NOTES FOR MATH 227A: ALGEBRAIC TOPOLOGY

KO HONDA

1. Categories and Functors

1.1. **Categories.** A *category* $\mathcal{C}$ consists of:

(1) A collection $ob(\mathcal{C})$ of *objects*.
(2) A set $\mathrm{Hom}(A,B)$ of *morphisms* for each ordered pair (A,B) of objects. A morphism $f \in \mathrm{Hom}(A,B)$ is usually denoted by $f : A \to B$ or $A \xrightarrow{f} B$.
(3) A map $\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$ for each ordered triple (A,B,C) of objects, denoted $(f,g) \mapsto gf$.
(4) An identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A,A)$ for each object A.

The morphisms satisfy the following properties:

A. (Associativity) $(fg)h = f(gh)$ if $h \in \mathrm{Hom}(A,B)$, $g \in \mathrm{Hom}(B,C)$, and $f \in \mathrm{Hom}(C,D)$.
B. (Unit) $\mathrm{id}_B\, f = f = f\, \mathrm{id}_A$ if $f \in \mathrm{Hom}(A,B)$.

Examples: (HW: take a couple of examples and verify that all the axioms of a category are satisfied.)

1. $\mathbf{Top}$ = category of topological spaces and continuous maps. The objects are topological spaces X and the morphisms $\mathrm{Hom}(X,Y)$ are continuous maps from X to Y.

2. $\mathbf{Top}_\bullet$ = category of pointed topological spaces. The objects are pairs (X,x) consisting of a topological space X and a point $x \in X$. $\mathrm{Hom}((X,x),(Y,y))$ consist of continuous maps from X to Y that take x to y.
Similarly define $\mathbf{Top}^2$ = category of pairs (X,A) where X is a topological space and $A \subset X$ is a subspace. $\mathrm{Hom}((X,A),(Y,B))$ consists of continuous maps $f : X \to Y$ such that $f(A) \subset B$.

3. $\mathbf{hTop}$ = homotopy category of topological spaces. The objects are topological spaces X and $\mathrm{Hom}(X,Y)$ is the set of homotopy classes of continuous maps $f : X \to Y$. Recall that $f_0, f_1 : X \to Y$ are *homotopic* if there is a continuous map $F : X \times [0,1] \to Y$ such that $F|_{X\times\{i\}} = f_i$, $i = 0,1$. Homotopy induces an equivalence relation $\sim$ and the equivalence classes are called *homotopy classes*.

4. $\mathbf{Grp}$ = category of groups and group homomorphisms. The objects are groups G and the morphisms $\mathrm{Hom}(G,H)$ are group homomorphisms from G to H.

5. R–$\mathbf{Mod}$ = category of left R-modules of an associative ring R. The objects are left R-modules and the morphisms $\mathrm{Hom}(M,N)$ are left R-module maps. Also let $\mathbf{Mod}$–R be the category of right R-modules. We sometimes write $\mathbf{Ab}$ (category of abelian groups) for $\mathbb{Z}$–$\mathbf{Mod}$.

6. $\mathbf{Ch}(R$–$\mathbf{Mod})$ = category of chain complexes of (left) R-modules and chain maps.
Recall that a *chain complex* is a sequence C_* of R-module maps:

$$\longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow$$

such that $\partial_n\partial_{n+1} = 0$ and a *chain map* is a collection of R-module maps $C_n \xrightarrow{\phi_n} D_n$, often denoted $\phi : C_* \to D_*$, such that the diagram commutes:

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow \phi_{n+1} & & \downarrow \phi_n & & \downarrow \phi_{n-1} \\ D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} \end{array}$$

1.2. **Functors.** A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule which sends:

(1) an object $A \in ob(\mathcal{C})$ to $F(A) \in ob(\mathcal{D})$; and
(2) every morphism $f \in \mathrm{Hom}(A, B)$ in $\mathcal{C}$ to a morphism $F(f) \in \mathrm{Hom}(F(A), F(B))$ in $\mathcal{D}$,

such that:

A. $F(gf) = F(g)F(f)$.
B. $F(\mathrm{id}_A) = \mathrm{id}_{FA}$.

A *contravariant functor* F assigns to $f \in \mathrm{Hom}(A, B)$ an element $F(f) \in \mathrm{Hom}(F(B), F(A))$ satisfying $F(fg) = F(g)F(f)$.

Examples: (HW: verify the axioms of a functor)

1. The *singular chain complex functor* $\mathrm{Sing} : \mathbf{Top} \to \mathbf{Ch}(\mathbb{Z}\text{–}\mathbf{Mod})$, which is defined as follows:

Brief review of singular homology. Given $v_0, \dots, v_n \in \mathbb{R}^m$, let $[v_0, \dots, v_n]$ be the convex hull of $v_0, \dots, v_n$, i.e.,

$$[v_0, \dots, v_n] = \{\textstyle\sum_{i=0}^n t_i v_i \in \mathbb{R}^m \mid \sum_{i=0}^n t_i = 1, t_i \geq 0, i = 0, \dots, n\}.$$

The *standard n-simplex* Δ^n is $[e_0, \dots, e_n] \subset \mathbb{R}^{n+1}$.

Let $X \in ob(\mathbf{Top})$. A continuous map $\sigma : \Delta^n \to X$ is called a *singular n-simplex*. We define $C_n(X)$ to the free $\mathbb{Z}$-module generated by singular n-simplices; an element of $C_n(X)$ is called *singular n-chain*. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is given by:

$$\partial_n \sigma = \textstyle\sum_{i=0}^n (-1)^n \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]}.$$

Here $\sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]}$ means the composition of $[e_0, \dots, e_{n-1}] \to [e_0, \dots, \hat{e}_i, \dots, e_n]$ given by the canonical linear homeomorphism, followed by σ restricted to $[e_0, \dots, \hat{e}_i, \dots, e_n]$.

One can verify that $\partial_n\partial_{n+1} = 0$, i.e., $C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$ is a chain complex.

Given a continuous map $f : X \to Y$, we define the induced map $f_n : C_n(X) \to C_n(Y)$ by mapping $\sigma : \Delta^n \to X$ to $f \circ \sigma : \Delta^n \to Y$ and extending linearly. One can verify that $\partial_{n+1} f_{n+1} = f_n \partial_{n+1}$.

1'. Similarly, define the *relative singular chain complex functor* $\mathrm{Sing} : \mathbf{Top}^2 \to \mathbf{Ch}(\mathbb{Z}\text{–}\mathbf{Mod})$ which sends (X, A) to

$$C_*(X, A) = (\to C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \to),$$

where $C_n(X, A) := C_n(X)/C_n(A)$ and ∂_n is the map $C_n(X, A) \to C_{n-1}(X, A)$ induced from $\partial_n : C_n(X) \to C_{n-1}(X)$.

2. The *homology functor* $H_i : \mathbf{Ch}(R\text{–}\mathbf{Mod}) \to R\text{–}\mathbf{Mod}$ which assigns to a chain complex C_* the *homology*

$$H_i(C_*) := \ker(d_i)/\operatorname{im}(d_{i+1}),$$

and to a chain map $\phi : C_* \to D_*$ the induced homomorphism $\phi_* : H_i(C_*) \to H_i(D_*)$.

3. The composition $H_i \circ \mathrm{Sing} : \mathbf{Top} \to \mathbb{Z}\text{–}\mathbf{Mod}$ is the *singular homology functor* which takes $X \mapsto H_i^{\mathrm{sing}}(X; \mathbb{Z})$.

4. The *fundamental group functor* $\pi_1 : \mathbf{Top}_\bullet \to \mathbf{Grp}$ sends $(X, x) \mapsto \pi_1(X, x)$.

5. Given a ring R and a right R-module N, there is a functor $N \otimes_R : R\text{–}\mathbf{Mod} \to \mathbb{Z}\text{–}\mathbf{Mod}$ given by $M \mapsto N \otimes_R M$.

5'. More generally, given rings R, S, and an $S - R$ bimodule N (this means that S acts on N from the left and R from the right), we have the functor $N \otimes_R : R\text{–}\mathbf{Mod} \to S\text{–}\mathbf{Mod}$ given by $M \mapsto N \otimes_R M$. We can also extend $N \otimes_R$ to $\mathbf{Ch}(R\text{–}\mathbf{Mod}) \to \mathbf{Ch}(S\text{–}\mathbf{Mod})$.

6. Given an R-module N, there is a covariant functor $\mathrm{Hom}(N, -) : R\text{–}\mathbf{Mod} \to R\text{–}\mathbf{Mod}$ given by $M \mapsto \mathrm{Hom}_R(N, M)$ and a contravariant functor $\mathrm{Hom}(-, N) : R\text{–}\mathbf{Mod} \to R\text{–}\mathbf{Mod}$ given by $M \mapsto \mathrm{Hom}_R(M, N)$. We can also extend the Hom functors to $\mathbf{Ch}(R\text{–}\mathbf{Mod}) \to \mathbf{Ch}(R\text{–}\mathbf{Mod})$.

1.3. **Adjoint functors.** Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are *adjoint* if for each $A \in ob(\mathcal{C})$ and $B \in ob(\mathcal{D})$ there is a bijection

$$\tau_{AB} : \mathrm{Hom}_{\mathcal{D}}(FA, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A, GB),$$

subject to the following naturality condition: For any $f \in \mathrm{Hom}(A, A')$ and $g \in \mathrm{Hom}(B, B')$, the diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}}(FA', B) & \xrightarrow{Ff^*} & \mathrm{Hom}_{\mathcal{D}}(FA, B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{D}}(FA, B') \\
\downarrow{\scriptstyle \tau_{A'B}} & & \downarrow{\scriptstyle \tau_{AB}} & & \downarrow{\scriptstyle \tau_{AB'}} \\
\mathrm{Hom}_{\mathcal{C}}(A', GB) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{C}}(A, GB) & \xrightarrow{Gg_*} & \mathrm{Hom}_{\mathcal{C}}(A, GB').
\end{array}$$

We say that F is a *left adjoint of* G and G is a *right adjoint of* F.

Example: Given an $R - S$-bimodule N, the functors $\otimes_R N : \mathbf{Mod}\text{–}R \to \mathbf{Mod}\text{–}S$ and $\mathrm{Hom}_S(N, -) : \mathbf{Mod}\text{–}S \to \mathbf{Mod}\text{–}R$ are adjoint functors.

1.4. **Natural transformations.** Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ associates a morphism $\eta_A : F(A) \to G(A)$ for each $A \in ob(\mathcal{C})$ such that for each morphism $f : A \to A'$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc}
F(A) & \xrightarrow{Ff} & F(A') \\
\downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_{A'}} \\
G(A) & \xrightarrow{Gf} & G(A').
\end{array}$$

Example: A right R-module map $\phi : N \to N'$ gives a natural transformation $\eta : N\otimes_R \Rightarrow N'\otimes_R$ where $N\otimes_R, N'\otimes_R : R$–$\mathbf{Mod} \to \mathbb{Z}$–$\mathbf{Mod}$: Given a left R-module map $f : M \to M'$,

$$\begin{array}{ccc} N \otimes_R M & \xrightarrow{(N\otimes_R)f} & N \otimes_R M' \\ \big\downarrow \eta_M & & \big\downarrow \eta_{M'} \\ N' \otimes_R M & \xrightarrow{(N'\otimes_R)f} & N' \otimes_R M'. \end{array}$$

2. Homology with coefficients

2.1. **Definitions.** Let X be a topological space and G be a $\mathbb{Z}$-module (aka abelian group). An *n-chain on X with coefficients in G* is finite formal sum $\sum_i n_i\sigma_i$, where σ_i is a singular n-simplex on X and $n_i \in G$. The set $C_n(X;G)$ of n-chains on X with coefficients in G can be written as

$$C_n(X;G) = C_n(X) \otimes_{\mathbb{Z}} G.$$

The functor $\otimes G : \mathbf{Ch} \to \mathbf{Ch}$, where $\mathbf{Ch} = \mathbf{Ch}(\mathbb{Z}\text{–}\mathbf{Mod})$, sends $C_*(X)$ to $C_*(X;G)$. (Note that $\mathbb{Z}$ is commutative, so left and right modules are the same; we are also tensoring over $\mathbb{Z}$.) In particular this means that $C_n(X) \xrightarrow{\partial} C_{n-1}(X)$ passes to $C_n(X;G) \xrightarrow{\partial\otimes \mathrm{id}_G} C_{n-1}(X;G)$ and $(\partial \otimes \mathrm{id}_G)^2 = 0$. The homology of $C_*(X;G)$ is written as $H_*(X;G)$ and is called the *homology of X with coefficients in G*. This is the result of the composition

$$H_n \circ \otimes G \circ \text{Sing} : \mathbf{Top} \to \mathbb{Z}\text{–}\mathbf{Mod}.$$

Similarly, we can define $C_n(X,A;G) = C_n(X,A) \otimes G$. The *relative homology of (X,A) with coefficients in G* is its homology group.

Remark: If we take homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Q}$, then everything involved become vector spaces, and the calculations are often easier.

Question: Is $H_n(X;G) \simeq H_n(X;\mathbb{Z}) \otimes G$?

In order to answer this question, we take a detour in homological algebra.

2.2. **Right exactness of $\otimes$.** Let N be a left R-module and

$$0 \to A \to B \to C \to 0$$

be an exact sequence of right R-modules. Then we have the following:

Fact: The functor $\otimes_R N : \mathbf{Mod}\text{–}R \to \mathbb{Z}\text{–}\mathbf{Mod}$ is *right exact*, i.e.,

$$A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$$

is an exact sequence. HW: check this!

However, $A \otimes N \to B \otimes N$ is not always injective. (We omit R from the notation from now on.)

Prototypical Example:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

If we tensor with $\mathbb{Z}/n\mathbb{Z}$, then we have

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

exact, but the left arrow is not injective.

To continue this exact sequence to the left, we introduce the *derived functors* of $\otimes N$.

2.3. **Projective resolutions.**

Definition 2.3.1. A (right) R-module P is *projective* if it satisfies the following lifting property: given a map $a : P \to N$ and a surjection $M \xrightarrow{b} N \to 0$, there exists a lift $\overline{a} : P \to M$ such that $a = b\overline{a}$.

HW: Prove that P is projective if and only if it is a direct summand of a free R-module.

Definition 2.3.2. A *projective resolution* of an R-module M is an exact sequence

$$\cdots \to P_2 \xrightarrow{i_2} P_1 \xrightarrow{i_1} P_0 \xrightarrow{i_0} M \to 0, \tag{2.3.1}$$

where P_i are projective R-modules. (We often write $P_* \to M$.) As a special case, if we take the P_i to be free, we have a *free resolution*.

Lemma 2.3.3. *A $\mathbb{Z}$-module M has a free resolution* $0 \to F_1 \to F_0 \to M \to 0$.

Proof. Let $\{a_\alpha\}$ be a generating set for M and F_0 be the free abelian group generated by $\{a_\alpha\}$. Then the kernel of the natural map $F_0 \to M$ is a subgroup of a free abelian group, and hence is free. (Note this is not obvious.) This gives the injection $F_1 \to F_0$. □

Remark: The same proof applies to show that any R-module M admits a free resolution.

Example: If $A = \mathbb{Z}/m\mathbb{Z}$, then

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

is a free resolution of $\mathbb{Z}/m\mathbb{Z}$. We could also have taken

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}^2 \xrightarrow{j} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0,$$

where $i : 1 \mapsto (0,1)$ and $j : (1,0) \mapsto m$, $(0,1) \mapsto 0$.

In particular the example shows that projective/free resolutions are not unique!

Lemma 2.3.4.

(1) *Given projective resolutions P_* of A and P'_* of B and a map $\phi_{-1} : A \to B$, there is a chain map $\phi : P_* \to P'_*$ which extends ϕ_{-1}, i.e., the following diagram commutes:*

$$\begin{array}{ccccccc} \longrightarrow & P_1 & \xrightarrow{i_1} & P_0 & \xrightarrow{i_0} & A & \longrightarrow 0 \\ & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_{-1}} & \\ \longrightarrow & P'_1 & \xrightarrow{i'_1} & P'_0 & \xrightarrow{i'_0} & B & \longrightarrow 0 \end{array}$$

(2) *Any two chain maps $\phi, \psi : P_* \to P'_*$ extending ϕ_{-1} are chain homotopic.*

Recall that two chain maps $\phi, \psi : (C_*, \partial) \to (C'_*, \partial')$ are *chain homotopic* if there exist maps $h_i : C_i \to C'_{i+1}$ such that $\phi_i - \psi_i = \partial'_{i+1} h_i + h_{i-1}\partial_i$. Moreover, chain homotopic chain maps induce the same maps on homology.

Proof. (1) The map ϕ_0 is defined using the lifting property: $i'_0\phi_0 = \phi_{-1}i_0$. For ϕ_1, we show that $\phi_0 i_1(P_1) \subset \operatorname{Im} i'_1 = \ker i'_0$. Indeed, $i'_0\phi_0 i_1 = \phi_{-1}i_0 i_1 = 0$. Hence ϕ_1 can be defined using the lifting property.

(2) Since $i'_0(\phi_0 - \psi_0) = 0$, it follows that $(\phi_0 - \psi_0)(P_0) \subset \operatorname{Im}(i'_1)$. Hence h_0 can be defined using the lifting property. Next observe that $i'_1(\phi_1 - \psi_1 - h_0 i_1) = (\phi_0 - \psi_0 - i'_1 h_0)i_1 = 0$. Hence h_1 can also be defined using the lifting property. □

2.4. **Definition of Tor.** Given a projective resolution $P_* \to M$, we tensor with a right R-module N to obtain

$$\cdots \xrightarrow{i_2} P_1 \otimes N \xrightarrow{i_1} P_0 \otimes N \to 0,$$

which is no longer exact but is a chain complex. Its homology is denoted by $\mathrm{Tor}_i(M, N)$.

Theorem 2.4.1. $\mathrm{Tor}_i(M, N)$ *only depends on* M *and* N. *In particular, it is independent of the projective resolution used for* M.

