Legendrian immersions $V^n \to (W^{2n+1}, \xi_W)$.*

In particular, since a generic Legendrian immersion is a Legendrian embedding by dimension reasons, if $f : V \to W$ is an embedding and df is formally homotopic to a Legendrian embedding $TV \to TW$ on the bundle level, then f is homotopic (through immersions) to a Legendrian embedding $f' : V \to W$. Note that f and f' may not be isotopic as embeddings.

Also if L is open (i.e., the core of L is $\leq (n-1)$-dimensional), then by dimensional reasons

(1) $f : V \to W$ can be isotoped to a Legendrian embedding through embeddings and
(2) any two Legendrian embeddings $f_1, g_1 : V \to W$ that are formally homotopic as Legendrian embeddings are isotopic through Legendrian embeddings.

If we want to understand Legendrian submanifolds, it remains to understand what happens to the n-dimensional handles/disks.

6. CONVEX INTEGRATION

We now discuss the other important technique for proving h-principles: convex integration. Let us start with the following illustrative example (cf. [EM, Figure 17.1]).

Model example. Given a path $f : [0,1] \to \mathbb{R}^2$ with $|f'(t)| < 1$, can we find $\widetilde{f} : [0,1] \to \mathbb{R}^2$ which is C^0-close to f such that $|\widetilde{f}'(t)| = 1$? Consider the graph of the path f in $[0,1] \times \mathbb{R}^2$. Then the tangent line to f at (t_0, x_0, y_0) lies inside the "light cone" $(t-t_0)^2 > (x-x_0)^2 + (y-y_0)^2$. We want to modify to $\widetilde{f}$ such that the tangent line lies on the light cone $(t-t_0)^2 = (x-x_0)^2 + (y-y_0)^2$. This is easily done by "spiraling around" the graph of the original function f. HW: prove this precisely.

Although we will not prove it in class, convex integration can be used to prove the following:

Theorem 6.0.1 (Nash-Kuiper)**.** *Given an immersion $f : (V^n, g) \to (\mathbb{R}^q, g_{std})$, $n < q$, such that $|df(v)| < |v|$ for all tangent vectors v, there exists a C^1-map $\widetilde{f} : V^n \to \mathbb{R}^q$ which is C^0-close to f and satisfies $|d\widetilde{f}(v)| = |v|$ for all v.*

For today assume that $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ is an open differential relation.

Recalling the projection $p_{1,0} : J^1(\mathbb{R}^n, \mathbb{R}^q) \to J^0(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q$, let us write $\mathcal{R}_{t,y} = \mathcal{R} \cap p_{1,0}^{-1}(t,y)$, i.e., the fiber over (t,y). Given a section $F : \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R}^q)$, we write it as $F = (f, \phi)$, where f is the map from $\mathbb{R}^n$ to $J^0(\mathbb{R}^n, \mathbb{R}^q)$ and ϕ is the map from $\mathbb{R}^n$ to the fiber $\mathbb{R}^{nq}$.

6.1. **One-dimensional convex integration.** Let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$.

Definition 6.1.1. Given a subset $\Omega \subset \mathbb{R}^q$ and $y \in \Omega$, let $\operatorname{Conn}_y \Omega$ be the path component of Ω containing y. We say (i) Ω is *ample* if for any $y \in \Omega$ the convex hull of $\operatorname{Conn}_y \Omega$ is all of $\mathbb{R}^q$ and (ii) $\mathcal{R}$ is *ample* if $\mathcal{R}_{t,y}$ is ample for every $(t,y) \in \mathbb{R} \times \mathbb{R}^q$.

Remark 6.1.2. If $\Omega = \varnothing \subset \mathbb{R}^q$, then it is ample by definition.

Definition 6.1.3. A formal solution $F = (f, \phi)$ of $\mathcal{R}$ is *short* if f is a genuine solution of

$$\operatorname{Conv}_F \mathcal{R} = \cup_{t \in \mathbb{R}} \text{Convex hull of } \operatorname{Conn}_{F(t)} \mathcal{R}_{f(t)}.$$

Note that if $\mathcal{R}$ is ample, then every formal solution of $\mathcal{R}$ is short.

Lemma 6.1.4 (One-dimensional convex integration)**.** *Let $F = (f, \phi) : [0,1] \to \mathcal{R}$ be a short formal solution of $\mathcal{R}$. Then there is a family of short formal solutions $F_\tau = (f_\tau, \phi_\tau) : [0,1] \to \mathcal{R}$, $\tau \in [0,1]$, such that:*

1. *$F_0 = F$, F_1 is holonomic,*
2. *f_τ is arbitrarily C^0-close to f for all $\tau \in [0,1]$,*
3. *$F_\tau|_{\partial[0,1]}$ is constant if $F|_{\partial[0,1]}$ is holonomic.*

We write f° for the composition of f with the projection to $\mathbb{R}^q$.

Idea of proof. Subdivide the interval $[0,1]$ into sufficiently small intervals and construct the homotopy over each small interval. Over each small interval, $f^\circ(t)$ and $\phi(t)$ are almost constant. Since we are doing "integration" (i.e., something analogous to Riemann sums and taking a limit as interval sizes go to zero), might as well assume that $f^\circ(t)$ and $\phi(t)$ are constant. Since F is short, $\phi(t)$ is in the convex hull of some $A := \{a_1, \dots, a_k\}$, where a_i are in the same path component of $\mathcal{R}_{t,f(t)}$. We then approximate f in C^0 by a piecewise linear function whose derivatives are in A. For more details see [EM, Section 17.3]. □

We also have the following parametric version:

Lemma 6.1.5 (Parametric one-dimensional convex integration)**.** *Let $\mathcal{R}(p) \subset J^1(\mathbb{R}, \mathbb{R}^q)$ be a family of open differential relations parametrized by $p \in [0,1]^\ell$ and let $F(p,t) = (f(p,t), \phi(p,t)) : [0,1] \to \mathcal{R}(p)$ be a short formal solution smoothly parametrized by $p \in [0,1]^\ell$. Then there is a family of short formal solutions*

$$F_\tau(p) = (f_\tau(p), \phi_\tau(p)) : [0,1] \to \mathcal{R}(p), \quad \tau \in [0,1],$$

parametrized by p, which satisfies (1)–(3) of Lemma 6.1.4 with the parameter p, as well as:

(4) $|\frac{\partial f_1}{\partial p_i} - \frac{\partial f}{\partial p_i}| < \varepsilon$ *for* $i = 1, \dots, \ell$.