Proof. This follows from Lemma 2.3.4: Given projective resolutions $P_*, P'_* \to M$, there exist chain maps $\phi : P_* \to P'_*$ and $\psi : P'_* \to P_*$ such that $\psi\phi, \mathrm{id} : P_* \to P'_*$ are chain homotopic. Tensoring with N still preserves this property. $\square$

Remark: Given a projective resolution P_* of M, there is a chain map

$$\begin{array}{ccccccc} \longrightarrow & P_1 & \xrightarrow{i_1} & P_0 & \longrightarrow & 0 \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle i_0} & & \\ \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

which induces an isomorphism on homology. A chain map $f : C_* \to D_*$ which induces an isomorphism on all homology groups is called a *quasi-isomorphism*. Hence P_* *is a quasi-isomorphic replacement of* M.

Properties of Tor**:**

(1) $\mathrm{Tor}_i(M, N) = 0$ if $i > 1$ and $R = \mathbb{Z}$. (This follows from the existence of a free resolution $0 \to F_1 \to F_0 \to M \to 0$.)
(2) $\mathrm{Tor}_0(M, N) = M \otimes N$. (If $P_1 \to P_0 \to M \to 0$ is exact, then $P_1 \otimes N \xrightarrow{i_1} P_0 \otimes N \to M \otimes N \to 0$ is exact. Hence, $\mathrm{Tor}_0(M, N) = P_0 \otimes N / \operatorname{Im} i_1 \simeq M \otimes N$.)
(3) $\mathrm{Tor}_i(M, N) = \mathrm{Tor}_i(N, M)$ if R is commutative. (Proof is not obvious.)
(4) If M is projective, then $\mathrm{Tor}_i(M, N) = 0$ for $i \geq 1$. (If M is projective, then $0 \to M \to M \to 0$ is the projective resolution for M. Hence $0 \to M \otimes N \to 0$ is the chain complex which computes Tor.)
(5) $\mathrm{Tor}_i(M \oplus M', N) \simeq \mathrm{Tor}_i(M, N) \oplus \mathrm{Tor}_i(M', N)$.
(6) $\mathrm{Tor}_1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \ker(Z/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$, where $k = \mathrm{GCD}(m, n)$. In particular, $\mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. (Tensor the truncated free resolution $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$ to obtain $0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z} \to 0$.)

2.5. **More on the Tor functor.**

Theorem 2.5.1. *Given an exact sequence* $0 \to A \to B \to C \to 0$ *of right* R*-modules, there exist projective resolutions* $F_* \to A$, $G_* \to B$, $H_* \to C$ *and exact sequences* $0 \to F_* \xrightarrow{i} G_* \xrightarrow{j} H_* \to 0$ *which make* *the following diagram commutative:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_{-1}} & & \\
 & \longrightarrow & G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{g_0} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle j_{-1}} & & \\
 & \longrightarrow & H_1 & \xrightarrow{h_1} & H_0 & \xrightarrow{h_0} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. Take projective resolutions $F_* \to A$ and $H_* \to C$. Then we set $G_i = F_i \oplus H_i$ and $i : F_i \to G_i$ and $j : G_i \to H_i$ to be the usual inclusion and projection maps. The goal is to define the maps g_0, g_1, etc. For g_0, lift h_0 to a map $\overline{h}_0 : H_0 \to B$. Then let $g_0 = (i_{-1}f_0, \overline{h}_0) : F_0 \oplus H_0 \to B$. We verify that g_0 is surjective: Given $b \in B$, there exists $b' \in B$ such that $j_{-1}(b' - b) = 0$ and $b' = \overline{h}_0(y)$ for some $y \in H_0$. Now $b' - b \in \operatorname{Im} i_{-1}$, so there exists $x \in F_0$ such that $i_{-1}f_0(x) = b' - b$. Thus g_0 maps $(x, y) \mapsto b$. The definitions of g_1 etc. are similar and it remains to verify that $\operatorname{Im} g_{i+1} = \ker g_i$ (HW). □

We now apply the functor $\otimes M$ where M is a left R-module.

Fact: The sequence $0 \to F_i \otimes M \to G_i \otimes M \to H_i \otimes M \to 0$ is exact since $0 \to F_i \to G_i \to H_i \to 0$ splits and $G_i \simeq F_i \oplus H_i$. (HW: check this!)

Hence we have an exact sequence of chain complexes

$$0 \to F_* \otimes M \to G_* \otimes M \to H_* \otimes M \to 0.$$

The corresponding long exact sequence is:

$$\cdots \to \operatorname{Tor}_1(A, M) \to \operatorname{Tor}_1(B, M) \to \operatorname{Tor}_1(C, M) \to$$
$$\to A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

Example: Applying $\otimes \mathbb{Z}/m\mathbb{Z}$ to

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0,$$

we obtain

$$0 \to \operatorname{Tor}_1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z} \xrightarrow{0} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z} \to 0.$$

2.6. **Universal Coefficient Theorem.** We now restrict to $\mathbb{Z}$-modules.

Theorem 2.6.1 (Universal Coefficient Theorem for homology)**.** *There is an exact sequence*

$$0 \to H_n(X) \otimes M \to H_n(X; M) \to \operatorname{Tor}_1(H_{n-1}(X), M) \to 0,$$

and the exact sequence splits, albeit noncanonically. Therefore,

$$H_n(X;M) \simeq (H_n(X) \otimes M) \oplus \mathrm{Tor}_1(H_{n-1}(X), M).$$

HW: Show that if an exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ splits, i.e., there exists $k : C \to B$ such that $jk = \mathrm{id}$, then $B \simeq A \oplus C$.

Proof. Start with the exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$

where $C_n = C_n(X)$, $Z_n = \ker \partial_n$, and $B_{n-1} = \mathrm{Im}\, \partial_n$. The exact sequence can be viewed as a row in an exact sequence of chain complexes $0 \to Z_* \to C_* \to B_* \to 0$, where the boundary maps for Z_* and B_* are zero. Now apply $\otimes M$: since B_{n-1} is a submodule of C_{n-1}, it is free, and $\mathrm{Tor}_1(B_{n-1}, M) = 0$. Hence the exact sequence remains exact under $\otimes M$, i.e.,

$$0 \to Z_n \otimes M \to C_n \otimes M \to B_{n-1} \otimes M \to 0 \tag{2.6.1}$$

is exact. The corresponding long exact sequence is:

$$B_n \otimes M \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes M \to H_n(X;M) \to B_{n-1} \otimes M \xrightarrow{i_{n-1} \otimes \mathrm{id}} Z_{n-1} \otimes M.$$

(HW: check that the connecting homomorphism $B_n \to Z_n$ actually agrees with the inclusion map i_n.) Hence

$$0 \to (Z_n \otimes M)/\,\mathrm{Im}(i_n \otimes \mathrm{id}) \to H_n(X;M) \to \ker(i_{n-1} \otimes \mathrm{id}) \to 0.$$

We also have the exact sequence $0 \to B_n \to Z_n \to H_n(X) \to 0$, whose derived exact sequence is:

$$0 \to \mathrm{Tor}_1(H_n(X), M) \to B_n \otimes M \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes M \to H_n(X) \otimes M \to 0.$$

We obtain $H_n(X) \otimes M \simeq (Z_n \otimes M)/\,\mathrm{Im}(i_n \otimes \mathrm{id})$ and $\mathrm{Tor}_1(H_{n-1}(X), M) \simeq \ker(i_{n-1} \otimes \mathrm{id})$; this proves the first assertion of the theorem.

To prove the noncanonical splitting, notice that there is a splitting $j : B_{n-1} \otimes M \to C_n \otimes M$ of Equation (2.6.1) since Z_n, C_n, B_{n-1} are all free $\mathbb{Z}$-modules. Restrict j to j' on

$$\mathrm{Tor}_1(H_{n-1}(X), M) \simeq \ker(i_{n-1} \otimes \mathrm{id}) \subset B_{n-1} \otimes M.$$

Then by the definition of the connecting homomorphism $i_{n-1} \otimes \mathrm{id}$ for Equation (2.6.1), we see that $(\partial_n \otimes \mathrm{id})j'(x) = 0 \in C_{n-1} \otimes M$ for $x \in \mathrm{Tor}_1(H_{n-1}(X), M)$. Hence j' descends to a map $\overline{j} : \mathrm{Tor}_1(H_{n-1}(X), M) \to H_n(X;M)$. The fact that j is a splitting implies that $\overline{j}$ is a splitting (check this!). □

Example: $X = \mathbb{RP}^2$. Recall that $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/2\mathbb{Z}$, and $H_2(X) = 0$. Then we compute that

$$H_0(X;\mathbb{Z}/2\mathbb{Z}) = H_0(X) \otimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z},$$
$$H_1(X;\mathbb{Z}/2\mathbb{Z}) = (H_1(X) \otimes \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$
$$H_2(X;\mathbb{Z}/2\mathbb{Z}) = \mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

3. Cohomology

3.1. **Definitions.** Given an abelian group G, recall the contravariant functor $\mathrm{Hom}(-,G) : \mathbf{Ab} \to \mathbf{Ab}$ such that $A \mapsto \mathrm{Hom}(A,G)$ and $f \in \mathrm{Hom}(A,B)$ is mapped to $f^* : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$ which sends ϕ to $\phi \circ f$.

There is also a contravariant functor $\mathrm{Hom}(-,G) : \mathbf{Ch} \to \mathbf{Ch}$ which sends a chain complex

$$\cdots \to C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0 \to 0,$$

to the *cochain complex*

$$\cdots \leftarrow \mathrm{Hom}(C_n,G) \xleftarrow{\delta_{n-1}} \cdots \xleftarrow{\delta_0} \mathrm{Hom}(C_0,G) \leftarrow 0,$$

where the cochain map is given by $\delta_{n-1} = \partial_n^*$ (note slightly awkward indexing), and $f : C_* \to D_*$ to $\mathrm{Hom}(-,G)f : \mathrm{Hom}(D_*,G) \to \mathrm{Hom}(C_*,G)$.

Now we compose with the functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{Ch}$: We denote $\mathrm{Hom}(C_n,G)$ by C^n or $C^n(G)$ or $C^n(X;G)$ if $C_n = C_n(X)$. Unwinding the definition of $\delta_{n-1} = \partial_n^*$, given $\phi \in C^{n-1}$ and $\alpha \in C_n$, we have $\delta_{n-1}\phi(\alpha) = \phi(\partial_n \alpha)$.

Claim: $\delta^2 = 0$.

Proof. This is $\partial^2 = 0$ dualized: $(\delta^2\phi)(\alpha) = \delta\phi(\partial\alpha) = \phi(\partial^2\alpha) = 0$. □

The homology of this chain complex $H^n(C;G) = \ker\delta/\operatorname{Im}\delta$ is the *nth cohomology group*.

3.2. **Interpreting cohomology.** The cochain group $C^n(X;G) = \mathrm{Hom}(C_n(X),G)$ is the set of functions

$$\phi : \{\text{singular } n\text{-simplices of } X\} \to G.$$

In particular, $\phi \in C^0(X;\mathbb{R})$ assigns a real number to each point of X and $\delta\phi \in C^1(X,\mathbb{R})$ assigns a real number to each arc $\alpha : \Delta^1 = [0,1] \to X$ as follows:

$$\delta\phi(\alpha) = \phi(\partial\alpha) = \phi(\alpha(1)) - \phi(\alpha(0)).$$

For example, if $a, b \in X$, α is a path from a to b, and $\phi(a) = r, \phi(b) = s$, then $\delta\phi(\alpha) = \phi(b) - \phi(a) = s - r$. Note that $\delta\phi = 0$ means ϕ assigns the same value to all the points in a connected component of X.

Next, if $\phi \in C^1(X;\mathbb{R})$, then $\delta\phi \in C^2(X;\mathbb{R})$ is given by:

$$\delta\phi(\alpha) = \phi(\alpha|_{[v_1,v_2]}) - \phi(\alpha|_{[v_0,v_2]}) + \phi(\alpha|_{[v_0,v_1]}).$$

Observe that $\delta\phi = 0$ means $\phi(\alpha|_{[v_0,v_2]}) = \phi(\alpha|_{[v_0,v_1]}) + \phi(\alpha|_{[v_1,v_2]})$.

One way of constructing a cochain $\phi \in C^n_{sm}(X;\mathbb{R})$, if X is a manifold and all the simplices are smooth, is to integrate an n-form ω, i.e., define

$$\phi(\beta) = \int_\beta \omega.$$

Then $\delta\phi(\alpha) = \int_{\partial\alpha}\omega = \int_\alpha d\omega$ by Stokes' Theorem. (Here d is the exterior derivative.) Hence there is a chain map of (co)-chain complexes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega^n(X;\mathbb{R}) & \xrightarrow{d} & \Omega^{n+1}(X;\mathbb{R}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & C^n_{sm}(X;\mathbb{R}) & \xrightarrow{\delta} & C^{n+1}_{sm}(X;\mathbb{R}) & \longrightarrow & \cdots \end{array}$$

Here $\Omega^n(X;\mathbb{R})$ is the space of smooth n-forms on X.

3.3. **Universal coefficient theorem.** Just as the functor $\otimes G$ is only right exact, the $\mathrm{Hom}(-,G)$ functor is only left exact:

Fact: If $A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact, then $0 \to \mathrm{Hom}(C,G) \xrightarrow{j^*} \mathrm{Hom}(B,G) \xrightarrow{i^*} \mathrm{Hom}(A,G)$ is exact.

Proof. j^* is injective: Since $j : B \to C$ is surjective, given $f \in \mathrm{Hom}(C,G)$, the composition $f \circ j(x) = 0$ for all $x \in B$ implies that $f(y) = 0$ for all $y \in B$.

Im $j^* = \ker i^*$. $\subset$: Given $f \in \mathrm{Hom}(C,G)$, $i^*j^*f = (ji)^*f = 0$. $\supset$: If $i^*g : A \to B \to G$ is zero, then we can take the quotient map $C = B/A \to G$. □

Prototypical Example: Given $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$, applying $\mathrm{Hom}(-,\mathbb{Z})$ gives:

$$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \simeq 0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z} \xrightarrow{n} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}.$$

As in the case of Tor, take a projective resolution $P_* \to A$ and apply $\mathrm{Hom}(-,G)$ to obtain:

$$\mathrm{Hom}(P_*,G) := (\cdots \leftarrow \mathrm{Hom}(P_1,G) \leftarrow \mathrm{Hom}(P_0,G) \leftarrow 0).$$

The (co)-homology of this chain complex is denoted by $\mathrm{Ext}^i(A,G)$. As before, $\mathrm{Ext}^i(A,G)$ does not depend on the choice of projective resolution.

Properties of Ext:

1. $\mathrm{Ext}^0(A,G) = \mathrm{Hom}(A,G)$. (Follows from the left exactness of the Hom functor.)
2. $\mathrm{Ext}^i(A,G) = 0$ for $i > 1$ if $R = \mathbb{Z}$. (Use the free resolution $0 \to F_1 \to F_0 \to 0$ for A.)
3. $\mathrm{Ext}^i(A,G) = 0$ if A is projective. (Use the resolution $0 \to A \to 0$ for A.)
4. $\mathrm{Ext}^i(A \oplus B,G) = \mathrm{Ext}^i(A,G) \oplus \mathrm{Ext}^i(B,G)$.
5. $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. (Take the resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0$ for $\mathbb{Z}/n\mathbb{Z}$, and dualize to obtain $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0$. Then $\mathrm{Ext}^0(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = 0$ and $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.)

Now, given an exact sequence of abelian groups $0 \to A \to B \to C \to 0$, take the corresponding exact sequence of projective resolutions $0 \to P_* \to Q_* \to R_* \to 0$ and apply $\mathrm{Hom}(-,G)$. Its long exact sequence is:

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to$$
$$\to \mathrm{Ext}^1(C,G) \to \mathrm{Ext}^1(B,G) \to \mathrm{Ext}^1(A,G) \to \dots$$

We have the following (proof similar to homology case):

Theorem 3.3.1 (Universal Coefficient Theorem for cohomology)**.** *If a chain complex C_* of free abelian groups have homology groups $H_n(C_*)$, then the cohomology groups $H^n(C_*;G)$ are given by:*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C_*),G) \to H^n(C_*;G) \to \mathrm{Hom}(H_n(C_*),G) \to 0.$$

The sequence splits, albeit noncanonically.

Example: $X = \mathbb{RP}^2$. $H^0(X;\mathbb{Z}) \simeq \mathrm{Hom}(H_0(X;\mathbb{Z}),\mathbb{Z}) = \mathbb{Z}$. (In fact, $H^0(X;\mathbb{Z}) \simeq H_0(X;\mathbb{Z})$ always.) $H^1(X;\mathbb{Z}) \simeq \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z})$ always, since $H_0(X;\mathbb{Z})$ is always free. Hence $H^1(X;\mathbb{Z}) \simeq \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) = 0$. Finally,

$$H^2(X;\mathbb{Z}) \simeq \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1(H_1(X;\mathbb{Z}),\mathbb{Z}) = \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

Remark: The Universal Coefficient Theorem basically states that $H^*(X;G)$ can be computed from the knowledge of $H_*(X;G)$. Then why do we care about cohomology?

3.4. **Properties of cohomology.** The cohomology groups satisfy all the properties enjoyed by homology, but with the arrows in the other direction. This includes the long exact sequence for relative homology, homotopy invariance, excision, Mayer-Vietoris, etc. For a while we suppress the coefficient module G.

Relative groups. Starting with $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0$, applying $\mathrm{Hom}(-,G)$ yields:

$$0 \to C^n(X,A) \xrightarrow{j^*} C^n(X) \xrightarrow{i^*} C^n(A) \to 0,$$

where $C^n(X,A) := \mathrm{Hom}(C_n(X,A),G)$. The sequence is exact since $C_n(X,A)$ is free.

Note that $\phi \in C^n(X,A)$ can be viewed as a map $\phi : \{\text{singular } n\text{-simplices of } X\} \to G$ such that $\phi(\sigma) = 0$ if $\mathrm{Im}\,\sigma \subset A$.

We also define $\delta_{X,A} : C^n(X,A) \to C^{n+1}(X,A)$ as the dual of the map $\partial_{X,A} : C_{n+1}(X,A) \to C_n(X,A)$. (For the time being also write δ_X and δ_A for the duals of ∂_X and ∂_A.)

By dualizing all the arrows for the case of homology, we obtain the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^n(X,A) & \xrightarrow{j^*} & C^n(X) & \xrightarrow{i^*} & C^n(A) & \longrightarrow & 0 \\
 & & \downarrow \delta_{X,A} & & \downarrow \delta_X & & \downarrow \delta_A & & \\
0 & \longrightarrow & C^{n+1}(X,A) & \xrightarrow{j^*} & C^{n+1}(X) & \xrightarrow{i^*} & C^{n+1}(A) & \longrightarrow & 0
\end{array}$$

and a long exact sequence in relative cohomology:

$$\to H^n(X,A) \to H^n(X) \to H^n(A) \xrightarrow{c} H^{n+1}(X,A) \to .$$

HW: Verify that $\delta_{X,A}$ agrees with the restriction of $\delta_X : C^n(X) \to C^{n+1}(X)$ to $C^n(X,A) \to C^n(X,A)$ (this is with respect to the inclusion $j^* : C^n(X,A) \to C^n(X)$).

HW: Verify that the diagram commutes, where the maps h_A and $h_{X,A}$ arise in the Universal Coefficient Theorem and the map d^* is the dual of the connecting homomorphism $b : H_{n+1}(X,A) \to H_n(A)$:

$$\begin{array}{ccc}
H^n(A) & \xrightarrow{c} & H^{n+1}(X,A) \\
\downarrow h_A & & \downarrow h_{X,A} \\
\mathrm{Hom}(H_n(A),G) & \xrightarrow{b^*} & \mathrm{Hom}(H_{n+1}(X,A),G).
\end{array}$$

Homotopy invariance. This follows from observing that if $\phi, \psi : C_* \to D_*$ are chain homotopic chain maps, i.e., there exists $h : C_* \to D_{*+1}$ such that $\phi - \psi = h\partial_C + \partial_D h$, then $\phi^*, \psi^* : \mathrm{Hom}(D_*,G) \to \mathrm{Hom}(C_*,G)$ are chain homotopic as cochain maps, i.e., there exists $h^* : \mathrm{Hom}(D_{*+1},G) \to \mathrm{Hom}(C_*,G)$ such that $\phi^* - \psi^* = \partial_C^* h^* + h^* \partial_D^*$.

Excision. Suppose $Z \subset A \subset X$ and the closure of Z is in the interior of A. Recall that the inclusion $i : (X - Z, A - Z) \to (X,A)$ induces an isomorphism on homology. The Universal Coefficient Theorem is natural in the following way (i.e., the following diagram commutes):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1(H_{n-1}(X,A),G) & \longrightarrow & H^n(X,A) & \longrightarrow & \mathrm{Hom}(H_n(X,A),G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}^1(H_{n-1}(X-Z,A-Z),G) & \longrightarrow & H^n(X-Z,A-Z) & \longrightarrow & \mathrm{Hom}(H_n(X-Z,A-Z),G) & \longrightarrow & 0. \end{array}$$

HW: Check the commutativity of the diagram.

The left and right arrows are isomorphisms; hence by the five lemma the middle one is also.

HW: Hatcher, Section 3.1: 3,7,11.

4. CUP PRODUCTS

Today we highlight the key difference between homology and cohomology: cohomology has a ring structure.

4.1. **Definition of cup product.** Given *cochains* $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ (here R is a commutative ring), define the *cup product* $\phi \cup \psi \in C^{k+l}(X;R)$ by evaluating on a singular $(k+l)$-simplex $\sigma : \Delta^{k+l} \to X$, and extending bilinearly:

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[e_0,\dots,e_k]})\psi(\sigma|_{[e_k,\dots,e_{k+l}]}).$$

Lemma 4.1.1. $\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi$.

Proof.

$$\begin{aligned}
\delta(\phi \cup \psi)(\sigma) &= (\phi \cup \psi)(\partial\sigma) = (\phi \cup \psi)(\textstyle\sum_i (-1)^i \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_{k+l+1}]}),\\
(\delta\phi \cup \psi)(\sigma) &= \delta\phi(\sigma|_{[e_0,\dots,e_{k+1}]})\psi(\sigma|_{[e_{k+1},\dots,e_{k+l+1}]})\\
&= \phi(\textstyle\sum_i (-1)^i \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_{k+1}]})\psi(\sigma|_{[e_{k+1},\dots,e_{k+l+1}]})\\
(\phi \cup \delta\psi)(\sigma) &= \phi(\sigma|_{[e_0,\dots,e_k]})\delta\psi(\sigma|_{[e_k,\dots,e_{k+l+1}]})\\
&= \phi(\sigma|_{[e_0,\dots,e_k]})(-1)^k\psi(\textstyle\sum_{i=k}^{k+l+1}(-1)^i\sigma|_{[e_k,\dots,\hat{e}_i,\dots,e_{k+l+1}]})
\end{aligned}$$

□

A *graded ring/algebra* A *over* R comes with a decomposition $A = \oplus_{i=0}^\infty A_i$ of R-modules and a multiplication $A_i \times A_j \to A_{i+j}$ which is R-bilinear. If an element $a \in A_i$ we write $|a| = i$ and call it the *degree of* a. The graded algebra A is a *differential graded algebra (dga)* if it has a differential $d : A_i \to A_{i+1}$ such that $d(ab) = (da)b + (-1)^{|a|} a db$. Hence $C^*(X;R) := \oplus_{i=0}^\infty C^i(X;R)$ is a dga with multiplication $\cup$ and differential δ.

HW: Verify that the cup product on $C^*(X;R)$ is associative.

If $\delta\phi = 0$ and $\delta\psi = 0$, i.e, both ϕ and ψ are *closed*, then $\delta(\phi \cup \psi) = 0$ by the lemma. Also, if $\phi = \delta\eta$ and $\delta\psi = 0$, i.e., ϕ is *exact* and ψ is *closed*, then $\delta(\eta \cup \psi) = \delta\eta \cup \psi = \phi \cup \psi$, i.e., $\phi \cup \psi$ is exact. Therefore, the cup product on the chain level induces a map

$$H^k(X;R) \times H^l(X;R) \stackrel{\cup}{\to} H^{k+l}(X;R).$$

Since the cup product is bilinear, by the universal property of tensor product, we have a map

$$H^k(X;R) \otimes H^l(X;R) \stackrel{\cup}{\to} H^{k+l}(X;R).$$

Hence $H^*(X;R) := \oplus_{i=0}^\infty H^i(X;R)$ is an associative graded ring.

Lemma 4.1.2. *Given $f : X \to Y$, the induced map $f^* : C^n(X;R) \to C^n(Y;R)$ satisfies*

$$f^*(\phi \cup \psi) = f^*\phi \cup f^*\psi.$$

Proof. Just unwind definitions. Given cocycles $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$,

$$\begin{aligned}
f^*(\phi \cup \psi)(\sigma) &= (\phi \cup \psi)(f \circ \sigma) = \phi(f \circ \sigma|_{[e_0,\dots,e_k]})\psi(f \circ \sigma|_{[e_k,\dots,e_{k+l}]})\\
&= f^*\phi(\sigma|_{[e_0,\dots,e_k]})f^*\psi(\sigma|_{[e_k,\dots,e_{k+l}]}) = (f^*\phi \cup f^*\psi)(\sigma).
\end{aligned}$$

□

4.2. Supercommutativity.

Theorem 4.2.1. $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$, *if* $\alpha \in H^k(X;R)$ *and* $\beta \in H^l(X;R)$.

Such a relation is called *graded commutative*, *skew-commutative*, or *supercommutative*. Note that the cup product is *NOT* supercommutative on the chain level. The proof (which is rather involved) will be postponed until next time.

Comparison with de Rham theory: If we have $\omega \in \Omega^k(X)$ and $\eta \in \Omega^l(X)$, then we have the *wedge product* $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$, already on the chain level! This is because the skew-commutativity was built into the definition: $\omega(x)(v_1, \dots, v_k) = (-1)^{|\sigma|} \omega(x)(\sigma(v_1, \dots, v_k))$, where $|\sigma|$ is the sign of a permutation σ.

Example: Consider $X = T^2$ given as in Figure 1. We will do calculations in simplicial (co)-homology. Suppose $\phi \in C^1(X)$ is given by $\phi(a) = \phi(c) = 1$ and $\phi(b) = 0$. Then $\delta\phi(A) = \delta\phi(B) = 0$. Hence

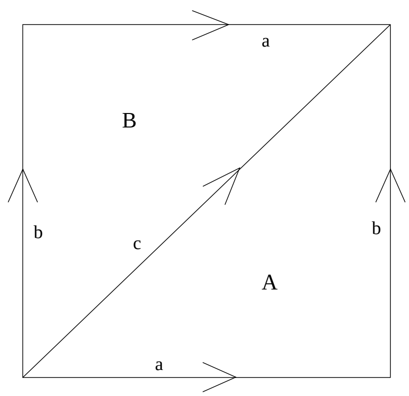

FIGURE 1.

$\delta\phi = 0$. Similarly, $\psi \in C^1(X)$ is given by $\psi(b) = \psi(c) = 1$ and $\psi(a) = 0$, and $\delta\psi = 0$. Now, $(\phi \cup \psi)(A) = \phi(a)\psi(b) = 1$ and $(\phi \cup \psi)(B) = \phi(b)\psi(a) = 0$. Since $A - B$ generates $H_2(X) = \mathbb{Z}$, we have $(\phi \cup \psi)(A - B) = 1$, in other words, $\phi \cup \psi$ generates $H^2(X) = \mathbb{Z}$. We also verify that $(\psi \cup \phi)(A - B) = -1$, although the skew-commutativity does not hold on the chain level!

Finally, we discuss a little bit of Poincaré duality. Observe that ϕ could have been defined as follows: take a closed curve b' parallel to and oriented in the same direction as b. Then $\phi(x)$ is the oriented intersection number of x with b'. Similarly ψ can be defined by taking the oriented intersection number with $-a'$. The curves b' and $-a'$ are said to be the *Poincaré duals* of ϕ and ψ.

4.3. Proof of Theorem 4.2.1.

The proof is given in several steps.

Step 1. We denote by $[e_{j_0}, \dots, e_{j_m}]$ the restriction to $\Delta^m \to \Delta^n$ of the linear map $\mathbb{R}^m \to \mathbb{R}^n$ that sends e_i, $i = 0, \dots, m$, to e_{j_i}. A particular case we are interested in is $[e_n, \dots, e_0] : \Delta^n \to \Delta^n$ which sends e_i to e_{n-i}. This induces the map

$$\rho : C^n(X) \to C^n(X),$$
$$\sigma \mapsto \varepsilon_n \sigma[e_n, \dots, e_0],$$

where ε_n is the constant $(-1)^{n(n+1)/2}$.

Step 2. ρ is a chain map, i.e., $\partial\rho = \rho\partial$.

For $\sigma : \Delta^n \to X$,

$$\partial\rho(\sigma) = \partial(\varepsilon_n \sigma[e_n, \dots, e_0]) = \varepsilon_n \sum_i (-1)^i \sigma[e_n, \dots, \widehat{e}_{n-i}, \dots, e_0],$$
$$\rho\partial(\sigma) = \rho(\sum_i (-1)^i \sigma[e_0, \dots, \widehat{e}_i, \dots, e_n]) = \varepsilon_{n-1} \sum_i (-1)^i \sigma[e_n, \dots, \widehat{e}_i, \dots, e_0]$$
$$= \varepsilon_{n-1} \sum_i (-1)^{n-i} [e_n, \dots, \widehat{e}_{n-i}, \dots, e_n],$$

and $\partial\rho = \rho\partial$ holds by observing that $\varepsilon_n = (-1)^n \varepsilon_{n-1}$.

Step 3. ρ and id are chain homotopic, i.e., there exists an operator $P : C_*(X) \to C_{*+1}(X)$ such that $\rho - \mathrm{id} = \partial P + P\partial$.

The chain homotopy operator P is a variation of the prism operator. Consider the prism $\pi : \Delta^n \times [0,1] \to \Delta^n$; label its vertices $v_i = e_i \times \{0\}$ and $w_i = e_i \times \{1\}$, $i = 0, \dots, n$. We then define

$$P\sigma = \sum_i (-1)^i \varepsilon_{n-i} \sigma\pi[v_0, \dots, v_i, w_n, \dots, w_i],$$

where $[v_0, \dots, v_i, w_n, \dots, w_i]$ refers to composition with $\Delta^n \to \Delta^n \times [0,1]$ sending e_j to the jth vertex in the list. (Recall the prism operator maps $\sigma \mapsto \sum_i (-1)^i (\sigma \times \mathrm{id})[v_0, \dots, v_i, w_i, \dots, w_n]$.)

Omitting $\sigma\pi$ from the notation, we compute:

$$\partial P = \sum_i (-1)^i \varepsilon_{n-i} \partial[v_0, \dots, v_i, w_n, \dots, w_i]$$
$$= \sum_{i \geq j} (-1)^{i+j} \varepsilon_{n-i} [v_0, \dots, \widehat{v}_j, \dots, v_i, w_n \dots, w_i] + \sum_{i \leq j} (-1)^{n-j+1} \varepsilon_{n-i} [v_0, \dots, v_i, w_n, \dots, \widehat{w}_j, \dots, w_i],$$
$$P\partial = P(\sum_j (-1)^j [v_0, \dots, \widehat{v}_j, \dots, v_n])$$
$$= \sum_{i<j} (-1)^j (-1)^i \varepsilon_{n-i-1} [v_0, \dots, v_i, w_n, \dots, \widehat{w}_j, \dots, w_i]$$
$$+ \sum_{i>j} (-1)^j (-1)^{i+1} \varepsilon_{n-i} [v_0, \dots, \widehat{v}_j, \dots v_i, w_n, \dots, w_i]$$
$$= \sum_{i>j} (-1)^{i+j+1} \varepsilon_{n-i} [v_0, \dots, \widehat{v}_j, \dots v_i, w_n, \dots, w_i]$$
$$+ \sum_{i<j} (-1)^{i+j} \varepsilon_{n-i-1} [v_0, \dots, v_i, w_n, \dots, \widehat{w}_j, \dots, w_i].$$

Apart from the $i = j$ terms in ∂P, we have cancellation by observing that $\varepsilon_{n-i} = (-1)^{n-i} \varepsilon_{n-i-1}$. The remaining terms give $\varepsilon_n [w_n, \dots, w_0] - [v_0, \dots, v_n]$ (check this!); this prove the chain homotopy.

Step 4. Recall that if two chain maps $f, g : C_* \to D_*$ are chain homotopic, then they induce the same map on cohomology $f^*, g^* : H^*(D_*) \to H^*(C_*)$. In particular, this holds for $\mathrm{id}, \rho : C_*(X) \to C_*(X)$.

Given cocycles $\phi \in C^k(X)$, $\psi \in C^l(X)$, $\rho^*(\phi \cup \psi)$ is cohomologous to $\phi \cup \psi$ and

$$\rho^*(\phi \cup \psi)(\sigma) = (\phi \cup \psi)(\rho\sigma) = \varepsilon_{k+l} \phi(\sigma[e_{k+l}, \dots, e_l]) \psi(\sigma[e_l, \dots, e_0])$$
$$= \varepsilon_{k+l} \varepsilon_l \varepsilon_k \psi(\rho(\sigma[e_0, \dots, e_l])) \phi(\rho(\sigma[e_l, \dots, e_{k+l}]))$$
$$= (-1)^{kl} \rho^*\psi \cup \rho^*\phi(\sigma).$$

by noting that $\varepsilon_{k+l} = (-1)^{kl} \varepsilon_l \varepsilon_k$. This shows that $[\phi] \cup [\psi] = (-1)^{kl} [\psi] \cup [\phi]$.

5. Computations

HW: Hatcher, Section 3.2: 1–8.

5.1. **Preliminaries.** Recall the cup product on the level of chains: If $\phi \in C^k(X)$, $\psi \in C^l(X)$, and $\sigma \in C_{k+l}(X)$, then

$$(\phi \cup \psi)(\sigma) = \phi(\sigma[e_0, \dots, e_k])\psi(\sigma[e_k, \dots, e_{k+l}]).$$

The cup product descends to cohomology and gives:

$$H^k(X) \times H^l(X) \to H^{k+l}(X),$$
$$H^k(X, A) \times H^l(X) \to H^{k+l}(X, A),$$
$$H^k(X, A) \times H^l(X, A) \to H^{k+l}(X, A).$$

The latter two are straightforward when we recall that $\phi \in C^k(X, A)$ is $\phi \in C^k(X)$ such that $\phi(C_k(A)) = 0$.

Lemma 5.1.1. *The cup product descends to a map:*

$$H^k(X, A) \times H^l(X, B) \to H^{k+l}(X, A \cup B),$$

if A, B are open subsets of X or subcomplexes of CW complexes of X.

Proof. Note that it is not clear whether $\phi \cup \psi(\sigma) = 0$ for $\phi \in C^k(X, A)$, $\psi \in C^l(X, B)$, $\sigma \in C_{k+l}(A \cup B)$.

Hence we consider the replacement

$$C^k(X, A) \times C^l(X, B) \to C^{k+l}(X, A + B),$$

where elements of $C^{k+l}(X, A + B)$ vanish on sums of chains in A and chains in B.