6.2. **Iterated convex integration.** We now consider $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$. Writing

$$J^1(\mathbb{R}^n, \mathbb{R}^q) = \oplus_{i=1}^n J^1_{x_i}(\mathbb{R}^n, \mathbb{R}^q),$$

where $J^1_{x_i}$ refers to the components corresponding to the derivative in the x_i-direction, we say that $\mathcal{R}$ is *ample in the coordinate directions* if the intersection of $\mathcal{R}$ with every fiber of

$$\Pi_k : \oplus_{i=1}^n J^1_{x_i}(\mathbb{R}^n, \mathbb{R}^q) \to \oplus_{i=1, i\neq k}^n J^1_{x_i}(\mathbb{R}^n, \mathbb{R}^q)$$

is ample for every k.

Example 6.2.1. Verify the following/give more details for HW:

(1) The immersion relation $\mathcal{R}_{\text{imm}} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ is ample in the coordinate directions if and only if $n < q$. (Note that $n = q$ is not ample in the coordinate directions.) Brief explanation: Let $(t, x, v_1, \dots, \widehat{v}_k, \dots, v_n)$ be an element of $\oplus_{i\neq k} J^1_{x_i}(\mathbb{R}^n, \mathbb{R}^q)$, where $(t,x) \in J^0(\mathbb{R}^n, \mathbb{R}^q)$. We consider two cases: (i) $v_1 \wedge \dots \wedge \widehat{v}_k \wedge \dots \wedge v_n \neq 0$ and (ii) $v_1 \wedge \dots \wedge \widehat{v}_k \wedge \dots \wedge v_n = 0$. For (ii), the intersection $\mathcal{R} \cap \Pi_k^{-1}(t, x, v_1, \dots, \widehat{v}_k, \dots, v_n) = \varnothing$, and the ampleness is automatically satisfied. For (i), $\mathcal{R} \cap \Pi_k^{-1}(t, x, v_1, \dots, \widehat{v}_k, \dots, v_n)$ is the set of v_k such that $v_1 \cdots \wedge \dots \wedge v_n \neq 0$, which is ample if and only if $n < q$. (When $n = q$, then the intersection has two components, the upper and lower half planes, and neither component has the entire plane as convex hull.)

(2) The submersion relation $\mathcal{R}_{\text{sub}} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ is not ample in the coordinate directions for $n \geq q$.

Lemma 6.2.2 (Iterated convex integration)**.** *If $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ is open and ample in the coordinate directions, then the same conclusion holds as before.*

Sketch of proof. Let $\phi = (\phi^1, \dots, \phi^n)$. We integrate $F = (f, \phi)$ one coordinate at a time using Lemma 6.1.5.

Step 1. First view $[0,1]^n$ as a family of intervals $[0,1] \times \{p\}$, $p \in [0,1]^{n-1}$ and apply Lemma 6.1.5 parametrically to $F(p)$. This yields the formal solution

$$F^1 = (f^1, \frac{\partial f^1}{\partial x_1}, \phi^2, \dots, \phi^n)$$

of $\mathcal{R}$ which is homotopic to F in $\mathcal{R}$, such that $|f^1 - f^0|_{C^0} < \varepsilon$.

Step 2. Next view $[0,1]^n$ as a family of intervals parallel to the x_2-axis and apply Lemma 6.1.5 to construct the formal solution

$$F^2 = (f^2, \frac{\partial f^1}{\partial x_1}, \frac{\partial f^2}{\partial x_2}, \phi^3, \dots, \phi^n).$$

By Lemma 6.1.5(4), $\frac{\partial f^1}{\partial x_1}$ and $\frac{\partial f^2}{\partial x_1}$ are C^0-close and hence we can deform F^2 to

$$\widetilde{F}^2 = (f^2, \frac{\partial f^2}{\partial x_1}, \frac{\partial f^2}{\partial x_2}, \phi^3, \dots, \phi^n),$$

and so on. $\square$

Leaving the definition of ampleness for $\mathcal{R} \subset X^{(1)}$ to your imagination, we have:

Theorem 6.2.3. *If $\mathcal{R} \subset X^{(1)}$ is an open ample differential relation, then all forms of the h-principle hold for $\mathcal{R}$.*

The theorem is an immediate consequence of Lemma 6.2.2.

Examples. We are considering $J^1(V, W)$ where W may have an auxiliary symplectic structure ω or almost complex structure J.

(1) $\mathcal{R}_{\mathrm{imm}}$ is open and ample for $n < q$.
(2) $\mathcal{R}_{\mathrm{Lag}} \subset \mathcal{R}_{\mathrm{imm}}$ (corresponding to Lagrangian n-planes in $Gr_n(W)$) is ample but closed.
(3) $\mathcal{R}_{\mathrm{Lag}^\varepsilon}$ with $\varepsilon > 0$ small (corresponding to n-planes in $Gr_n(W)$ that make an angle of at most $\varepsilon > 0$ with a Lagrangian n-plane) is ample and open.
(4) A subspace $S \subset \mathbb{C}^n$ is *totally real* if $S \cap JS = 0$. $\mathcal{R}_{\mathrm{Real}}$ (corresponding to totally real n-planes in $Gr_n(W)$) is ample and open.

Hence Theorem 6.2.3 applies to $\mathcal{R}_{\mathrm{imm}}$, $\mathcal{R}_{\mathrm{Lag}^\varepsilon}$, and $\mathcal{R}_{\mathrm{Real}}$.