Now recall the inclusion $i : C_n(A + B) \to C_n(A \cup B)$. It was shown to have a chain homotopy inverse in Math 225C. Hence, dualizing, we obtain a quasi-isomorphism $i^* : C^n(A \cup B) \to C^n(A + B)$. This in turn implies that the map $C^n(X, A \cup B) \to C^n(X, A + B)$ is a quasi-isomorphism: Apply the five lemma to:

$$\begin{array}{ccccc}
H^n(X, A \cup B) & \longrightarrow & H^n(X) & \longrightarrow & H^n(A \cup B) \\
\downarrow & & \downarrow \text{id} & & \downarrow \simeq \\
H^n(X, A + B) & \longrightarrow & H^n(X) & \longrightarrow & H^n(A + B).
\end{array}$$

The lemma then follows. □

We also have a natural pairing $\langle , \rangle : H^i(X, A) \times H_i(X, A) \to R$ induced from

$$\langle , \rangle : C^i(X, A) \times C_i(X, A) \mapsto R,$$
$$(\phi, \sigma) \mapsto \langle \phi, \sigma \rangle = \phi(\sigma).$$

(Check this is well-defined!) Given $f : X \to Y$, there are maps $f_* : H_i(X) \to H_i(Y)$ and $f^* : H^i(Y) \to H^i(X)$ satisfying the adjoint condition:

$$\langle \phi, f_* \sigma \rangle = \langle f^* \phi, \sigma \rangle.$$

5.2. **Basic calculation.** Given $i+j=n$, $\mathbb{R}^n=\mathbb{R}^i\times\mathbb{R}^j$, $\mathbb{R}^i=\mathbb{R}^i\times\{0\}$, $\mathbb{R}^j=\{0\}\times\mathbb{R}^j$,

$$H^i(\mathbb{R}^n,\mathbb{R}^n-\mathbb{R}^j)\times H^j(\mathbb{R}^n,\mathbb{R}^n-\mathbb{R}^i)\xrightarrow{\cup}H^n(\mathbb{R}^n,\mathbb{R}^n-\{0\})$$

takes generator times generator to generator. Note that $\mathbb{R}^n-\{0\}=(\mathbb{R}^n-\mathbb{R}^j)\cup(\mathbb{R}^n-\mathbb{R}^i)$.

Intuition: The basic calculation is the Poincaré dual of that fact that two planes $\mathbb{R}^j$ and $\mathbb{R}^i$ of complementary dimension intersect at a point.

Consider the projections

$$\pi_1:(\mathbb{R}^n,\mathbb{R}^n-\mathbb{R}^j)\to(\mathbb{R}^i,\mathbb{R}^i-\{0\})$$
$$\pi_2:(\mathbb{R}^n,\mathbb{R}^n-\mathbb{R}^i)\to(\mathbb{R}^j,\mathbb{R}^j-\{0\})$$

that project out the $\mathbb{R}^j$-and $\mathbb{R}^i$-directions. Observe that $H_i(\mathbb{R}^i,\mathbb{R}^i-\{0\})$ is generated by an affine linear simplex $\sigma_i:\Delta^i\hookrightarrow\mathbb{R}^i$ which passes through $\{0\}$ in its interior. Let $\widetilde{\sigma}_i$ be a lift of σ_i, i.e., $\pi_1\widetilde{\sigma}_i=\sigma_i$. Let $\phi\in H^i(\mathbb{R}^i,\mathbb{R}^i-\{0\})$ be a cocycle such that $\langle\phi,\sigma_i\rangle=1$; its existence is guaranteed for example by the Universal Coefficient Theorem. Then $\langle\pi_1^*\phi,\widetilde{\sigma}_i\rangle=1$. Similarly, there exist σ_j and ψ such that $\langle\pi_2^*\psi,\widetilde{\sigma}_j\rangle=1$. Finally, it is not hard to find an n-simplex $\widetilde{\sigma}_n$ for $H_n(\mathbb{R}^n,\mathbb{R}^n-\{0\})$ whose front i face and back j face agree with some $\widetilde{\sigma}_i$ and $\widetilde{\sigma}_j$. This implies that $\langle\pi_1^*\phi\cup\pi_2^*\psi,\widetilde{\sigma}_n\rangle=1$.

5.3. **Calculation of $H^*(\mathbb{CP}^n;\mathbb{Z})$.**

Theorem 5.3.1. *$H^*(\mathbb{CP}^n;\mathbb{Z})=\mathbb{Z}[\alpha]/(\alpha^{n+1})$, and the degree of α is 2.*

Proof. We suppress $\mathbb{Z}$-coefficients and write $\mathbb{P}^n$ for $\mathbb{CP}^n$. Recall that $\mathbb{P}^n$ has a single i-cell for $i=0,2,4,\dots,2n$. Hence $H^i(\mathbb{P}^n)=\mathbb{Z}$ for $i=0,2,\dots,2n$ and 0 otherwise.

We argue by induction on n. For $n=1$, $H^*(\mathbb{P}^1)=\mathbb{Z}[\alpha]/(\alpha^2)=0$. Suppose $H^*(\mathbb{P}^{n-1})=\mathbb{Z}[\alpha]/(\alpha^n)$. We consider the inclusion $i:\mathbb{P}^{n-1}\to\mathbb{P}^n$ and the induced map $i^*:H^*(\mathbb{P}^n)\to H^*(\mathbb{P}^{n-1})$, which is an algebra homomorphism and is an isomorphism for H^i with $i\leq 2n-2$. Hence if α is a generator for $H^2(\mathbb{P}^n)$ then α^{n-1} generates $H^{2n-2}(\mathbb{P}^n)$. It remains to show that α^n generates $H^{2n}(\mathbb{P}^n)$.

Let $\mathbb{P}^i$ and $\mathbb{P}^j$, $i+j=n$, be projective planes in $\mathbb{P}^n$ that intersect transversely (i.e., in a point). The fact that they intersect in a point will be translated into $\alpha^i\cup\alpha^j$ generating $H^{2n}(\mathbb{P}^n)$.

We have the diagram

$$\begin{array}{ccc}
H^i(\mathbb{P}^n)\times H^j(\mathbb{P}^n) & \xrightarrow{\cup} & H^n(\mathbb{P}^n)\\
\uparrow a & & \uparrow b\\
H^i(\mathbb{P}^n,\mathbb{P}^n-\mathbb{P}^j)\times H^j(\mathbb{P}^n,\mathbb{P}^n-\mathbb{P}^i) & \xrightarrow{\cup} & H^n(\mathbb{P}^n,\mathbb{P}^n-\{pt\})\\
\downarrow c & & \downarrow d\\
H^i(\mathbb{C}^n,\mathbb{C}^n-\mathbb{C}^j)\times H^j(\mathbb{C}^n,\mathbb{C}^n-\mathbb{C}^i) & \xrightarrow{\cup} & H^n(\mathbb{C}^n,\mathbb{C}^n-\{0\}).
\end{array}$$

This commutes by the naturality of the cup product.

It remains to show that the maps a,b,c,d are isomorphisms. b is an isomorphism by the relative exact sequence for the pair $(\mathbb{P}^n,\mathbb{P}^n-\{pt\})$. d is an isomorphism by excision (excise $Z=\mathbb{P}^{n-1}$). For one of the components of a, we consider the relative sequence for $(\mathbb{P}^n,\mathbb{P}^n-\mathbb{P}^j)$, observing that $H^i(\mathbb{P}^n-\mathbb{P}^j)=0$ since $\mathbb{P}^n-\mathbb{P}^j$ deformation retracts onto $\mathbb{P}^{i-1}$ (HW!). c is similar. $\square$

6. EILENBERG-ZILBER THEOREM

This presentation follows Greenberg-Harper, "Algebraic Topology".

6.1. **Statement of theorem.**
Given chain complexes (C_*, ∂) and (D_*, ∂'), their tensor product can also be viewed as a chain complex where $(C_* \otimes D_*)_n := \oplus_i (C_i \otimes D_{n-i})$ and the boundary map is

$$(\partial \otimes \partial')(x \otimes y) = \partial x \otimes y + (-1)^i x \otimes \partial' y,$$

where $x \in C_i$ and $y \in D_{n-i}$.

Question: Do you need the $(-1)^i$?

Goal: Given topological spaces X and Y, relate $C_*(X) \otimes C_*(Y)$ and $C_*(X \times Y)$.

There is a map going one way, called the *Eilenberg-Zilber map*:

$$A : C_n(X \times Y) \mapsto (C_*(X) \otimes C_*(Y))_n,$$

$$\sigma = (\pi_X \sigma, \pi_Y \sigma) \mapsto \sum_{i=0}^{n} \pi_X \sigma[e_0, \dots, e_i] \otimes \pi_Y \sigma[e_i, \dots, e_n],$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are projections.

HW: Verify that A is a chain map.

We now view this more categorically: Let $\mathbf{Top} \times \mathbf{Top}$ be the product of $\mathbf{Top}$ with itself, i.e., the objects are pairs (X, Y) of spaces and the morphisms are pairs of maps $(X, Y) \xrightarrow{(f,g)} (X', Y')$. Then there exist two functors $F, F' : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ch}$,

$$F(X, Y) = C_*(X \times Y), \quad F'(X, Y) = C_*(X) \otimes C_*(Y).$$

HW: Check these are actually functors.

Lemma 6.1.1. *$A : F \Rightarrow F'$ is a natural transformation.*

Proof. We unwind definitions: Given $(X, Y) \xrightarrow{(f,g)} (X', Y')$, we verify the commutativity of the diagram

$$\begin{array}{ccc} C_n(X \times Y) & \xrightarrow{F(f,g)} & C_n(X' \times Y') \\ \downarrow A & & \downarrow A \\ (C_*(X) \otimes C_*(Y))_n & \xrightarrow{F'(f,g)} & (C_*(X') \otimes C_*(Y'))_n. \end{array}$$

One way maps

$$\sigma = (\pi_X \sigma, \pi_Y \sigma) \overset{A}{\mapsto} \sum_i \pi_X \sigma[e_0, \dots, e_i] \otimes \pi_Y \sigma[e_i, \dots, e_n]$$

$$\overset{F'(f,g)}{\mapsto} \sum_i f \pi_X \sigma[e_0, \dots, e_i] \otimes g \pi_Y \sigma[e_i, \dots, e_n].$$

The other way gives the same result. □

Theorem 6.1.2 (Eilenberg-Zilber)**.** *There exists a natural transformation $B : F' \Rightarrow F$ such that $AB : F' \Rightarrow F'$ and $BA : F \Rightarrow F$ are chain homotopic.*

What do we mean by "chain homotopic" natural transformations? Given functors $F, F' : \mathcal{C} \to \mathbf{Ch}$, two natural transformations $A, B : G \Rightarrow G'$ are *chain homotopic* if there exist $H(X) : F(X) \mapsto F'(X)$, where $F_i(X)$ is mapped to $F'_{i+1}(X)$ such that $A(X) - B(X) = \partial H(X) - H(X)\partial$, and such that the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{Ff} & F(Y) \\ \Big\downarrow{\scriptstyle H(X)} & & \Big\downarrow{\scriptstyle H(Y)} \\ F'(X) & \xrightarrow{F'f} & F'(Y). \end{array}$$

In particular, $A : C_*(X \otimes Y) \to (C_*(X) \otimes C_*(Y))$ is a chain homotopy equivalence.

Remark 6.1.3. The homotopy inverse $B : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ can actually be given explicitly using "shuffle homomorphisms".

Corollary 6.1.4 (Künneth formula)**.** *If R is a principal ideal domain (PID), then*

$$H_n(X \times Y) \simeq (\oplus_{i=0}^n H_i(X) \otimes H_{n-i}(Y)) \oplus (\oplus_{i=0}^n \operatorname{Tor}_1(H_i(X), H_{n-i-1}(Y))).$$

The proof is very similar to the proof of the Universal Coefficient Theorem and can be found on pp. 253-256 of Greenberg-Harper.

6.2. **Acyclic models.** A *category with models* $(\mathcal{A}, \mathcal{M})$ is a category $\mathcal{A}$ with a set of objects $\mathcal{M}$, called the set of *models*.

If $F : \mathcal{A} \to \mathbf{Ch}$ is a functor, then let $F_i : \mathcal{A} \to R - \mathbf{Mod}$ be the functor which sends X to the degree i part $(F(X))_i$ of $F(X)$. (For today assume that $\mathbf{Ch} = \mathbf{Ch}_{\geq 0}$, i.e., chain complexes such that $C_i = 0$ for $i < 0$.)

Definition 6.2.1. A *basis for* F_i is a collection $\{d_N \in F_i(N) \mid N \in \mathcal{N}_i\}$, where $\mathcal{N}_i \subset \mathcal{M}$, such that, for any $X \in ob(X)$, $F_i(X)$ is the free R-module generated by

$$\{F_i(u)(d_N) \mid N \in \mathcal{N}_i, u \in \operatorname{Hom}(N, X)\}.$$

We say F is *free* if all the F_i have bases for $i \geq 0$.

Examples. The functors $F, F' : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ch}$

$$F(X, Y) = C_*(X \times Y), \quad F'(X, Y) = C_*(X) \otimes C_*(Y).$$

from the previous subsection have models $\mathcal{M} = \{(\Delta^i, \Delta^j), i, j \geq 0\}$ and are free.

1. For F_n, choose $\mathcal{N}_n = \{(\Delta^n, \Delta^n)\}$ and $d_i \in F_n(\Delta^n, \Delta^n) = C_n(\Delta^n \times \Delta^n)$ given by the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$. Given $F_n(X, Y) = C_n(X \times Y)$, consider $\sigma : \Delta^n \to X \times Y$. It can be written as a composition

$$\Delta^n \xrightarrow{d_n} \Delta^n \times \Delta^n \overset{u=(\pi_X\sigma,\pi_Y\sigma)}{\longrightarrow} X \times Y,$$

so that $\sigma = F_n(\pi_X \sigma, \pi_Y \sigma)(d_n)$.

2. For F'_n, choose $\mathcal{N}_n = \{(\Delta^i, \Delta^j) \mid i + j = n\}$ and $\delta_i \otimes \delta_j \in F'_n(\Delta^i, \Delta^j) = (C_*(\Delta^i) \otimes C_*(\Delta^j))_n$, such that $\delta_i = \mathrm{id} : \Delta^i \to \Delta^i$ and $\delta_j = \mathrm{id} : \Delta^j \to \Delta^j$. Given $\sigma \otimes \tau \in F'_n(X, Y) = (C_*(X) \otimes C_*(Y))_n$ where $\sigma : \Delta^i \to X$ and $\tau : \Delta^j \to Y$, $\sigma \otimes \tau = F'_n(\sigma \times \tau)(\delta_i \otimes \delta_j)$.

An *augmented chain complex* is a chain complex with $C_i = 0$ for $i < -1$, $C_{-1} = R$, and $C_0 \xrightarrow{\varepsilon} C_{-1}$ surjective. Recall that given $C_*(X)$ we can extend it to an augmented chain complex $\cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} R \to 0$ by setting $\varepsilon(\sum_i a_i\sigma_i) = \sum_i a_i$. Its homology is the *reduced homology of* X.

Let $\mathbf{Ch}' \subset \mathbf{Ch}_{\geq -1}$ be the (full) subcategory of augmented chain complexes. We can view F and F' as functors to $\mathbf{Ch}'$ instead by augmenting $C_*(X \times Y)$ and $C_*(X) \otimes C_*(Y)$.

Definition 6.2.2. A functor $F : \mathcal{A} \to \mathbf{Ch}'$ is *acyclic* if for every $H_*(F(M)) = 0$ for every $M \in \mathcal{M}$.

Examples. F and F' are both acyclic. F is clear, since $\Delta^i \times \Delta^j$ is contractible and hence its reduced homology $\widetilde{H}_*(\Delta^i \times \Delta^j) = 0$. The case of F' is HW.

Theorem 6.2.3. *Let $(\mathcal{A}, \mathcal{M})$ be a category with models and $F, F' : \mathcal{A} \to \mathbf{Ch}'$ be functors such that F is free and F' is acyclic. Then there is a natural transformation $\Phi : F \Rightarrow F'$ which is unique up to chain homotopy.*

Corollary 6.2.4. *If F and F' are both free and acyclic, then F, F' are chain homotopy equivalent.*

Proof of Theorem 6.2.3. We will prove the existence of the natural transformation Φ. The definition of the chain homotopy H is similar.

For each $X \in ob(\mathcal{A})$ we want to define $\Phi(X) : F(X) \to F'(X)$. Let $\{d_N \in F(N)\}_{N \in \mathcal{N}_i}$ be a basis for F_i. Then $F_i(X)$ is freely generated by $F_i(u)(d_N)$ as d_N ranges in $\mathcal{N}_i$ and u ranged in $\mathrm{Hom}(N, X)$.

Step 1. Suppose we have defined $\Phi_i(N)(d_N)$ for all $N \in \mathcal{N}_i$. Then $\Phi_i(X) : F_i(X) \to F_i'(X)$ is determined by the following commutative diagram:

$$\begin{array}{ccc} F_i(N) & \xrightarrow{\Phi_i(N)} & F_i'(N) \\ {\scriptstyle F_i(u)}\downarrow & & \downarrow{\scriptstyle F_i'(u)} \\ F_i(X) & \xrightarrow{\Phi_i(X)} & F_i'(X), \end{array}$$

i.e., $\Phi_i(X)(F_i(u)(d_N)) = F_i'(u)(\Phi_i(N)(d_N))$.

We need to verify the naturality of $\Phi_i : F_i \Rightarrow F_i'$, i.e., for $f : X \to Y$ with $\Phi_i(X)$, $\Phi_i(Y)$ that we just defined, the diagram is commutative:

$$\begin{array}{ccc} F_i(X) & \xrightarrow{\Phi_i(X)} & F_i'(X) \\ {\scriptstyle F_i(f)}\downarrow & & \downarrow{\scriptstyle F_i'(f)} \\ F_i(Y) & \xrightarrow{\Phi_i(Y)} & F_i'(Y). \end{array}$$

This is HW. Note that this step only uses the freeness of F.

Step 2. By induction on i we define $\Phi_i(X)$ for all $X \in ob(\mathcal{A})$ so that $\Phi_i : F_i \Rightarrow F_i'$ is a natural transformation. Suppose for all $j < i$ we have defined (the unique) $\Phi_j(X)$ for all $X \in ob(\mathcal{A})$ so that $\Phi_j : F_j \Rightarrow F_j'$ is a natural transformation.

We define $\Phi_i(N)(d_N)$, $N \in \mathcal{N}_i$, to be any element such that $\partial\Phi_i(N)(d_N) = \Phi_{i-1}(N)\partial d_N$; such an element exists by the acyclicity of F' and the fact that $\Phi_{i-1}(N)\partial d_N$ is closed.

$$\begin{array}{ccc} F_i(N) & \xrightarrow{\partial} & F_{i-1}(N) \\ {\scriptstyle \Phi_i(N)}\downarrow & & \downarrow{\scriptstyle \Phi_{i-1}(N)} \\ F'_i(N) & \xrightarrow{\partial} & F'_{i-1}(N). \end{array}$$

Then by Step 1 we can define $\Phi_i(X)$ for all $X \in ob(\mathcal{A})$.

When $i = 0$, we similarly define $\Phi_0(N)(d_N)$ to be any element such that $\varepsilon'\Phi_0(N)(d_N) = \varepsilon d_N$:

$$\begin{array}{ccc} F_0(N) & \xrightarrow{\varepsilon} & R \\ {\scriptstyle \Phi_0(N)}\downarrow & & \downarrow\simeq \\ F'_0(N) & \xrightarrow{\varepsilon'} & R, \end{array}$$

and extend to $\Phi_0(X)$ using Step 1.

Step 3. To combine the natural transformations Φ_i into Φ, we need to verify that $\partial\Phi_i(X) = \Phi_{i-1}(X)\partial$:

$$\begin{aligned} \partial\Phi_i(X)(F_i(u)(d_N)) &= \partial(F'_i(u)\Phi_i(N)d_N) = F'_{i-1}(u)\partial(\Phi_i(N)d_N) = F'_{i-1}(u)\Phi_{i-1}(N)\partial d_N \\ &= \Phi_{i-1}(X)F_{i-1}(u)\partial d_N = \Phi_{i-1}(X)\partial(F_i(u)d_N). \end{aligned}$$

□

7. Orientations

Today our topological spaces are topological manifolds M of dimension n, i.e., Hausdorff topological spaces covered by open sets homeomorphic to $\mathbb{R}^n$.

7.1. **Definitions.**

Basic Calculation. Given $x \in M$,

$$H_n(M, M - x) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - x) \simeq H_{n-1}(\mathbb{R}^n - x) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z}.$$

We write $H_n(M|A) := H_n(M, M - A)$.

Definition 7.1.1.

(1) An *orientation at* x is a choice of generator of $H_n(M|x)$.
(2) An *orientation of* M is a function $x \mapsto \mu_x \in H_n(M|x)$ with local consistency: for each $x \in M$ there exists an open ball $B \ni x$ and $\mu_B \in H_n(M|B)$ such that $\mu_y = \phi_{B,y}\mu_B$ for all $y \in B$ under the natural map $\phi_{B,y} : H_n(M|B) \to H_n(M|y)$.
(3) If an orientation exists for M, then M is *orientable*.

Recall that if M is smooth, then M is orientable if the "determinant line bundle" $\Lambda^n T^* M$ is trivial (take this as the definition for M smooth), which is equivalent to the existence of a nonvanishing section (aka volume form) $M \to \Lambda^n T^* M$. This is equivalent to a smooth (with respect to M) choice of orientation of $T_x^* M$.

Orientation double cover. Denote the units of $H_n(M|x)$ by $H_n(M|x)^\times$. Then

$$\widetilde{M} = \coprod_{x \in M} H_n(M|x)^\times$$

and there is a map $\pi : \widetilde{M} \to M$ which sends $H_n(M|x)^\times \to x$. We topologize $\widetilde{M}$ by choosing a basis $\{U_{B,\mu_B}\}$ so that $\pi : \widetilde{M} \to M$ is a covering space: Given an open ball $B \ni x$ and a generator μ_B of $H_n(M|B)$, let $U_{B,\mu_B} := \{\phi_{B,y}(\mu_B) \mid y \in B\}$.

Claim/HW: $\widetilde{M}$ is orientable.

Lemma 7.1.2. *If M is connected, then $\widetilde{M}$ is orientable if and only if $\widetilde{M}$ has 2 connected components.*

Proof. If $\widetilde{M}$ has 2 components, then each sheet is homeomorphic to M, and hence M is orientable. If M is orientable, then it has 2 orientations and each orientation defines a component. □

Analogously, we can define R-orientations for R a commutative ring with identity in a similar manner. Note that $H_n(M|x; R) \simeq H_n(M|x) \otimes R \simeq R$. Then:

- an *R-orientation at* x is a choice of unit $u \in H_n(M|x; R)^\times$,
- there is a covering space $M_{R^\times} \to M$ with fibers $H_n(M|x; R)^\times$, and
- an *R-orientation of* M is a section of $M_{R^\times} \to M$.

7.2. **Fundamental class.**

Theorem 7.2.1. *Let M be a closed (= compact without boundary), connected n-manifold.*

(A) *If M is R-orientable, then $H_n(M;R) \to H_n(M|x;R)$ is an isomorphism for all $x \in M$.*
(B) *If M is not R-orientable, then $H_n(M;\mathbb{R}) \to H_n(M|x;R) \simeq R$ is an injection with image $\{r \in R \mid 2r = 0\}$.*
(C) *$H_i(M;R) = 0$ for $i > n$.*

A generator of $H_n(M;R) \simeq R$ is called a *fundamental class* for M with respect to R.

Proof. We'll prove (A) and (C), leaving (B) for HW. If M is R-orientable, then there is a section σ of $M_{R^\times} \to M$.

We inductively prove that for all $A \subset M$ compact,

(i) there exists a unique class $\sigma_A \in H_n(M|A,R)$ such that $\phi_{A,x}(\sigma_A) = \sigma(x)$, and
(ii) $H_i(M|A;R) = 0$ for $i > n$.

Suppressing R, we show the following:

(1) If (i) and (ii) hold for compact sets A, B, $A \cap B$, then they hold for $A \cup B$.
(2) Write A as a union of $A_1, \dots, A_n$, where each A_i is contained in an open ball. Since $H_n(M|A_i) \simeq H_n(\mathbb{R}^n|A_i)$ by excision, we reduce to the case $M \simeq \mathbb{R}^n$.
(3) If A is a closed ball in $M = \mathbb{R}^n$, (i) and (ii) are immediate.
(4) Argue for an arbitrary compact set $A \subset \mathbb{R}^n$.

(1) follows from the Mayer-Vietoris sequence

$$0 \longrightarrow H_n(M|A \cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A \cup B),$$

where $\Phi(\alpha) = (\alpha, -\alpha)$ and $\Psi(\alpha, \beta) = \alpha + \beta$. In particular $\Psi(\sigma_A, -\sigma_B) = \sigma_A - \sigma_B = 0$, and comes from some $\sigma_{A\cup B}$ under the map Φ. (Check consistency + uniqueness!)

(4) If A is an arbitrary compact set in $\mathbb{R}^n$, suppose $\alpha \in H_i(\mathbb{R}^n|A)$ is represented by a relative cycle z. Then $\mathrm{Supp}(\partial z)$ is a compact subset of $\mathbb{R}^n - A$, and hence has positive finite distance to A. We can then cover A with closed balls $B_1, \dots, B_k$ disjoint from $\mathrm{Supp}(\partial z)$ and view it as an element of $H_i(\mathbb{R}^n| \cup_i B_i)$ which maps to α under $H_i(\mathbb{R}^n| \cup_i B_i) \to H_i(\mathbb{R}^n|A)$. If $i > n$, then (3) implies that $\alpha = 0$. If $i = n$, then existence in (i) is clear. (Check uniqueness!) □

8. POINCARÉ DUALITY

HW: Hatcher, Section 3.3: 1,2,5,6,16,22,26.

8.1. **Cap product.** The *cap product* is defined as follows:

$$\cap : C_k(X) \times C^l(X) \to C_{k-l}(X)$$
$$(\sigma, \phi) \mapsto \phi(\sigma[e_0, \dots, e_l])\sigma[e_l, \dots, e_k].$$

Lemma 8.1.1. $\partial(\sigma \cap \phi) = (-1)^l(\partial\sigma \cap \phi - \sigma \cap \delta\phi)$.

This is left for HW; it's the same kind of verification as for the cup product. The lemma implies that $\cap$ descends to:

$$\cap : H_k(X) \times H^l(X) \to H_{k-l}(X).$$

Naturality. Given a map $f : X \to Y$, consider the diagram

$$\begin{array}{ccc} H_k(X) \times H^l(X) & \xrightarrow{\cap} & H_{k-l}(X) \\ \downarrow f_* \quad f^* \uparrow & & \downarrow f_* \\ H_k(Y) \times H^l(Y) & \xrightarrow{\cap} & H_{k-l}(Y). \end{array}$$

Lemma 8.1.2. $f_*\sigma \cap \phi = f_*(\sigma \cap f^*\phi)$.

Proof. Follows from $\phi(f\sigma[e_0, \dots, e_l])f\sigma[e_l, \dots, e_k] = f^*\phi(\sigma[e_0, \dots, e_l])f\sigma[e_l, \dots, e_k]$. □

8.2. **Statement of Poincaré duality.**

Theorem 8.2.1. *Let M be a closed R-orientable n-manifold and $[M]$ an R-fundamental class. Then*

$$[M]\cap : H^k(M; R) \to H_{n-k}(M; R)$$

is an isomorphism.

The most important thing to know about Poincaré duality is that it is really a *local result.*

Theorem 8.2.2. *There exists a cohomology theory $H^i_c(X; R)$ called* compactly supported cohomology *and a duality map*

$$D_M : H^k_c(M; R) \xrightarrow{\sim} H_{n-k}(M; R).$$

In particular, we may take $M = \mathbb{R}^n$.

8.3. **Compactly supported cohomology.** Suppress the coefficient ring R. The chain complex $C^i_c(X)$ consists of $\phi \in C^i(X)$ such that there exists a compact set $K \subset X$ for which $\phi(\sigma) = 0$ for all σ with support on $X - K$. Since $\delta\phi$ also has the same property, $\delta\phi \in C^{i+1}_c(X)$ and $(C^*_c(X), \delta)$ is a cochain complex. Its cohomology is $H^i_c(X)$, called *compactly supported cohomology*.

Comparison with de Rham theory. Recall from Math 225B the de Rham complex:

$$\dots \to \Omega^i(M) \xrightarrow{d} \Omega^{i+1}(M) \to \dots.$$

In this setting we defined the compactly supported i-forms by

$$\Omega^i_c(M) = \{\omega \in \Omega^i(M) \mid \text{Supp}(\omega) \text{ is compact}\},$$

and its cohomology is $H^i_{c,dR}(M)$.

Omitting "dR", the version of Poincaré duality for de Rham theory is that

$$H^i_c(M) \times H^{n-i}(M) \to H^n_c(M) \xrightarrow{\sim} \mathbb{R}$$
$$(\omega, \eta) \mapsto \omega \wedge \eta \mapsto \int_M \omega \wedge \eta$$

is a nondegenerate pairing, i.e., for each $\omega \in H^i_c(M)$ there exists $\eta \in H^{n-i}(M)$ such that $\int \omega \wedge \eta \neq 0$.

Compare this to the situation in singular theory (ignoring signs):

$$H^i_c(M) \times H^{n-i}(M) \overset{([M]\cap, \mathrm{id})}{\longrightarrow} H_{n-i}(M) \times H^{n-i}(M) \overset{\langle,\rangle}{\longrightarrow} R,$$
$$(\phi, \psi) \mapsto ([M] \cap \phi, \psi) \mapsto \langle [M] \cap \phi, \psi \rangle = \langle [M], \phi \cup \psi \rangle.$$

Hence $[M]\cap$ corresponds to integration in de Rham theory.

We will now explain a more algebraic definition of $H^i_c(X)$ in terms of direct limits. The initial observation is the following: Each $\phi \in C^i_c(X)$ belongs to some $C^i(X|K)$ for $K \subset X$ compact. Also if $K \subset L$ there is an inclusion of chain complexes

$$\phi_{K,L} : C^i(X|K) \to C^i(X|L)$$

and a corresponding map on cohomology

$$\phi_{K,L} : H^i(X|K) \to H^i(X|L).$$

Definition 8.3.1. A *directed system* is a partially ordered set $(I, \leq)$ such that given $\alpha, \beta \in I$ there exists γ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

In our case the directed system I is the set of compact subsets of X and $\leq$ is given by inclusion.

A directed system $(I, \leq)$ can be viewed as a category $\mathcal{I}$ such that $ob(\mathcal{I}) = I$ and $\mathrm{Hom}(\alpha, \beta)$ is one element if $\alpha \leq \beta$ and empty otherwise.

Definition 8.3.2. Given a functor $F : \mathcal{I} \to \mathcal{C}$ (so this means we have assignments $\alpha \mapsto F(\alpha)$ and morphisms $F_{\alpha\beta} : F(\alpha) \to F(\beta)$), its *direct limit* or *colimit* is an object C in $\mathcal{C}$ together with morphisms $\{\phi_\alpha : F(\alpha) \to C\}$ such that if $\alpha \leq \beta$ then $\phi_\alpha = \phi_\beta F_{\alpha\beta}$, and it satisfies the universal property: If A is any other object in $\mathcal{C}$ together with morphisms $\psi_\alpha : F(\alpha) \to A$ satisfying $\psi_\alpha = \psi_\beta F_{\alpha\beta}$, then there is a unique morphism $f : C \to A$ such that $\psi_\alpha = f\phi_\alpha$.

Slightly confusing notation: $\lim_\to F(\alpha)$ or $\mathrm{colim}_\to F(\alpha)$.

Lemma 8.3.3. *Colimits exist in* $\mathbf{Ab}$ *or* $R - \mathbf{Mod}$.

Proof. This is an explicit construction: Take $\sqcup_\alpha F(\alpha)/\sim$, where $x_\alpha \in F(\alpha)$ is identified with $F_{\alpha\beta}(x_\alpha) \in F(\beta)$. Check the universal property! ∎

HW: Show that taking direct limits commutes with taking homology in $R - \mathbf{Mod}$. In particular, the direct limit of exact sequences is exact.

Claim 8.3.4. $\lim_\to H^i(X|L) = H^i_c(X)$.

Proof. This is a straightforward unwinding of the definitions. An element of $\lim_\to H^i(X|L)$ is represented by a cocycle $\phi \in C^i(X|K)$ for some K and $[\phi] = 0 \in \lim_\to H^i(X|L)$ if and only if ϕ is a coboundary in some $C^i(X|L)$ for $K \subset L$. ∎

8.4. **Definition of D_M.** We now define $D_M : H^i_c(M) \to H_{n-i}(M)$ for an R-oriented M. For each compact set $K \subset M$, we have the "fundamental class" μ_K and we define

$$\mu_K \cap : H^i(M|K) \to H_{n-i}(M).$$

When $K \subset L$ compact, we want to show that $\mu_K \cap = \mu_L \cap \circ \phi^*_{K,L}$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} H^i(M|K) & \xrightarrow{\mu_K \cap} & H_{n-i}(M) \\ \downarrow \phi^*_{K,L} & \nearrow \mu_L \cap & \\ H^i(M|L) & & \end{array}$$

This follows from $\mu_K \cap \phi = \phi_{L,K}(\mu_L) \cap \phi = \mu_L \cap \phi^*_{L,K}(\phi)$. By the universal property of direct limits, we have a unique map

$$D_M : H^i_c(M) \to H_{n-i}(M).$$

8.5. **Proof of Poincaré duality.** The proof is given in several steps.

Step 1. The base case $M = \mathbb{R}^n$. Let $B_j \subset \mathbb{R}^n$ be a closed ball of radius j about the origin. Then $H^i_c(\mathbb{R}^n) = \lim_{j\to\infty} H^i(R^n|B_j)$. Using excision etc., we find that $H^i(\mathbb{R}^n|B_j) \simeq R$ if $i = n$ and 0 otherwise; also $H^i(\mathbb{R}^n|B_j) \xrightarrow{\sim} H^i(\mathbb{R}^n|B_{j+1})$. Hence $H^i_c(\mathbb{R}^n) \simeq R$ if $i = n$ and 0 otherwise. Since $H_{n-i}(M) \simeq R$ if $i = n$ and 0 otherwise, it suffices to check

$$\mu_{B_j} \cap : H^n(\mathbb{R}^n|B_j) \xrightarrow{\sim} H_0(\mathbb{R}^n).$$

This is immediate from taking μ_{B_j} to be an "embedded" n-simplex which contains B_j and taking the generator of $H^n(\mathbb{R}^n|B_j)$ to evaluate to 1 on μ_{B_j}.

Step 2. Assume the following diagram is sign-commutative: For U, V open and $M = U \cup V$,

(8.5.1)
$$\begin{array}{ccccccc} H^k_c(U\cap V) & \longrightarrow & H^k_c(U) \oplus H^k_c(V) & \longrightarrow & H^k_c(M) & \longrightarrow & H^{k+1}_c(U\cap V) \\ \downarrow D_{U\cap V} & & \downarrow D_U \oplus -D_V & & \downarrow D_M & & \downarrow D_{U\cap V} \\ H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U\cap V). \end{array}$$

By the five lemma, if $D_U, D_V, D_{U\cap V}$ are isomorphisms, then so is D_M.

Step 3. Suppose there is a filtration $U_1 \subset U_2 \subset \dots$ of open sets such that $\cup_i U_i = M$ and D_{U_i} is an isomorphism, then so is D_M: Observe that

$$H^k_c(U_i) = \lim_{K\subset U_i} H^k(U_i|K) \simeq \lim_{K\subset U_i} H^k(M|K)$$

and hence

$$H^k_c(M) = \lim_{i\to\infty} H^k_c(U_i).$$

Similarly we can prove that $H_{n-k}(M) = \lim_{i\to\infty} H_{n-k}(U_i)$ (HW). It is also easy to show (HW) the commutativity of:

$$\begin{array}{ccc} H^k_c(U_i) & \xrightarrow{D_{U_i}} & H_{n-k}(U_i) \\ \downarrow & & \downarrow \\ H^k_c(U_{i+1}) & \xrightarrow{D_{U_{i+1}}} & H_{n-k}(U_{i+1}). \end{array}$$

Hence D_{U_i} isomorphism for each i implies D_M isomorphism.

Step 4. For M an open subset of $\mathbb{R}^n$, cover M by countably many open balls V_i; their finite intersections are convex open balls $\simeq \mathbb{R}^n$. Then by repeated application of Step 2, for $U_i := \cup_{j\le i} V_j$, D_{U_i} is an isomorphism. Next Step 3 applied to $U_1, U_2, \dots$ implies that D_M is an isomorphism. For second countable M, cover M by countably many open balls V_i; their finite intersections are open subsets of $\mathbb{R}^n$ for which D_* is an isomorphism. Hence the same exhaustion procedure shows that D_M is an isomorphism for M second countable. In general, need to use Zorn's lemma....