7. Wrinkling

For more details, see Eliashberg-Mishachev [EM2], "Wrinkled embeddings". Let $\psi_\delta : \mathbb{R} \to \mathbb{R}^2$ be the map

$$z \mapsto (z^3 - 3\delta z, \int_0^z (z^2-\delta)^2 dz = \tfrac{z^5}{5} - \tfrac{2}{3}\delta z^3 + \delta^2 z).$$

The precise choice of constants in the definition of ψ_δ is not important, but the rough shape of the image of the function is, i.e., it is a zigzag for $\delta > 0$.

Definition 7.0.1. A smooth map $f : V^n \to W^m$, $n < m$, is a *wrinkled embedding* if

(1) f is a topological embedding;
(2) each connected component S_i of the singular set $\Sigma(f) = \{x \in V \mid df(x) \text{ is not injective}\}$ is diffeomorphic to the standard $(n-1)$-sphere S^{n-1} and bounds an n-disk $D^n \subset V$;
(3) the map $f|_{Op(S_i)}$ is equivalent to the map $Z(n,m) : Op_{\mathbb{R}^n} S^{n-1} \to \mathbb{R}^m$ given by:

$$(x_1, \dots, x_{n-1}, z) \mapsto (x_1, \dots, x_{n-1}, \psi_{1-|x|^2}(z), 0, \dots, 0).$$

Note that $f|_{D^n}$ may have singularities in the interior.

Let $S_i' \subset S_i$ be the equator given by $\{z = 0\}$ in the local model. Then $S_i = (S_i - S_i') \cup S_i'$, where $S_i - S_i'$ is a 2-fold cuspidal edge and S_i' is the set of 3-fold corners. We refer to S_i as the *wrinkles* and S_i' as the *unfurled swallowtail set*.

When $n = 2$, the prototypical model for f is a family $f_x : \mathbb{R} \to \mathbb{R}^3$, $x \in [-2,2]$, where $f_x(s) = (s, \gamma_x(s))$, we view γ_x as Legendrians in the front projection $\mathbb{R}^2$, $\gamma_{-2}(s) = \gamma_2(s) = (s,0)$, a zigzag is created at γ_{-1} and destroyed at γ_1. A picture would be good here....

Also, for families of wrinkled embeddings we allow *embryo singularities* given by the model

$$f_t(x,z) = (x, \psi_{t-|x|^2}(z)), \quad t \in [-1,1].$$

The singularity is at $t = 0$, $x = 0$, $z = 0$. It represents the creation or annihilation of a small wrinkle.

The main theorem of [EM2] is the following:

Theorem 7.0.2. *Let $G_t : V \to Gr_n W$ be a tangential homotopy covering an embedding $i : V \to W$ with $G_0 = Gdi$. Then there is a homotopy of wrinkled embeddings $f_t : V \to W$, $f_0 = i$, such that $Gdf_t : V \to Gr_n W$ is C^0-close to G_t.*

The parametric version of the theorem also holds.

We briefly describe the idea for $[-1,1] \to \mathbb{R}^2$ given by $x \mapsto (x,0)$. Consider the tangential homotopy $T[-1,1] \to T\mathbb{R}^2$ given by uniform counterclockwise rotation from an angle of 0 to an angle of $\frac{\pi}{4}$. Then by approximating using wrinkled embeddings such that the wrinkles are constructed at smaller and smaller scales as we progress in the tangential homotopy, we obtain the desired homotopy in Theorem 7.0.2.

8. Loose Legendrian knots

This material is based on Murphy [M], "Loose Legendrian embeddings in high-dimensional contact manifolds". Let (M^{2n+1},ξ) be a contact manifold.

For today we assume L is a connected manifold of dimension n.

Definition 8.0.1. A *formal Legendrian embedding* is a pair consisting of an embedding $f: L\to M$ and a homotopy of bundle monomorphisms $F_t: TL\to TM$, $t\in[0,1]$, covering f, such that $F_0=df$ and $F_1: TL\to\xi\subset TM$ is Legendrian.

For example, see Appendix A of [M] or Appendix B in the Cieliebak-Eliashberg book [CE] "From Stein to Weinstein and back" for (topological) invariants of formal Legendrian isotopy classes.

We define a class of Legendrian submanifolds that exhibit flexible properties. Recall the standard contact $(\mathbb{R}^{2n+1},\alpha=dz-\sum_{i=1}^n y_i dx_i)$. Its front projection is the projection to $\mathbb{R}^{n+1}$ with coordinates $(z,x_1,\dots,x_n)$. Let us write $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$, $x'=(x_2,\dots,x_n)$, $y'=(y_2,\dots,y_n)$.

Definition 8.0.2. A *loose Legendrian submanifold* $L\subset(M^{2n+1},\xi)$ is a Legendrian submanifold that admits a *loose chart*, i.e., there exists a neighborhood of M of the form $Q\times P$, where

$$Q=\{|x_1|,|y_1|,|z|\le\tfrac{1}{2}\}\subset\mathbb{R}^3_{z,x_1,y_1},\ P=\{|x'|\le\rho,|y'|\le\rho\}\subset\mathbb{R}^{2n-2}_{x',y'},$$

$L\cap(Q\times P)=L_0\times\{y'=0,|x'|\le\rho\}$, and we have a quantitative constraint $\rho>1$. Here $L_0\subset Q$ agrees with $y_1=z=0$ near the boundary and projects to a zigzag in the front projection $\mathbb{R}^2_{z,x_1}$.

Remark 8.0.3. Alternatively, we can use L_0' that "looks like a fish" instead of L_0. The quantitative constraint I think is also $\rho>1$ (check this).