8.6. **Proof of** (8.5.1)**.** We need to verify three things:

(a) The exactness of the top row.
(b) The commutativity of the left two squares.
(c) The commutativity of the right square.

(a) follows from the exactness of

$$\dots \longrightarrow H^k(M|K\cap L) \longrightarrow H^k(M|K)\oplus H^k(M|L) \longrightarrow H^k(M|K\cup L) \longrightarrow \dots,$$

where $K\subset U$ and $L\subset V$ compact, identifications $H^k(M|K\cap L)\simeq H^k(U\cap V|K\cap L)$ and $H^k(M|K)\simeq H^k(U|K)$, and the fact that the direct limit of an exact sequence is exact.

(b) follows from the commutativity of

$$\begin{array}{ccc} H^k(M|K\cap L) & \longrightarrow & H^k(M|K) \\ \downarrow{\scriptstyle \mu_{K\cap L}\cap} & & \downarrow{\scriptstyle \mu_K\cap} \\ H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U). \end{array}$$

Since $\phi_{K,K\cap L}\mu_K = \mu_{K\cap L}$, it suffices to show that $i_{U\cap V,U}(\phi_{K,K\cap L}\mu_K\cap\phi) = \mu_K\cap\phi^*_{K,K\cap L}(\phi)$, where $\phi\in H^k(M|K\cap L)$. Take a representative a of μ_K with support in U and a representative α of ϕ with support on $U\cap V$. Then both sides give $\langle a,\alpha\rangle$. The three other squares are analogous.

(c) is the hardest part:

$$\begin{array}{ccccc} H^k(M|K\cup L) & \xrightarrow{r} & H^{k+1}(M|K\cap L) & \xrightarrow{\sim} & H^{k+1}(U\cap V|K\cap L) \\ \downarrow{\scriptstyle \mu_{K\cup L}\cap} & & & & \downarrow{\scriptstyle \mu_{K\cap L}\cap} \\ H_{n-k}(M) & & \xrightarrow{\qquad s \qquad} & & H_{n-k-1}(U\cap V) \end{array}$$

Write $A = M-K$ and $B = M-L$. We first unwind the definitions of the connecting homomorphisms r, s. For r, consider the short exact sequence:

$$0 \longrightarrow C^*(M,A+B) \longrightarrow C^*(M,A)\oplus C^*(M,B) \longrightarrow C^*(M,A\cap B) \longrightarrow 0.$$

where $C^*(M, A+B)$ means cochains vanishing on chains in A and on chains in B. Starting with $\phi \in C^i(M, A\cap B)$ with $\delta\phi = 0$, we pick $(\phi_A, -\phi_B) \in C^i(M,A) \oplus C^i(M,B)$ so that $\phi = \phi_A - \phi_B$. Then $r(\phi) = \delta\phi_A = \delta\phi_B$. Similarly, starting with $z \in C_i(M)$ with $\partial z = 0$, we pick $(z_U, -z_V) \in C_i(U) \oplus C_i(V)$ such that $z = z_U - z_V$. Then $s(z) = z_U = z_V$.

We also decompose $\mu_{K\cup L} = \alpha_{U-L} + \alpha_{U\cap V} + \alpha_{V-K}$ (why can we do this?) with supports on $U-L, U\cap V, V-K$, respectively. Draw a picture of this! Note that $\alpha_{U-L} + \alpha_{U\cap V}$ represents μ_K.

We then compute

$$\begin{aligned} s(\mu_{K\cup L} \cap \phi) &= s(\alpha_{U-L} \cap \phi + (\alpha_{U\cap V} + \alpha_{V-K}) \cap \phi) \\ &= \partial(\alpha_{U-L} \cap \phi) \\ &= \pm\partial\alpha_{U-L} \cap \phi = \pm\partial\alpha_{U-L} \cap (\phi_A - \phi_B) \\ &= \pm\partial\alpha_{U-L} \cap \phi_A \\ &= \pm\partial\alpha_{U\cap V} \cap \phi_A, \end{aligned}$$

where the first line to the second follows from observing that α_{U-L} is supported in U and $\alpha_{U\cap V} + \alpha_{V-K}$ is supported in V; the third line to the fourth follows since ϕ_B is zero on $U - L$; the the fourth to fifth follows since ϕ_A is zero on $V - K$ and $\partial(\alpha_{U-L} + \alpha_{U\cap V})$ vanishes on $M - K$.

We also compute

$$\mu_{K\cap L} \cap r(\phi) = \mu_{K\cap L} \cap \delta\phi_A = \alpha_{U\cap V} \cap \delta\phi_A \sim \pm\partial\alpha_{U\cap V} \cap \phi_B,$$

where $\sim$ means "cohomologous to".

8.7. **Lefschetz duality.**

Let M be a compact R-oriented n-manifold with boundary ∂M.

Theorem 8.7.1 (Lefschetz duality)**.** *The following diagram commutes:*

$$\begin{array}{ccccccc} H^k(M,\partial M) & \longrightarrow & H^k(M) & \longrightarrow & H^k(\partial M) & \longrightarrow & H^{k+1}(M,\partial M) \\ \downarrow{\scriptstyle [M]\cap} & & \downarrow{\scriptstyle [M]\cap} & & \downarrow{\scriptstyle [\partial M]\cap} & & \downarrow{\scriptstyle [M]\cap} \\ H_{n-k}(M) & \longrightarrow & H_{n-k}(M,\partial M) & \longrightarrow & H_{n-k-1}(\partial M) & \longrightarrow & H_{n-k-1}(M), \end{array}$$

and each of the vertical arrows is an isomorphism.

For a slightly more general statement, see Hatcher, Theorem 3.43. We'll explain the terms in the diagram and indicate the proof:

(1) Fact: There is a collared neighborhood of $\partial M \subset M$ of the form $(-\epsilon, 0] \times \partial M$ where $\partial M = \{0\} \times \partial M$.

(2) We define the fundamental class $[M] \in H_n(M, \partial M)$ as:

$$\mu_{A_\epsilon} \in H_n(M - \partial M | A_\epsilon) \simeq H_n(M, \partial M),$$

where $A_\epsilon = M - ((-\epsilon, 0] \times \partial M)$.

(3) $H^k_c(M - \partial M) = \lim_{\epsilon\to 0} H^k(M - \partial M | A_\epsilon) \simeq H^k(M, \partial M)$ since the direct limit stabilizes. Hence $[M]\cap : H^k(M, \partial M) \to H_{n-k}(M)$ is really $\mu_{A_\epsilon}\cap : H^k(M - \partial M|A_\epsilon) \to H_{n-k}(M)$.

(4) The first, third, and fourth vertical arrows are isomorphisms by Poincaré duality; hence the second one is also by the five lemma.

(5) The second and third squares are consequences of $\partial([M] \cap \phi) = \pm(\partial[M] \cap \phi - [M] \cap \delta\phi)$. *The main thing to check is that* $\partial[M] = [\partial M]$*, which is HW.*

8.8. **Alexander duality.** Refer to Greenberg-Harper, Section 2.7 since we omit the proof.

Theorem 8.8.1 (Alexander duality)**.** *Let M be a compact R-oriented n-manifold and $A \subset M$ a closed subset. Then there is an isomorphism*

$$D_A : \lim_{U \supset A} H^k(U) \xrightarrow{\sim} H_{n-k}(M, M - A).$$

The direct limit $\lim_{U\supset A} H^k(U)$ is with respect to the directed system which consists of open sets $U \supset A$ directed by reverse inclusion: $V \subset U$ implies $U \leq V$.

9. Homotopy coexactness

9.1. **Basic constructions.** We list some basic constructions. Let X, Y be topological spaces.

1. The *cone* CX is $X \times [0,1]/(x,1) \sim (x',1)$ for all $x, x' \in X$.

2. The *suspension* SX is

$$X \times [-1,1]/(x,1) \sim (x',1), (x,-1) \sim (x',-1)$$

for all $x, x' \in X$. For example, $SS^n = S^{n+1}$.

C and S induce functors $\mathrm{Top} \to \mathrm{Top}$ (these are called "endofunctors"). In the case of S, $X \mapsto SX$ and $f : X \to Y$ is mapped to $Sf : SX \to SY$.

3. The *join* $X * Y$ is $X \times Y \times [0,1]/\sim$, $(x,y,0) \sim (x,y',0), (x,y,1) \sim (x',y,1)$.

The following are operations in the category Top$_\bullet$ of pointed topological spaces. Let (X, x_0) and (Y, y_0) be pointed topological spaces.

4. The *reduced cone* $C^{\mathrm{red}}X = CX/(x_0,t) \sim (x_0,t')$ for all $t, t' \in [0,1]$. The basepoint is the equivalence class of $(x_0, 0)$.

5. The *reduced suspension* $\Sigma X = SX/(x_0,t) \sim (x_0,t')$ for all $t, t' \in [-1,1]$. The basepoint is the equivalence class of $(x_0, 0)$. We also have $\Sigma S^n = S^{n+1}$.

Similarly, C^{red} and Σ induce endofunctors $\mathrm{Top}_\bullet \to \mathrm{Top}_\bullet$.

6. The *smash product* of (X, x_0) and (Y, y_0) is $X \wedge Y = X \times Y/(X \times \{y_0\}) \cup (\{x_0\} \times Y)$. The basepoint is the equivalence class of (x_0, y_0) This is the replacement for $X \times Y$ in the pointed category.

9.2. **Mapping cones and mapping cylinders.** For more details see Spanier, Chapter 7, Section 1.

Given $f : X \to Y$, we can form the *mapping cylinder* $M_f = Y \sqcup (X \times [0,1])/(x,1) \sim f(x)$ and the *mapping cone* $C_f = M_f/(x,0) \sim (x',0)$. In Top$_\bullet$ we can analogously form the *reduced mapping cylinder* and the *reduced mapping cone* by collapsing $\{x_0\} \times [0,1]$.

Remark 9.2.1. Unfortunately, people usually use the same notation for both the reduced and nonreduced objects; we need to be careful about which category we're in.

Recall that $[X, Y]$ is the homotopy class of maps $X \to Y$. In $[(X, x_0), (Y, y_0)]$ there is a distinguished homotopy class, denoted 0, i.e., the homotopy class of maps that are nullhomotopic to the constant map to y_0.

For the moment let us work in Top$_\bullet$ or hTop$_\bullet$.

Definition 9.2.2. The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *coexact* if for all W the sequence

$$[Z, W] \xrightarrow{g^*} [Y, W] \xrightarrow{f^*} [X, W]$$

is an exact sequence of "pointed sets", i.e., $\mathrm{Im}(g^*) = (f^*)^{-1}(0)$.

Theorem 9.2.3. *Given $f : X \to Y$, the sequence $X \xrightarrow{f} Y \xrightarrow{i} C_f$ is coexact. Here $i : Y \to C_f$ is the obvious inclusion.*

Proof. Let us first unwind definitions: Consider $[C_f, W] \xrightarrow{i^*} [Y, W] \xrightarrow{f^*} [X, W]$. We want to show that $\operatorname{Im} i^* = (f^*)^{-1}(0)$.

Note that $h : C_f \to W$ is equivalent to the following:

- $h \circ i : Y \to W$, and
- a nullhomotopy from $h \circ i : X \to W$ to the constant map which maps to the basepoint $w_0 \in W$.

This interpretation immediately implies that $\operatorname{Im} i^* = (f^*)^{-1}(0)$. □

Corollary 9.2.4. *Given $f : X \to Y$, there exists a long coexact sequence:*

(9.2.1) $$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{j} C_i \xrightarrow{f^1} C_j \xrightarrow{i^1} C_{f^1} \dots$$

HW: There is also a relative version: Given $f = (f_0, f_1) : (X, A) \to (Y, B)$, its mapping cone is (C_{f_0}, C_{f_1}) and we have the coexact sequence

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{i} (C_{f_0}, C_{f_1}).$$

9.3. **Puppe sequence.**

Theorem 9.3.1 (Puppe sequence)**.** *The long coexact sequence* (9.2.1) *can be written as:*

(9.3.1) $$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{j} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i} \Sigma C_f \longrightarrow \dots.$$

Proof. Starting with Equation (9.2.1), we show that (i) $C_i \simeq \Sigma X$, (ii) $C_j \simeq \Sigma Y$, and (iii) $f^1 \simeq \Sigma f$. For Hatcher's picture proof see p.397.

(i) Note that $C_i = C_f \cup_Y CY$, where we are identifying the "bases" of the two cones, and hence C_i is the union of CX and CY glued with $f : X \to Y$. Let $r : C_i \to C_i/CY = \Sigma X$ be the quotient map. Define $s : \Sigma X \to C_i$, where $\Sigma X = X \times [-1, 1]/\sim$ and s maps $X \times (0, 1]/\sim \to CX$ by inclusion $(x, t) \mapsto (x, t)$ and $X \times [-1, 0] \to Y \times [-1, 0]$ by $(x, t) \mapsto (f(x), t)$. HW: Write down precise formulas that show r and s are homotopy inverses.

(ii) Now $C_j = C_i \cup C(C_f)$ glued along C_f. It is not hard to see that $C(C_f)$ deformation retracts to CY and we are left with $CY \cup CY$ glued along Y, which is ΣY. Let $t : C_j \to \Sigma Y$ be the corresponding map.

(iii) It is not hard to see that $tf^1 s : \Sigma X \to \Sigma Y$ is equal to Σf.

□

9.4. **Homological algebra version of Puppe sequence.**

Consider a chain map $\phi : (C_*, \partial_C) \to (D_*, \partial_D)$. Just as the mapping cone C_f enabled us to put any map $f : X \to Y$ into a "coexact" sequence, there is a way to put ϕ_* into a long exact sequence in homology.

We define the chain complex $(\operatorname{Cone}(\phi), d)$, called the *mapping cone of f*, as follows:

$$\operatorname{Cone}(\phi)_k = C_{k-1} \oplus D_k$$

and

$$d : C_{k-1} \oplus D_k \to C_{k-2} \oplus D_{k-1}$$

maps $d(x, y) = (\partial_C x, \phi(x) - \partial_D y)$. We check that

$$d^2(x, y) = (\partial_C^2 x, \phi\partial_C(x) - \partial_D\phi(x) + \partial_D^2 y) = 0.$$

The mapping cone $\operatorname{Cone}(\phi)$ fits into the following short exact sequence of chain complexes:

$$0 \to D_* \to \operatorname{Cone}(\phi) \to C_{*-1} \to 0.$$

HW: (1) Show that its long exact sequence has the form

$$H_i(D_*) \to H_i(\mathrm{Cone}(\phi)) \to H_{i-1}(C_*) \stackrel{\phi_*}{\to} H_{i-1}(D_*),$$

where the connecting homomorphism is ϕ_*.

(2) If $f_* : C_*(X) \to D_*$ is the induced map from $f : X \to Y$, then $\mathrm{Cone}(f_*)$ is quasi-isomorphic to $C_*(C_f)$.

10. Higher homotopy groups

Today we introduce the higher homotopy groups $\pi_n(X, x_0)$ of a pointed topological spaces (X, x_0) and describe their basic properties.

10.1. **Definitions.** Let $I^n = [0,1] \times \cdots \times [0,1]$ be the n-dimensional unit cube. Denote its boundary ∂I^n as the set of points in I^n where at least one coordinate is 0 or 1. As a set, let

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

When $n = 0$, we take I^0 to be a point and $\partial I^0 = \varnothing$. Hence $\pi_0(X, x_0)$ is the set of path components of X.

Given $f, g : (I^n, \partial I^n) \to (X, x_0)$, define $fg : (I^n, \partial I^n) \to (X, x_0)$ via

$$(s_1, s_2, \dots, s_n) \mapsto \begin{cases} f(2s_1, s_2, \dots, s_n), & 0 \leq s_1 \leq 1/2 \\ g(2s_1 - 1, s_2 \dots, s_n), & 1/2 \leq s_1 \leq 1 \end{cases}$$

The product descends to a product on $\pi_n(X, x_0)$ — the proof is identical to the case of $\pi_1(X, x_0)$.

Remark: There is an apparent asymmetry in the definition, as the first coordinate s_1 is preferred. For HW show the independence of fg on the choice of coordinate; this is similar to the proof below of the fact that $\pi_n(X)$ is abelian for $n \geq 2$.

The nth homotopy group $\pi_n(X, x_0)$ is equivalent to $[(S^n, *), (X, x_0)]$ (verify this for HW), by recalling that S^n is the quotient of D^n, where ∂D^n is identified to a point.

10.2. **Properties of the homotopy groups.**

Lemma 10.2.1. *π_n is a functor $\mathbf{Top}_\bullet \to \mathbf{Grp}$ for $n \geq 1$ and $\mathbf{Ab}$ for $n \geq 2$.*

Proof. Given $\phi : (X, x_0) \to (Y, y_0)$, the induced homomorphism $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is given by $f \mapsto \phi \circ f$. It is immediate that:

(1) $\mathrm{id}_* : \pi_n(X, x_0) \to \pi_n(X, x_0)$ is the identity map.
(2) $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.

We give a pictorial proof that $fg \simeq gf$ for $n \geq 2$. For HW, make it rigorous. Let $r_C : I^n \hookrightarrow I^n$ be an embedding with image a small square C in the interior. First homotop f to f' so that $f'(x) = x_0$ for $x \notin C$ and $f'(x) = f r_C^{-1}(x)$ for $s \in C$. Similarly define g'. Then we homotop $f'g'$ to $g'f'$ by precomposing with an isotopy of I^n that switches the orders of the squares C_1 and C_2 for f and g, respectively. □

HW: Prove that $\pi_n(X \times Y, (x_0, y_0)) \simeq \pi_n(X, x_0) \times \pi_n(Y, y_0)$.

HW: Prove that $\pi_n(X, x_0) = 0$ for $n \geq 1$, if X is *contractible*, i.e., X has the homotopy type of a point.

HW: Let $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path-connected, locally path-connected covering space. Then $\pi_n(X, x_0) = \pi_n(\tilde{X}, \tilde{x}_0)$ for $n \geq 2$.

It follows that, for $n \geq 2$, $\pi_n(\mathbb{RP}^m) = \pi_n(S^m)$ and $\pi_n(S^1) = \pi_n(\mathbb{R}) = 0$.