Certain stabilization operations yield loose Legendrians:

(1) If we spin a zigzag in the front projection around (think of the zigzag as a geodesic emanating from the north pole in standard S^n and spinning around the north pole), the result is a loose Legendrian.
(2) In the front projection $\mathbb{R}^{n+1}_{z,x_1,\dots,x_n}$ suppose we have two sheets $z=0$ and $z=1$. On $|x|\le1$ we replace $z=0$ by $z=f(x)$, supported on $|x|\le1$, so that some portion of f is greater than 1. This stabilization operation does change the isotopy type of the embedding, but can be used in conjunction with a careful bookkeeping of crossing changes.

Theorem 8.0.4 (Murphy)**.** *Let $n\ge2$. For each formal Legendrian isotopy class $\mathcal{L}$ there exists a loose Legendrian in $\mathcal{L}$. Moreover, loose Legendrians in $\mathcal{L}$ are unique up to contact isotopy.*

The idea of the proof is summarized in the following progression:

$$\text{formal Legendrian}\overset{\text{convex}\int}{\longrightarrow}\varepsilon\text{-Legendrian}\longrightarrow\text{wrinkled Legendrian}\overset{\text{resolve sing.}}{\longrightarrow}\text{Legendrian}.$$

Step 1. Using convex integration, given a formal Legendrian embedding $(f: L\to M, F_t: TL\to TM)$, a C^0-small isotopy takes it to $f: L\to M$ (keeping the same name f) which is ε-close to a Legendrian for small $\varepsilon>0$. [Recall we discussed ε-Lagrangian embeddings; the ε-Legendrian case is analogous. Check this for HW.]

Step 2. Using this, we can cover $f(L)$ by sufficiently small Darboux charts $(\mathbb{R}^{2n+1}_{z,x,y},dz-\sum_i y_i dx_i)$ such that $f(L)$ projected to $\mathbb{R}^{n+1}_{z,x}$ (the front projection) is graphical over $\mathbb{R}^n_x$. We also subdivide L so that each cell is small and lies in a Darboux chart as above. Now we view $\mathbb{R}^{2n+1}=J^1(\mathbb{R}^n_x,\mathbb{R}_z)$ and note that a holonomic section of $J^1(\mathbb{R}^n, \mathbb{R})$ is a Legendrian submanifold with respect to $dz - \sum_i y_i dx_i$. Hence we can apply the usual holonomic approximation theorem to each i-cell where $i < n$ to make $f(L)$ Legendrian on the $(n-1)$-skeleton and each $f(D^n)$ graphical in a front projection. [It is clear that $f(\partial D^n)$ is graphical in some front projection, but it is crucial to show that $f(\partial D^n)$ is graphical in every front projection that it nontrivially intersects. It is also not a priori obvious since holonomic approximation is a C^0-small approximation result and wiggles the map a bit! For HW verify that if the subdivision of L is sufficiently small then $f(\partial D^n)$ is graphical in every chart.]

Now we apply Theorem 7.0.2 to $f(D^n) \subset J^1(\mathbb{R}^n_x, \mathbb{R}_z)$ to obtain a wrinkled embedding in the front projection $\mathbb{R}^{n+1}_{z,x}$ that is C^1-close to the front projection for $f(D^n)$ (and hence gives rise to a wrinkled Legendrian embedding). The important thing to observe is that the wrinkles are contained in Darboux charts.

Remark 8.0.5. Let $\mathcal{S}$ be the unfurled swallowtail set of a wrinkled Legendrian L. Then L does not have a well-defined tangent space at $\mathcal{S}$, so a priori we can't even talk about its formal homotopy class; there is some modification we can do to L inside the Darboux charts to make sense of the formal homotopy class.

Step 3. In order to recover a Legendrian from a wrinkled Legendrian, we keep track of *twist markings*.

Definition 8.0.6. A *twist marking* $\mathcal{M}$ on a wrinkled Legendrian L is an embedded $(S^{n-1}-\text{fin. many disks}) \subset L$ which bounds $\mathcal{S}$. We also require each component of $\mathcal{M}$ to lie in a Darboux chart.

By creating a family of zigzags along $\mathcal{M}$, we can resolve $\mathcal{S}$. This gives us a Legendrian, but we might still need to adjust its formal Legendrian isotopy class by applying stabilizations and using the discussion of formal Legendrian isotopy classes.

Step 4. Given two loose Legendrians L_0 and L_1 in the same formal homotopy class, they are isotopic through wrinkled Legendrians L_t. We would like to get L_t such that L_t is never smooth (i.e., always has wrinkles). To do this, we need to convert L_0 and L_1 right away to wrinkled Legendrians; this is where the loose condition plays a crucial role. The loose chart can be converted into an "inside out" wrinkle, which we assume persists for all L_t. The "inside out" wrinkle naturally comes with a small twist marking $\mathcal{M}_0$ (or $\mathcal{M}_1$). It then remains to find a family $\mathcal{M}_t$ of twist markings so that when we cross t_i with an embryonic singularity, $\mathcal{M}_{t_i+\varepsilon}$ is obtained from $\mathcal{M}_{t_i-\varepsilon}$ by removing a disk so that the newly created boundary of $\mathcal{M}_{t_i+\varepsilon}$ bounds the newly created unfurled swallowtail set, or vice versa. The family of genuine Legendrians obtained by resolving L_t along $\mathcal{M}_t$ are all Legendrian isotopic.

For HW, provide more details!

9. FLEXIBLE WEINSTEIN MANIFOLDS

For more details see Chapters 9 and 14 of [CE]. Unfortunately, we again need to introduce a lot of terminology before we get to the main point....

A *Morse cobordism* (W^{2n}, ϕ) is a compact manifold with boundary $\partial W = \partial_- W \cup \partial_+ W$ such that $\phi : W \to \mathbb{R}$ is a Morse function, $\partial_- W$ and $\partial_+ W$ are regular level sets of ϕ, and $\phi(\partial_- W) < \phi(\partial_+ W)$.

A *Weinstein cobordism* (W^{2n}, β, ϕ) is a Morse cobordism (W, ϕ) such that:

(1) the Liouville vector field given by $i_X d\beta = \beta$ points inward along $\partial_- W$ and outward along $\partial_+ W$;
(2) X is a gradient-like vector field of ϕ.

If $\partial_- W = \varnothing$, then (W, β, ϕ) is a *Weinstein domain*. Recall that $(\partial_\pm W, \beta|_{\partial_\pm W})$ is contact, where the orientations of $\partial_+ W$ (resp. $\partial_- W$) agrees with (resp. is opposite that of) $\beta \wedge d\beta^{n-1}$.

9.1. **Weinstein homotopy.**

Definition 9.1.1.

(1) (W, ϕ, X) is a *Smale cobordism* if (W, ϕ) is a Morse cobordism, X is gradient-like for ϕ, and X points inward along $\partial_- W$ and outward along $\partial_+ W$.
(2) (W, ϕ, X) is an *elementary Smale cobordism* if X has no trajectories between distinct critical points.
(3) (W, ϕ_t), $t \in [0, 1]$, is a *Morse homotopy* if $\partial_\pm W$ are regular level sets of ϕ_t and ϕ_t is Morse except at finitely many times t_i where we have a birth/death (also called embryonic) singularity e_i. We also assume that $\phi_{t_i}(e_i)$ is not equal to ϕ_{t_i} of the other critical points of ϕ_{t_i}. Any ϕ_t in this family is often called a *generalized Morse function*. Note that a generic ϕ_t (rel endpoints and boundary) is a Morse homotopy.
(4) (W, ϕ_t, X_t), $t \in [0, 1]$, is a *Smale homotopy* if (W, ϕ_t) is a Morse homotopy and X_t is gradient-like for ϕ_t, $t \in [0, 1]$.
(5) (W, ϕ_t, X_t), $t \in [0, 1]$, is an *elementary Smale homotopy* if it is one of the following:
   - I An elementary Smale cobordism for all t.
   - IIb There exists $t_0 \in (0, 1)$ such that for $t < t_0$ ϕ_t has no critical points, for $t = t_0$ ϕ_t has a birth-type critical point, and for $t > t_0$ ϕ_t has two critical points of consecutive indices connected by a unique X_t-trajectory.
   - IId Same as IIb with time t reversed.

Definition 9.1.2. A *Weinstein homotopy* is a family (W, β_t, ϕ_t) such that (W, ϕ_t, X_t) is a *Smale homotopy*, where $i_{X_t} d\beta_t = \beta_t$.

9.2. **Flexible Weinstein structures.**

Definition 9.2.1. A Legendrian link $L = \cup_{i=1}^k L_i \subset (M^{2n-1}, \xi)$ (i.e., a Legendrian submanifold with k connected components) is *loose* if there is a pairwise disjoint collection of loose charts $U_i \subset M$ for L_i such that $U_i \cap L = U_i \cap L_i$.

Remark 9.2.2. The union of a collection of pairwise disjoint loose Legendrian knots is not necessarily a loose Legendrian link. HW: find an example!

Definition 9.2.3. A *partition of* (W, ϕ) is a subdivision of W into

$$\{c_0 \leq \phi \leq c_1\} \cup \cdots \cup \{c_{\ell-1} \leq \phi \leq c_\ell\},$$

where

$$c_0 = \phi(\partial_- W) < c_1 < \cdots < c_{\ell-1} < c_\ell = \phi(\partial_+ W)$$

and c_i are regular values of ϕ.

Definition 9.2.4. An elementary Weinstein cobordism (W, β, ϕ) is *flexible* if the attaching manifolds of all index n critical points form a loose Legendrian link in $\partial_- W$. A Weinstein cobordism (W, β, ϕ) is *flexible* if there exists a partition into elementary flexible cobordisms.

Theorem 9.2.5 (Eliashberg-Cieliebak)**.** *Suppose W has dimension $2n > 4$.*

(1) *If (W, β, ϕ) is a flexible Weinstein domain and ϕ_t, $t \in [0, 1]$, is a Morse homotopy without critical points of index $> n$ such that $\phi_0 = \phi$, then there exists a homotopy (W, β_t, ϕ_t), $t \in [0, 1]$, of flexible Weinstein structures such that $(W, \beta_0, \phi_0) = (W, \beta, \phi)$, fixed on $Op(\partial_- W)$ and fixed up to scaling on $Op(\partial_+ W)$.*

(2) *If (W, β_0, ϕ_0) and (W, β_1, ϕ_1) are flexible Weinstein domains, ϕ_t, $t \in [0, 1]$, is a Morse homotopy without critical points of index $> n$ connecting ϕ_0 to ϕ_1, and there is a homotopy rel $Op(\partial_- W)$ of nondegenerate 2-forms connecting $d\beta_0$ to $d\beta_1$, then there exists a homotopy (W, β_t, ϕ_t), $t \in [0, 1]$, of flexible Weinstein structures connecting (W, β_0, ϕ_0) to (W, β_1, ϕ_0), fixed on $Op(\partial_- W)$.*

Proof. For more details see Sections 14.2 and 14.3 of [CE] (which is a bit hard to read). We will just explain the main idea of the proof of (1). One of the key technical points is to introduce and keep track of gradient-like vector fields X_t for ϕ_t such that X_0 is the Liouville vector field for $\beta = \beta_0$. In the end X_1 may not match the Liouville vector field for β_1, but it's ok.

We subdivide the Morse homotopy (W, ϕ_t) according to time and then partition each subdivision so that each small piece is an elementary Smale cobordism.

Model situation. Suppose (W, β, ϕ, X) is a flexible Weinstein cobordism and Y is gradient-like for ϕ such that (W, ϕ, Y) is an elementary Smale cobordism. Then we claim there is a family X_t, $t \in [0, 1]$, of gradient-like vector fields for ϕ and Liouville forms β_t for X_t, $t \in [0, 1/2]$, such that:

(A) (W, β_t, ϕ, X_t), $t \in [0, 1/2]$, is a flexible Weinstein homotopy with $(W, \beta_0, \phi, X_0) = (W, \beta, \phi, X)$, which is fixed on $Op(\partial_- W)$.