Facts: $\pi_n(S^m) = 0$ if $n < m$, $= \mathbb{Z}$ if $n = m$, and is unknown in general for $n > m$. For example $\pi_3(S^2) = \mathbb{Z}$. (See Hatcher, Section 4.1 for a partial table.)

Open question: Give a general formula for $\pi_n(S^m)$.

10.3. **Relative homotopy groups.** We can generalize π_n to $\mathbf{Top}^2_\bullet \to \mathbf{Grp}$. To define $\pi_n(X, A, x_0)$ of a pair (X, A) with $x_0 \in A$, view I^{n-1} as part of ∂I^n with $s_n = 0$, define J^{n-1} as the closure of $\partial I^n - I^{n-1}$ and take homotopy classes of maps

$$(I^n, I^{n-1}, J^{n-1}) \to (X, A, x_0).$$

When $n = 0$, let $\pi_0(X, A, x_0)$ be the quotient set $\pi_0(X, x_0)/\pi_0(A, x_0)$.

The nth relative homotopy group $\pi_n(X, A, x_0)$ is equivalent to $[(D^n, \partial D^n, *), (X, A, x_0)]$.

Lemma 10.3.1. *$\pi_n(X, A, x_0)$ is a group for $n \geq 2$ and an abelian group for $n \geq 3$.*

Note in particular that $\pi_1(X, A, x_0)$ is not a group since we cannot compose: given $f : [0, 1] \to X$ with $f(0), f(1) \in A$ and $f(0) = x_0$, we need $f(1) = x_0$ in order to compose....

Exact sequence for (X, A). Analogous to the relative sequence in homology, we have:

Theorem 10.3.2. *There is a long exact sequence of homotopy groups*

$$\to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to .$$

Proof. This is an application of the Puppe sequence

$$(S^0, *) \to (S^0, S^0) \to (D^1, \partial D^1) \to (S^1, *) \to (S^1, S^1) \to (D^2, \partial D^2) \to .$$

Just apply $[\cdot, (X, A)]$. It is easy to see $[(S^n, *), (X, A)] = \pi_n(X)$ and $[(D^n, \partial D^n), (X, A)] = \pi_n(X, A)$. Finally we also see that $[(D^n, D^n), (X, A)] = \pi_n(A)$. ☐

10.4. **Compact-open topology and loop spaces.** Given topological spaces X, Y, we can define Y^X to be the space of continuous maps $f : X \to Y$. It can be topologized via the *compact-open* topology, defined by taking a *subbasis* consisting of sets $M(K, U) = \{f : X \to Y \text{ continuous } | f(K) \subset U\}$, where $K \subset X$ is a compact set and $U \subset X$ is open. Hence a basis is given by finite intersections of $M(K_i, U_i)$.

For example, if $X = I$, then two paths $f, g : I \to Y$ are "close" if there exists a subdivision $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ and open sets $U_i \subset Y$ so that f and g both map $[t_i, t_{i+1}]$ to U_i.

If Y is a metric space with metric d and X is compact, then the compact-open topology on Y^X coincides with the topology induced from $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. (Check!)

Given a continuous map $\phi : Z \to Y^X$, we can define $\psi : Z \times X \to Y$ by $\psi(z, x) = \phi(z)(x)$. We have the following:

Lemma 10.4.1. *Suppose X is a locally compact Hausdorff space. Then ϕ is continuous if and only if ψ is continuous.*

A topological space X is *locally compact* if $\forall x \in X$ and $U \ni x$ open there is a compact set K such that $x \in V \subset K \subset U$ for some open set V.

Now let (X, x_0) be a pointed topological space. Then the *based loop space* $(\Omega X, \tilde{x}_0)$ is the set of loops $f : (I, \partial I) \to (X, x_0)$, endowed with the compact-open topology. Its basepoint $\tilde{x}_0$ is the constant loop at x_0.

Corollary 10.4.2. *If X is locally compact and Hausdorff, then $[\Sigma X, Y] \simeq [X, \Omega Y]$. In particular, if $X = S^n$, then $\pi_{n+1}(Y) \simeq \pi_n(\Omega Y)$.*

Proof. Since X is locally compact and Hausdorff, there is a bijection between continuous maps $X \times [0, 1] \to Y$ and continuous maps $X \to Y^{[0,1]}$. Now observe that a map $f : \Sigma X \to Y$ is equivalent to a map $f : X \times [0, 1] \to Y$ such that $X \times \{0, 1\}$ and $\{x_0\} \times [0, 1]$ map to Y. It can therefore be viewed as a map $g : X \to \mathrm{Map}((I, \partial I), (X, x_0))$ such that $g(x_0)$ is the constant map $I \to X$ that maps to x_0. ☐

In categorical language, Ω is an endofunctor of $\mathbf{Top}_\bullet$ and Σ and Ω are adjoint functors in the full subcategory of $\mathbf{hTop}_\bullet$ whose objects are locally compact and Hausdorff.

10.5. **Proof of $\pi_n(S^m) = 0$ for $n < m$.** View a representative of $\pi_n(S^m)$ as

$$f : (I^n, \partial I^n) \to (\mathbb{R}^m \cup \{\infty\}, \infty).$$

The goal is to homotop f to f_1 so that 0 is not in the image of f. Once this is done, g maps to $\mathbb{R}^m$ and any map to $\mathbb{R}^m$ is contractible to a point.

Assuming $0 \in \mathrm{Im}(f)$, consider closed balls $B_1, B_2 \subset \mathbb{R}^m$ of radius 1 and 2 centered at 0. Their preimages $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are compact. Take a sufficiently fine cubical mesh of I^n. Let K_1 be the union of closed cubes that (nontrivially) intersect $f^{-1}(B_1)$ and let K_2 be the union of closed cubes intersecting K_1. Then $f(K_1) \subset B_2$. Subdivide each cube of K_2 into simplices (for example using the subdivision used in the definition of the prism operator). For each simplex Δ^n, we consider $f(e_0), \dots, f(e_n)$ and define $g(\sum a_i e_i) = \sum a_i f(e_i)$, i.e., g is linear on each simplex and agrees with f on each vertex. Also define a function $\phi : K_2 \to [0, 1]$ such that $\phi = 0$ on ∂K_2 and $\phi = 1$ on K_1.

We then define $f_t : K_2 \to \mathbb{R}^m$ by setting $f_t = (1 - t\phi)f + t\phi g$. On ∂K_2 we have $f = f_1$ and we can extend $f_1 = f$ to $I^n - K_2$. On K_1 we have $f_t = g$. Also $f_1(K_2 - K_1)$ will not pass through 0. For a generic choice of values of f(vertices), g will miss 0.

10.6. **Cellular approximation theorem.** Recall that a CW complex X is constructed by starting with a discrete set X^0 and inductively attaching n-cells e^n_α to the $(n-1)$-skeleton X^{n-1} via maps $\phi_\alpha : \partial e^n_\alpha \to X^{n-1}$ to obtain the n-skeleton X^n. Then $X = \cup_n X^{(n)}$. For more information on the topology of X, refer to the Appendix of Hatcher on the topology of cell complexes.

Definition 10.6.1. A map $f : X \to Y$ of CW complexes is *cellular* if $f(X^n) \subset Y^n$ for all n.

The above method can be used to prove the following:

Theorem 10.6.2 (Cellular approximation theorem)**.** *Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map.*

Proof. Suppose $f : X \to Y$ is cellular on X^{n-1}. Consider e^n_α. Since e^n_α is compact, $f(e^n_\alpha)$ intersects finitely many m-cells with $m > n$. (Check this!) By the method to prove $\pi_n(S^m) = 0$ for $m > n$, the image of $f(e^n_\alpha)$ misses a point on each m-cell of Y and hence can be homotoped into the n-skeleton Y^n. This works for finite CW complexes; for HW figure out how to deal with infinite CW complexes. $\square$

10.7. **Whitehead's theorem.**

Definition 10.7.1. A map $f : X \to Y$ is a *weak homotopy equivalence* if it induces isomorphisms $(f_*)_n : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $n \geq 0$ and all choices of basepoint x_0.

We have the following amazing theorem:

Theorem 10.7.2. *If $f : (X, x_0) \to (Y, y_0)$ is a weak homotopy equivalence between connected (=path connected) CW complexes, then f is a homotopy equivalence.*

Remark 10.7.3. The theorem does not say that two pointed topological spaces (X, x_0) and (Y, y_0) with isomorphic homotopy groups for all n are homotopy equivalent. You need a map from one to the other.

We'll give a proof for finite CW complexes. HW: Explain how to extend the proof to infinite CW complexes.

Proof. Suppose that f is an inclusion of a subcomplex. Then by the relative homotopy sequence, $\pi_n(Y,X) = 0$ for all n. Let e^n_α be an n-cell in Y with boundary in X. Then e^n_α can be homotoped into X relative to ∂e^n_α (note that this needs an argument) and the collection of these homotopies can be combined into a deformation retraction of Y onto X.

In the general case, first homotop $f : X \to Y$ so that f is a cellular map. Then replace $f : X \to Y$ by $g : X \to M_f$, where M_f is the mapping cylinder of f (which glues $X \times \{1\}$ to Y) and g is the identification of X with $X \times \{0\}$. Note that $Y \hookrightarrow M_f$ is a homotopy equivalence and g is an inclusion. It suffices to show that M_f deformation retracts onto X. But this is immediate from the previous paragraph because g is an inclusion of a subcomplex. □

10.8. **CW approximations.**

Definition 10.8.1. A *CW approximation* of a topological space X is a CW complex Z together with a weak homotopy equivalence $f : Z \to X$.

Theorem 10.8.2. *CW approximations exist for any pointed topological space* (X, x_0).

Proof. The point is to model the generators and relations of $\pi_n(X, x_0)$ using cellular attaching maps.

Step 1. Without loss of generality $\pi_0(X, x_0) = 1$. Start with one point $*$ and a map $f(*) = x_0$. Next choose a generating set $\{f^n_\alpha\}_{\alpha \in I_n}$ for $\pi_n(X, x_0)$. For each f^n_α, take an n-cell e^n_α that is attached to $*$ via the attaching map $\phi^n_\alpha : \partial e^n_\alpha \to *$. Define $f : e^n_\alpha \to X$ so it agrees with f^n_α. Then

$$f : Z_0 = \{*\} \cup (\cup_{n,\alpha} e^n_\alpha) \to X$$

is surjective on π_n for all n.

Step 2. Next we kill off the kernel of f_* (aka the relations). Suppose we have constructed up to Z_{n-1} and $(f_*)_i : \pi_i(Z_{n-1}, *) \to \pi_i(X, x_0)$ is an isomorphism up to $i = n - 1$. Attach $(n+1)$-cells e^{n+1}_β to Z_{n-1} for all generators β of $\ker(f_*)_n$ to obtain $Z_n = Z_{n-1} \cup (\cup_\beta e^{n+1}_\beta)$ and extend f to Z_n; this is possible since all the β's are nullhomotopic in (X, x_0).

We claim that $\pi_{n-1}(Z_{n-1}, *) \simeq \pi_{n-1}(Z_n, *)$, since any homotopy $S^{n-1} \times [0,1] \to X$ can be homotoped to a map to the n-skeleton X^n by the cellular approximation theorem. Also $(f_*)_n : \pi_n(Z_n, *) \xrightarrow{\sim} \pi_n(X, x_0)$: The map $\pi_n(Z_{n-1}, *) \to \pi_n(X, x_0)$ factors into $f_* i_*$, where i is the inclusion $Z_{n-1} \to Z_n$. Hence $(f_*)_n$ is surjective. The kernel of $\pi_n(Z_{n-1}, *) \to \pi_n(X, x_0)$ is killed by the cells e^{n+1}_β, and we have an isomorphism.

Observe that attaching e^{n+1}_β might make $\pi_m(Z_n, *)$ larger for $m > n$, but that is ok. □

Next we discuss the uniqueness of CW approximations:

Theorem 10.8.3 (Uniqueness of CW approximations)**.** *Given CW models $f : Z \to X$ and $f' : Z' \to X$ with Z, Z', X path-connected, there is a homotopy equivalence $h : Z \to Z'$.*

Proof. Consider the mapping cylinder $M_{f'}$ as a replacement for X. We view the composition $Z \xrightarrow{f} X \xrightarrow{i} M_{f'}$ as a map $(Z, z_0) \to (M_{f'}, Z')$. Since $\pi_n(M_{f'}) \simeq \pi_n(X) \simeq \pi_n(Z')$ for all n, $\pi_n(M_{f'}, Z') = 0$ for all n by the relative exact sequence. This implies that all the cells of Z can be compressed into Z', giving a homotopy of if to $h : Z \to M_{f'}$ with image in Z'. This implies that $h : Z \to Z'$ is a weak homotopy equivalence. Finally, by Whitehead's theorem, h is a homotopy equivalence since Z, Z' are CW complexes. □

Proposition 10.8.4. *A weak homotopy equivalence $f : Y \to Z$ induces:*

(1) *bijections* $f_* : [X,Y] \to [X,Z]$ *for all CW complexes* X*, and*
(2) *isomorphisms* $f_* : H_n(Y) \to H_n(Z)$.

Proof. We'll prove (1). See Hatcher, Prop. 4.21 for (2). The idea is similar to the uniqueness theorem. Replace Z by M_f and show that $f_* : [X,Y] \to [X,M_f]$ is a bijection: Given $g : X \to M_f$, view it as a map $(X,x_0) \to (M_f,Y)$. Since $\pi_n(M_f,Y) = 0$ for all n, g is homotopic to $h : X \to M_f$ with image in Y. This implies the surjectivity of f_*. To prove the injectivity, if we have $F : (X \times I, X \times \partial I) \to (M_f,Y)$, the same method shows that F can be homotoped to a map $G : (X \times I, X \times \partial I) \to (M_f,Y)$ with image in Y. □

Postnikov towers. Given a CW complex X, there exist spaces X_n with inclusion maps $X \stackrel{i_n}{\to} X_n$ such that $(i_n)_* : \pi_i(X) \stackrel{\sim}{\to} \pi_i(X_n)$ for $i \leq n$ and $\pi_i(X_n) = 0$ for $i > n$ and the X_n can be put in a sequence

$$\cdots \to X_3 \stackrel{s_3}{\to} X_2 \stackrel{s_2}{\to} X_1$$

with $s_n i_n = i_{n-1}$. The X_n are "truncations" of X with successively better approximations as $i \to \infty$. Starting with X, we can attach $(n+2)$-cells to kill π_{n+1}, then $(n+3)$-cells to kill π_{n+2}, etc. to obtain X_n. This gives the desired X_n with inclusion maps $i_n : X \to X_n$. It remains to define $s_{n+1} : X_{n+1} \to X_n$ as an extension of $i_n : X \to X_n$. Since X_{n+1} is obtained from X by attaching $(n+3)$-cells and higher and $\pi_{n+1}(X_n) = \pi_{n+2}(X_n) = \cdots = 0$, the attaching map of the cells are nullhomotopic. This implies that X can be extended to $(n+3)$-cells and higher.

Definition 10.8.5. (X,x_0) is n-*connected* if $\pi_i(X,x_0) = 0$ for all $0 \leq i \leq n$. (X,A) is n-*connected* if $\pi_i(X,A,x_0) = 0$ for all $0 \leq i \leq n$ and $x_0 \in A$. (Note that there is an extra condition that we need to take all $x_0 \in A$; this is to take care of the situation where A has multiple path components.)

Also recall that a subcomplex of a CW complex $A \subset X$ is a union of cells of X such that the closure of each cell of A is contained in A (i.e., for each cell the image of its attaching map is contained in A).

Definition 10.8.6. An n-*connected CW model for* (X,A) with A a nonempty CW complex is an n-connected CW pair (Z,A) and a map $f : Z \to X$ such that $f|_A = \mathrm{id}$ and $f_* : \pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and injective for $i = n$.

Since $\pi_i(Z)$ agrees with $\pi_i(A)$ for $i < n$ and with $\pi_i(X)$ for $i > n$, Z approximates A up to n and X after n. There is an analogous CW approximation theorem for n-connected models, whose proof we omit.

Theorem 10.8.7. *For each* (X,A) *with* A *a nonempty CW complex, there exists an* n*-connected CW model* $f : (Z,A) \to (X,A)$ *for all* $n \geq 0$*. We may also assume that* Z *is obtained from* A *by attaching cells of dimension* $> n$*. The* n*-connected CW model is unique up to homotopy equivalence.*

11. Excision

HW: Hatcher, Section 4.1: 1,2,8,9,10,11,18,19.

11.1. **Excision.**

Theorem 11.1.1 (Excision)**.** *Let X be a CW complex that can be written as $X = A \cup B$, where $A, B, C = A \cap B$ are subcomplexes and C is nonempty and connected. If (A, C) is m-connected and (B, C) is n-connected for $m, n \geq 0$, then $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < m + n$ and surjective for $i = m + n$.*

Sketch of proof. A corollary of the CW approximation theorem is the following HW:

Corollary 11.1.2. *If (X, A) is an n-connected CW pair, then there is a CW pair (Z, A) homotopy equivalent to (X, A) relative to A such that all the cells of $Z - A$ have dimension $> n$.*

Hence we may assume that all the cells of $A - C$ have dimension $> m$ and all the cells of $B - C$ have dimension $> n$.

The idea of the proof can be explained in the following simplified case where A is obtained from C by attaching a single $(m + 1)$-cell e^{m+1} and B is obtained from C by attaching a single $(n + 1)$-cell e^{n+1}.

To prove the surjectivity, consider a map

$$f : (I^i, \partial I^i, 0) \to (X, B, x_0).$$

Pick points $p \in \mathrm{int}(e^{m+1})$ and $q \in \mathrm{int}(e^{n+1})$. By the method of proof of $\pi_i(S^n) = 0$ for $i < n$, we can homotop f (while keeping the same name) so that f restricted to neighborhoods U_p and U_q of $f^{-1}(p)$ and $f^{-1}(q)$ are piecewise linear maps from a union of i-simplices.