(B) $X_1 = Y$ and the Smale cobordisms (W, ϕ, X_t) are elementary of Type I for $t \in [1/2, 1]$.

It's not hard to modify Y slightly (through elementary Smale cobordisms) so that Y agrees with X on small neighborhoods of the critical points of ϕ. Hence without loss of generality we assume that X and Y differ only on $\phi^{-1}([a, b]) = \Sigma \times [a, b]$ and there are no critical values of ϕ on $[a, b]$. Interpolating from X to Y via X_t, there are well-defined holonomy maps

$$\mathrm{Hol}_{X_t} : \Sigma \times \{a\} \xrightarrow{\sim} \Sigma \times \{b\}$$

given by backwards flow along X_t. We consider Λ, the intersection of the stable manifolds of the critical points with $\Sigma \times \{a\}$. By definition, $\mathrm{Hol}_{X_0}(\Lambda)$ is a loose Legendrian (or isotropic) link and the h-principle implies that $\mathrm{Hol}_{X_t}(\Lambda)$ can be C^0-approximated by an isotopy of loose Legendrians (or isotropics) $\widetilde{\Lambda}_t$ such that $\widetilde{\Lambda}_0 = \mathrm{Hol}_{X_0}(\Lambda)$. Handle attaching along $\widetilde{\Lambda}_t$ and rescaling $[0, 1] \to [0, 1/2]$ gives (A). The slight discrepancy between $\widetilde{\Lambda}_1$ and $\mathrm{Hol}_{X_1}(\Lambda)$ gives (B).

In the end an elementary Smale cobordism (W, ϕ_t, X_t), $t \in [0, 1]$, of Type I starting at a Weinstein cobordism $(W, \beta_0, \phi_0, X_0)$ can be readjusted using a diffeotopy $h_t : W \xrightarrow{\sim} W$, $t \in [0, 1]$, with $h_0 = \mathrm{id}$ and $(h_1)_* \phi_0 = \phi_1$. This gives the Weinstein cobordism $(W, (h_t)_* \beta_0, (h_t)_* \phi_0)$. □

10. OVERTWISTED CLASSIFICATION IN ALL DIMENSIONS

Reference: Borman-Eliashberg-Murphy, “Existence and classification of overtwisted contact structures in all dimensions”.

Let M be a closed $(2n+1)$-manifold. An *overtwisted (or BEM-overtwisted) disk* is a $2n$-dimensional disk D_{OT} with a certain contact germ ζ_{OT} on it, to be made precise later. Fix an embedding $\phi : D_{OT} \to M$. Let $\mathrm{Cont}(M,\phi)$ be the set of contact structures ζ on M for which $\phi : (D_{OT}, \zeta_{OT}) \to (M, \zeta)$ is a contact embedding, and let $\mathrm{ACont}(M,\phi)$ be the set of almost contact structures ξ on M for which $\phi : (D_{OT}, \zeta_{OT}) \to (M, \xi)$ is a contact embedding.

Theorem 10.0.1. *The inclusion* $\mathrm{Cont}(M,\phi) \to \mathrm{ACont}(M,\phi)$ *is a weak homotopy equivalence.*

Corollary 10.0.2. *On any closed manifold M any almost contact structure is homotopic to a BEM-overtwisted contact structure which is unique up to homotopy.*

Remark 10.0.3. There is a relative version of Theorem 10.0.1. For HW try to state it.

In this lecture we will explain the first part of Corollary 10.0.2, i.e., the existence of a BEM-overtwisted contact structure in a given homotopy class of almost contact structures.

10.1. **Reduction to saucers.** Given a contact manifold $(M^{2n+1}, \xi = \ker \alpha)$ and a hypersurface $\Sigma \subset M$, recall the *characteristic foliation* Σ_ξ which is the singular line field given by $\ker d\alpha|_{\xi \cap T\Sigma}$. For HW verify that if $N^{2n-1} \subset \Sigma$ satisfies $N \pitchfork \Sigma_\xi$, then N is a codimension 2 contact submanifold of M.

Definition 10.1.1. A *saucer* $B = \{(w,v) \in D^{2n} \times \mathbb{R} \mid f_-(w) \leq v \leq f_+(w)\}$, where $f_+, f_- : D^{2n} \to \mathbb{R}$ are smooth functions such that $f_- = f_+$ on ∂D^{2n}, and $f_- < f_+$ on $\mathrm{int}(D^{2n})$. We write $\partial_+ B$ and $\partial_- B$ for the portions of ∂B where $v = f_+(w)$ and $v = f_-(w)$.

Definition 10.1.2. A *regular semicontact saucer* is a saucer B such that each of $\partial_\pm B \simeq D^{2n}_{z,x,y'}$ admits a graphical embedding into $(D^{2n}_{z,x,y'} \times \mathbb{R}_{y_1}, dz - \sum_{i=1}^n y_i dx_i)$ such that $\partial(\partial_\pm B)$ maps to $\partial D^{2n} \times \{0\}$ and the characteristic foliations on $\partial_+ B$ and $\partial_- B$ are diffeomorphic to the characteristic foliation on the standard unit disk $\{y_1 = 0\}$.

Lemma 10.1.3. *An almost contact structure ξ on M is homotopic to ξ' such that there exist regular semicontact saucers B_i, $i = 1, \dots, k$, for which ξ' is genuine on $M - \cup_i B_i$.*

Proof. Use Gromov's h-principle for ξ restricted to the open manifold $M - B^{2n+1}$ to obtain ξ'' homotopic to ξ which is genuine on $M - B^{2n+1}$. We next view B^{2n+1} as $D^{2n} \times [0,1]_s$. Then applying Gromov's h-principle again (in parametric/relative form) there is a smooth family ξ''_s, $s \in [0,1]$, of contact germs on $D^{2n} \times \{s\}$ which agrees with ξ'' on $D^{2n} \times \{0,1\}$; such a family will be called a *semicontact structure*. For convenience we may also assume that the ξ''_s all agree on ∂D^{2n}.

By the compactness of the interval, there exists $\varepsilon > 0$ small such that each ξ''_s, $s \in [0,1]$, is defined on $D^{2n} \times [s-\varepsilon, s+\varepsilon]$. Now using the usual Moser technique, for $\delta > 0$ small, there is a diffeomorphism

$$\psi_{s\pm\delta,s} : (D^{2n} \times \{s \pm \delta\}, \xi''_{s\pm\delta}) \to (D^{2n} \times [s-\varepsilon, s+\varepsilon], \xi''_s),$$

whose image is graphical over D^{2n}. If $D_{s,\delta} := \mathrm{Im}(\psi_{s+\delta,s})$ lies above $D^{2n} \times \{s\}$, then the region $D^{2n} \times [s, s+\delta]$ can be filled with a genuine contact structure.

What's interesting is when $D_{s,\delta}$ does not lie above $D^{2n} \times \{s\}$. An important observation is that we may assume that a small neighborhood of the singular points of $(D_{s,\delta})_{\xi''_s}$ lies above $D^{2n} \times \{s\}$: We can use a

Reeb vector field of ξ''_s or its negative, depending on the sign of the singularity, to push the positive and negative singularities above $D^{2n} \times \{s\}$. For this we may need to make $\delta > 0$ even smaller. This means that the portion of $D_{s,\delta}$ below $D^{2n} \times \{s\}$ has nonsingular characteristic foliation. Let $W \subset D^{2n}$ be such that $\psi_{s+\delta,s}(W)$ has nonsingular characteristic foliation and contains all the points of $D_{s,\delta}$ below $D^{2n} \times \{s\}$. Then $W \times [s, s+\delta]$ can be subdivided into sufficiently small regular semicontact saucers. Repeating the procedure gives the lemma. □

10.2. **Circular shell model.** Observe that if we round the boundary ∂B of a regular semicontact saucer B, then its characteristic foliation $(\partial B)_\xi$ (after renaming contact structures) consists of two singularities: a source and a sink. We now recast the $(\partial B)_\xi$ in terms of a "circular shell model" (cf. Lemma 10.2.2).

Definition 10.2.1 (Domination)**.** Let ζ_1 and ζ_2 be contact germs on $Op(S^{2n})$. Then ζ_2 *dominates* ζ_1 if there exists a contact structure on $S^{2n} \times [1,2]$ which agrees with ζ_i on $S^{2n} \times \{i\}$ for $i = 1, 2$.

Let $\Delta^{2n-1}_{x,y,z}$ be a star-shaped domain (i.e., the intersection of Δ with every line through the origin is a single line segment through the origin) in $(\mathbb{R}^{2n-1}_{x,y,z}, \lambda = dz + \sum_{i=1}^{n-1} \frac{1}{2}(x_i dy_i - y_i dx_i))$ and let

$$K : \Delta_{x,y,z} \times S^1_t \to \mathbb{R}$$

be a "contact Hamiltonian" satisfying $K|_{\partial(\Delta \times S^1)} > 0$. We also write $\frac{1}{2}(x_i dy_i - y_i dx_i) = u_i d\phi_i$, where $u_i = x_i^2 + y_i^2$ and ϕ_i is the angular coordinate.

We consider a $(2n+1)$-disk B_K with boundary $\partial B_K = \Sigma_{K,1} \cup \Sigma_{K,2}$, where

$$\Sigma_{K,1} = \{v = K(x,y,z,t) \mid (x,y,z) \in \Delta, t \in S^1\} \subset (\Delta \times T^*S^1_{v,t}, \beta = \lambda + v dt),$$
$$\Sigma_{K,2} = \{v \leq K(x,y,z,t), x \in \partial\Delta\} \subset (\Delta \times \mathbb{C}, \beta = \lambda + v dt),$$

and the contact germ is induced from the inclusions. Here T^*S^1 has Liouville form vdt and we are viewing T^*S^1 as a subset of $\mathbb{C}$ where $(v,t) = (r^2, \phi)$ and (r, ϕ) are polar coordinates on $\mathbb{C}$. Here we are viewing a $(2n+1)$-disk $D^{2n+1} = D^2 \times D^{2n-1}$ with boundary

$$\partial(D^2 \times D^{2n-1}) = (S^1 \times D^{2n-1}) \cup (D^2 \cup \partial D^{2n-1}).$$

It's instructive the consider the case $n = 1$, where $\Delta = [-1,1]$, $\Sigma_{K,1} \simeq [-1,1] \times S^1$, and $\Sigma_{K,2} \simeq \{-1,1\} \times D^2$.

To understand ∂B_K better, let us calculate its characteristic foliation. Check for HW that the characteristic foliation on $\Sigma_{K,1}$ is directed by

$$Z = \partial_t + X_K - \tfrac{1}{2}\tfrac{\partial K}{\partial z} \textstyle\sum_i (x_i \partial_{x_i} + y_i \partial_{y_i}) + (\tfrac{1}{2} \textstyle\sum_i (x_i \tfrac{\partial K}{\partial x_i} + y_i \tfrac{\partial K}{\partial y_i}) - K)\partial_z, \tag{10.2.1}$$

where X_K is the Hamiltonian vector field of $K(x,y,z,t)$ on $\mathbb{R}^{2n-2}_{x,y}$ with respect to $\sum_{i=1}^{n-1} dx_i dy_i$ and with z, t treated as constants.

Note that ∂B_K also has two singularities — a source and a sink — and $\Sigma_{K,1}$ has no singularities.

Lemma 10.2.2. *The boundary of a regular semicontact saucer B dominates ∂B_K for some t-independent $K : \Delta \times S^1 \to \mathbb{R}$.*

The proof is omitted. The lemma implies that we can reduce from regular semicontact saucers to circular shells ∂B_K.

Examples. When $n = 1$, $Z = \partial_t - K\partial_z$. Ignoring the $\Sigma_{K,2}$ portion which is standard and contains the two singularities, we are reduced to consider the (partially defined) holonomy map $h_K : [-1,1] \dashrightarrow [-1,1]$

obtained by flowing along the characteristic foliation once in the S^1_t-direction and h_K is basically given by the function $-K$. If $K > 0$, then the holonomy is negative, and the contact structure can be extended as discussed in the first half of the course.

For any n, when K is a constant, then $Z = \partial_t - K\partial_z$. Also note that when Δ is small, then $Z \approx \partial_t + X_K - K\partial_z$; if Δ is a small D^{2n} times $[-1,1]$ (i.e., a cylinder), then the holonomy $h_K : D^{2n}\times[-1,1] \dashrightarrow D^{2n}\times[-1,1]$ is given by $-K$ in the z-direction and the Hamiltonian diffeomorphism obtained by integrating the time-dependent Hamiltonian vector field X_K in the x,y-directions.

10.3. **BEM overtwisted disk.** We take $\Delta_{cyl} = \{|z| \leq 1, u \leq 1\}$, where $u = \sum_i u_i$. Let $\varepsilon > 0$ be small. Define $k_\varepsilon : [0,1] \to \mathbb{R}$ such that $k_\varepsilon(x) = 0$ for $x \leq 1-\varepsilon$ and $k_\varepsilon(x) = x - (1-\varepsilon)$ for $x \geq 1-\varepsilon$. Letting $\Delta_\varepsilon = \{|z|, u \leq 1-\varepsilon\}$, define the S^1-invariant contact Hamiltonian

$$K_\varepsilon : \Delta_{cyl} \times S^1 \to \mathbb{R}$$

such that K_ε is dependent only on z and u and $K_\varepsilon(z,u) = \max(k_\varepsilon(u), k_\varepsilon(|z|))$ on $\Delta_{cyl} - \Delta_\varepsilon$ and $K_\varepsilon(z,u) < 0$ on $\mathrm{int}(\Delta_\varepsilon)$.

The *BEM overtwisted disk* $D^{2n}_{K_\varepsilon}$ is the subset of $\partial B_{K_\varepsilon}$ consisting of points such that $z \in [-1, 1-\varepsilon]$. Observe that $D^{2n}_{K_\varepsilon}$ has one singular point on $z = -1$.

Example. When $n = 1$, $\Delta_{cyl} = [-1,1]_z$ and $D^2_{K_\varepsilon}$ consists of $\{-1\} \times D^2 \subset \Sigma_{K,2}$ and $[-1, 1-\varepsilon] \times S^1 \subset \Sigma_{K,1}$, and contains the usual overtwisted disk.

10.4. **How to fill contact shells.** By now we are running out of time and getting to the most technical part of [BEM], so we will content ourselves with explaining what happens in dimension 3, i.e., $n = 1$.

Suppose we want to fill the contact shell ∂B_K where the holonomy map h_K can be positive. The BEM overtwisted disk $D^2_{K_\varepsilon}$ will "help create disorder" in the following way: $D^2_{K_\varepsilon}$ has an $[-1,1]$-invariant neighborhood; we "erase" the contact structure that has already been defined on $D^2_{K_\varepsilon} \times (-1,1)$. Next take a transverse arc γ from the (say negative) elliptic point of $D^2_{K_\varepsilon} \times \{-1\}$ to a (say positive) elliptic point of ∂B_K; we "erase" the contact structure on the neighborhood of γ. This gives us a connected sum $D' := (D^2_{K_\varepsilon} \times \{-1\})\#\partial B_K$. We now compare the characteristic foliations of $D := D^2_{K_\varepsilon} \times \{1\}$ and D'. The neighborhoods of the negative elliptic points of D and D' can be matched and we are left comparing the holonomy maps $h : [-1,1] \dashrightarrow [-1,1]$ and $h' : [-1,1] \dashrightarrow [-1,1]$ for D and D' The key point is that, even if we do not have $h' < h$ which is what we want, we can conjugate h', i.e., take $\phi \circ h' \circ \phi^{-1}$ where ϕ is a diffeomorphism of $[-1,1]$, so that $\phi \circ h' \circ \phi^{-1} < h$. *This is the miracle of creating disorder by introducing a portion with positive holonomy!* Once we have $\phi \circ h' \circ \phi^{-1} < h$, we can easily fill the region between D and D'.

10.5. **Comparison with other overtwisted objects.** There are other overtwisted objects. Unlike the BEM overtwisted disk, they are all compact $(n+1)$-dimensional objects N that admit singular foliations by Legendrian submanifolds.

- (PS) *plastikstufe*, due to Gromov and Niederkrüger, where we take an 2-dimensional overtwisted disk D^2 with the usual characteristic foliation and take the product with an $(n-1)$-manifold P so that the Legendrian leaves are leaves of D^2 times P, including $(\partial D^2) \times P$.
- (bLob) *bordered Legendrian open book*, due to Massot-Niederkrüger-Wendl, which is a partial open book with Legendrian pages S such that ∂S splits into disjoint unions of components $\partial_0 S$ and $\partial_1 S$ such that $\partial_0 S$ becomes the binding and $\partial_1 S \times S^1$ becomes the Legendrian boundary.

(OO) *Overtwisted orange*, due to Huang and myself, which has the topological type of $D^n \times S^1/\sim$, where the north pole of ∂D^n times S^1 is squashed to a point. The fibers $D^n \times \{\theta\}$ are Legendrian and the outer boundary is a singular Legendrian. The overtwisted orange is the only $(n+1)$-dimensional object that is contractible; it is also natural in the sense that it is obtained by gluing two higher-dimensional bypasses together.

Casals-Murphy and Huang showed that (BEM) $\Leftrightarrow$ (PS). It was shown by Huang and myself that (BEM) $\Leftrightarrow$ (OO). It is still not known whether (BEM) $\Leftrightarrow$ (bLob) in general, although this was shown by Huang for M^5.