This allows us to apply standard *transversality arguments*. In particular, $f^{-1}(p)$ is piecewise linear of dimension $i - (m + 1)$ and $f^{-1}(q)$ is piecewise linear of dimension $i - (n + 1)$; moreover we assume that they are generic. If

$$\mathrm{Im}(f^{-1}(p)) + \mathrm{Im}(f^{-1}(q)) = 2i - (m + n + 2) < i - 1,$$

then the projections of $f^{-1}(p)$ and $f^{-1}(q)$ to I^{i-1} are disjoint. This is equivalent to $i < m + n + 1$ or $i \leq m + n$.

Hence there exists a function $\phi : I^{i-1} \to [0, 1)$ such that $f^{-1}(q)$ lies below the graph of ϕ in $I = I^{i-1} \times I$, $f^{-1}(p)$ lies above the graph of ϕ, and $\phi = 0$ on ∂I^{i-1}. We then take the homotopy f_t, $t \in [0, 1]$, to be a map obtained by composing $g_t : I^n \xrightarrow{\sim} \{s_n \geq t\phi(s_1, \dots, s_{n-1})\}$ and f; view it as a map $(I^i, \partial I^i) \to (X, X - \{p\})$, noting that $(X, B) \simeq (X, X - \{p\})$. Then f_1 can be viewed as a map

$$(I^i, \partial I^i) \to (X - \{q\}, X - \{p, q\}) \simeq (A, C).$$

To prove the injectivity, we consider the homotopy

$$F : (I^i, \partial I^i, 0) \times [0, 1] \to (X, B, x_0)$$

between $f_0, f_1 : (I^i, \partial I^i, 0) \to (A, C, x_0)$. The proof is similar. □

11.2. **Stable homotopy groups.** An immediate corollary of the excision theorem is:

Corollary 11.2.1 (Freudenthal suspension theorem)**.** *The suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$ if X is $(n - 1)$-connected.*

Proof. Decompose SX into two cones C_+X and C_-X. Then $\pi_i(X) \simeq \pi_{i+1}(C_+X, X)$ by the relative sequence, $\pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X)$ is the excision map, and $\pi_{i+1}(SX, C_-X) \simeq \pi_{i+1}(SX)$ again by the relative sequence. The excision map is an isomorphism for $i+1 < 2n$ since (C_+X, X) is n-connected if X is $(n-1)$-connected by the relative sequence for (C_+X, X). (HW: check that the sequence of maps agrees with the suspension map.) □

This implies that $\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to \dots$ eventually stabilize (i.e., are isomorphisms): if X is m-connected, then SX is $(m+1)$-connected, and S^kX is $(m+k)$-connected, and eventually $i+k < 2(n+k)-1$. The direct limit is called the *stable homotopy group* $\pi_i^s(X)$. When $X = S^0$, $\pi_i^s(S^0) = \pi_{i+n}(S^n)$ for $n > i+1$. It is also abbreviated π_i^s and called the *stable i-stem*.

12. Hurewicz isomorphism theorem

12.1. **Some calculations.**

Example. $\pi_n(S^n) = \mathbb{Z}$ for $n \geq 2$.

Method 1. Consider the sequence of suspension maps

$$\pi_1(S^1) \xrightarrow{i_1} \pi_2(S^2) \xrightarrow{i_2} \pi_3(S^3) \to \dots.$$

By the Freudenthal suspension theorem, i_1 is surjective and $i_2, i_3, \dots$ are isomorphisms. Since $\pi_1(S^1) = \mathbb{Z}$, $\pi_2(S^2)$ is a quotient of $\mathbb{Z}$.

Now there exists a map, called the *Hurewicz map,*

$$H : \pi_n(X) \to H_n(X;\mathbb{Z}) \simeq \mathbb{Z}, \qquad (f : S^n \to X) \mapsto f_*\alpha,$$

where α is a generator of $H_n(S^n;\mathbb{Z})$. (HW: show this is a homomorphism!)

When $X = S^n$, H is surjective since id is mapped to α. Hence $\pi_2(S^2) = \mathbb{Z}$ and $\pi_n(S^n) = \mathbb{Z}$ for all $n \geq 2$.

Method 2. Given $f \in \pi_n(S^n, x_0)$ for $n \geq 2$, we can homotop f so that f restricted to a neighborhood U_p of $f^{-1}(p)$ is a piecewise linear map from a union of n-simplices. Hence by the usual transversality argument we may assume that $f^{-1}(p)$ is finite and intersects each n-simplex Δ in its interior. By a further homotopy and subdivision of the simplices, f on the complement of the $\mathrm{int}(\Delta)$'s map to x_0 and $f|_{\mathrm{int}(\Delta)}$ is a homeomorphism onto $S^n - \{x_0\}$; in other words, f is homotopic to the sum of standard $\deg = \pm 1$ homeomorphisms. The map H implies that $\pi_n(S^n) = \mathbb{Z}$.

The following two examples can be proved using Method 2.

Example. For $n \geq 2$, $\pi_n(\vee_\alpha S^n_\alpha)$ is free abelian with basis which consists of homotopy classes of inclusions $S^n_\alpha \hookrightarrow \vee_\alpha S^n_\alpha$.

Example. For $n \geq 2$, $\pi_{n+1}(X, \vee_\alpha S^n_\alpha)$, where X is obtained from $\vee_\alpha S^n_\alpha$ by attaching cells e^{n+1}_β via $\phi_\beta : \partial e^{n+1}_\beta \to \vee_\alpha S^n_\alpha$, is free abelian with basis in bijection with $\{e^{n+1}_\beta\}$.

12.2. **Change of basepoints for higher homotopy groups.** We now discuss the change-of-basepoint map $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$, $n \geq 2$, where γ is a path from x_1 to x_0. Viewing a representative of $\pi_n(X, x_1)$ as $f : (D^n, \partial D^n) \to (X, x_1)$, we define $\gamma f : (D^n, \partial D^n) \to (X, x_0)$ such that $\gamma f(x) = f(2x)$ if $|x| \leq \frac{1}{2}$ and $\gamma(2|x| - 1)$ if $\frac{1}{2} \leq |x| \leq 1$. Then $\beta_\gamma([f]) = [\gamma f]$.

Lemma 12.2.1. *β_γ is a group homomorphism and has inverse $\beta_{\gamma^{-1}}$.*

The first assertion is not obvious and is left as HW.

Hence $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$, i.e., $\pi_n(X, x_0)$ is a module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$.

Example. Consider $\pi_n(S^1 \vee S^n)$ for $n \geq 2$. The universal cover $\widetilde{M}$ of $M = S^1 \vee S^n$ is a line $\mathbb{R}$ with S^n's attached at all integers, and is homotopy equivalent to $\vee_{k\in\mathbb{Z}} S^n_k$. Hence $\pi_n(S^1 \vee S^n)$ is the free $\mathbb{Z}$-module with basis $\{S^n_k\}_{k\in\mathbb{Z}}$. On the other hand, $\pi_1(S^1 \vee S^n) \simeq \pi_1(S^1) \simeq \mathbb{Z}$, and its group ring is $\mathbb{Z}[t, t^{-1}]$. One can see that $\pi_n(S^1 \vee S^n)$ is the free $\mathbb{Z}[t, t^{-1}]$-module $\mathbb{Z}[t, t^{-1}]$.

12.3. **The Hurewicz theorem.**

Theorem 12.3.1. *If X is $(n-1)$-connected with $n \geq 2$, then the reduced homology groups $\widetilde{H}_i(X) = 0$ for $i < n$ and $\widetilde{H}_n(X) \simeq \pi_n(X)$. If (X, A) is $(n-1)$-connected with $n \geq 2$ and A simply connected and nonempty, then $H_i(X, A) = 0$ for $i < n$ and $H_n(X, A) \simeq \pi_n(X, A)$.*

Proof. We will only prove the first statement. In view of Proposition 10.8.4, we may assume X is a CW complex by taking a CW approximation. Also by Corollary 11.1.2 we may assume that X has a single 0-cell and no other i-cells with $i < n$. Hence $\widetilde{H}_i(X) = 0$ for $i < n$. We may further assume that X has no i-cells with $i \geq n+1$ since such cells do not affect both $\pi_n(X)$ and $H_n(X)$; this means that $X = (\vee_\alpha S^n_\alpha) \cup (\cup_\beta e^{n+1}_\beta)$. We have a commutative diagram where the top and bottom rows are exact:

$$\begin{array}{ccccccc}
\pi_{n+1}(X, \vee_\alpha S^n_\alpha) & \xrightarrow{\partial} & \pi_n(\vee_\alpha S^n_\alpha) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{n+1}(X, \vee_\alpha S^n_\alpha) & \xrightarrow{\partial} & H_n(\vee_\alpha S^n_\alpha) & \longrightarrow & H_n(X) & \longrightarrow & 0.
\end{array}$$

By the calculations in Section 12.1, the first two vertical arrows are isomorphisms. Hence the theorem follows from the five lemma. □

We'll state without proof a slightly more general version of the Hurewicz theorem: Let $\pi'_n(X, A, x_0)$ be the quotient of $\pi_n(X, A, x_0)$ obtained by identifying $[\gamma f] = [f]$ where $\gamma \in \pi_1(A, x_0)$. Then the homomorphism

$$h : \pi_n(X, A, x_0) \to H_n(X, A)$$

descends to $h' : \pi'_n(X, A, x_0) \to H_n(X, A)$.

Theorem 12.3.2 (Hurewicz, general version)**.** *If (X, A) is $(n-1)$-connected with $n \geq 2$, X, A are path-connected, and $A \neq \varnothing$, then $h' : \pi'_n(X, A, x_0) \to H_n(X, A)$ is an isomorphism and $H_i(X, A) = 0$ for $i < n$.*

13. FIBRATIONS

13.1. **Definitions.**

Definition 13.1.1. A map $p : E \to B$ satisfies the *lifting property with respect to the pair* (Z, A) if for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{g} & B, \end{array}$$

there exists a map $\tilde{g} : Z \to E$ such that $p\tilde{g} = g$ and $\tilde{g}i = f$.

Definition 13.1.2. A surjective map $p : E \to B$ is a *fibration* (resp. *Serre fibration*) if it satisfies the lifting property with respect to $(A \times I, A \times \{0\})$ for all topological spaces (resp. CW complexes) A.

Given a fibration $p : E \to B$, if B has a basepoint b_0, then $p^{-1}(b_0)$ is the *fiber* F *at* b_0. Picking $x_0 \in F$, we often denote the fibration by $(F, x_0) \stackrel{i}{\to} (E, x_0) \stackrel{p}{\to} (B, b_0)$.

Lemma 13.1.3. *A fiber bundle is a Serre fibration.*

A map $E \stackrel{p}{\to} B$ is a *fiber bundle* with fiber F if there is an open cover $\mathcal{U}$ of B and $\phi_U : p^{-1}(U) \stackrel{\sim}{\to} U \times F$ for all $U \in \mathcal{U}$ such that $p = \pi_U \phi_U$. Here $\pi_U, \pi_F : U \times F \to U$ are the projections to U and F.

Proof. We may reduce to the case where $A = I^n$ and $Z = I^n \times I$ since we can do homotopy lifting in stages using the CW structure. We may also subdivide I^n so that $f : Z \to B$ has image inside some U. Given a lift $\tilde{g} : I^n \times \{0\} \to p^{-1}(U) = U \times F$, we just need to extend $\pi_F \tilde{g}$ from $I^n \times \{0\}$ to all of $I^n \times I$; this is straightforward. □

13.2. **Fibration sequence.**

Theorem 13.2.1. *If* $p : E \to B$ *is a Serre fibration, then* $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ *is an isomorphism for all* $n \geq 1$. *Hence the relative homotopy sequence for* (E, F, x_0) *becomes the fibration sequence*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \dots$$

Proof. We first show that p_* is surjective: Let $f : (I^n, \partial I^n) \to (B, b_0)$ be a representative of $\pi_n(B, b_0)$. View it as a map $f : (I^n, J^{n-1}) \to (B, b_0)$. We want to lift it to a map $\tilde{f} : (I^n, J^{n-1}) \to (E, x_0)$. The constant map $J^{n-1} \to \{x_0\}$ is a lift of $f|_{J^{n-1}}$. Then since (I^n, J^{n-1}) and $(I^n, I^{n-1} \times \{0\})$ are homotopy equivalent, f can be lifted to $\tilde{f}$ on (I^n, J^{n-1}). We finally observe that $\tilde{f}|_{I^{n-1} \times \{1\}}$ maps to F since $f|_{I^{n-1} \times \{1\}}$ is the constant map to b_0. This gives us $\tilde{f} : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$.

The injectivity of p_* is similar: Given $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ and a homotopy $f_t : (I^n, \partial I^n) \to (B, b_0)$ from $f_0 = p_* \tilde{f}_0$ to $f_1 = p_* \tilde{f}_1$, the homotopy can be viewed as $f : (I^{n+1}, J^{n-1} \times I) \to (B, b_0)$. f is lifted to $\tilde{f}$ on $(I^n \times \{0, 1\}) \cup (J^{n-1} \times I) = J^n$ since $J^{n-1} \times I$ is mapped to x_0. Again, since $(I^{n+1}, J^{n-1} \times I)$ and (I^{n+1}, J^n) are homotopy equivalent, we can lift f to $\tilde{f}$ taking $\tilde{f}_0$ to $\tilde{f}_1$. □

13.3. **Examples.**

Example. The *Hopf fibration* $S^1 \to S^3 \to S^2$ is given as follows: Given the unit sphere $S^3 \subset \mathbb{C}^2$, S^1 acts freely on S^3 via $(e^{i\theta}, (z_1, z_2)) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$. The quotient is $\mathbb{CP}^1 = S^2$. It is not hard to see the quotient map is a fiber bundle $S^3 \to S^2$ with fiber S^1. We now consider the fibration sequence:

$$\pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) \to \pi_{n-1}(S^3).$$

Since $\pi_n(S^1) = 0$ for $n > 1$ and $\mathbb{Z}$ for $n = 1$, for $n \geq 3$ $\pi_n(S^3) \simeq \pi_n(S^2)$ and $\pi_2(S^2) \simeq \pi_1(S^1)$. In particular, $\pi_3(S^2) \simeq \mathbb{Z}$, generated by the Hopf fibration.

14. Homotopy and cohomology

14.1. **Eilenberg-MacLane spaces.** An *Eilenberg-MacLane space* $K(G,n)$ with G a group and $n>0$ is a connected CW complex such that $\pi_i(K(G,n))=G$ if $i=n$ and 1 if $i\neq n$.

Theorem 14.1.1. *A $K(G,n)$ exists (if $n>1$ we assume G is an abelian group) and any two $K(G,n)$'s are homotopy equivalent.*

A $K(G,n)$ can be constructed as in the CW approximation theorem using a presentation for G: Let $\{f_\alpha\}_{\alpha\in I}$ be the generators of G. Then starting with the wedge $\vee_{\alpha\in I}S^n_\alpha$, we attach e^{n+1}_β for the relations. We can then kill off π_i for $i>n$ by attaching $(n+2)$-cells and higher. Let us call this X. In order to prove the homotopy equivalence of any two $K(G,n)$'s, it suffices to show that there is a weak homotopy equivalence from X to any other $K(G,n)$. This follows from the uniqueness of CW models up to homotopy equivalence. (Check the details for HW.)

Note that $H_n(K(G,n);\mathbb{Z})\simeq\pi_n(K(G,n))\simeq G$ by the Hurewicz theorem. By taking a wedge of $K(G,n)$'s we can construct a CW complex with arbitrary homotopy groups.

14.2. **Spectra.**

Definition 14.2.1. An Ω-*spectrum* X is a sequence of based CW complexes $\{X_n\}_{n\in\mathbb{Z}}$ together with weak homotopy equivalences (the "structure maps") $\alpha_n=\alpha_n^X:X_n\to\Omega X_{n+1}$.

Recall that ΩX_n is the based loop space of X_n; by a theorem of Milnor, the loop space of a CW complex has the homotopy type of a CW complex. A map of Ω-spectra $f:X\to Y$ is a collection of based maps $f_n:X_n\to Y_n$ such that $\alpha_n^Y f_n=f_n\alpha_n^X$. Denote the category of Ω-spectra by $\mathcal{S}$. It has a zero object given by a sequence $*$ of 1-pointed spaces.

Remark 14.2.2. There are variants of this definition, depending on what the X_n are and what restriction to put on the maps α_n, and moreover the definitions are not all the same! The definition in Weibel is that for a *spectrum* we take X_n to be based topological spaces and α_n to be based homeomorphisms and for a *prespectrum* we take α_n to just be based maps.

Example. The *Eilenberg-MacLane spectrum* is an Ω-spectrum with $X_n=K(G,n)$. Recall the adjunction

$$[S^{i+1},K(G,n)]=[\Sigma S^i,K(G,n)]=[S^i,\Omega K(G,n)].$$

This implies that $\Omega K(G,n)$ is a $K(G,n-1)$. Let $\alpha_{n-1}:K(G,n-1)\to\Omega K(G,n)$ be a homotopy equivalence.

Example. The *suspension spectrum* $\Sigma^\infty X$ of a space X is given by $X_n=\Sigma^n X$ and $\alpha_n:\Sigma^n X\to\Omega\Sigma^{n+1}X$ which corresponds to $\epsilon_n=\mathrm{id}:\Sigma(\Sigma^n)X\to\Sigma^{n+1}X$ under the adjunction

$$[\Sigma^n X,\Omega\Sigma^{n+1}X]=[\Sigma(\Sigma^n X),\Sigma^{n+1}X].$$

Here α_n is not a weak homotopy equivalence (and hence $\Sigma^\infty X$ is a prespectrum). Refer to Hatcher, Section 4.J for more information on the topology of $\Omega\Sigma X$.

14.3. **From Ω-spectra to cohomology theories.**

Theorem 14.3.1. *If K is an Ω-spectrum, then the (contravariant) functors $X\mapsto h^n(X):=[X,K_n]$, $n\in\mathbb{Z}$, give a reduced cohomology theory on the category $\mathbf{CW}^\bullet$ of pointed connected CW complexes.*

Proof.

Step 0. The functoriality is clear: Given $\phi : X \to Y$, we have $\phi^* : [Y, K_n] \to [X, K_n]$ given by $f \mapsto f\phi$.

Step 1. We first give a group structure on $[X, K_n]$. This is a consequence of

$$[X, K_n] = [X, \Omega K_{n+1}] = [\Sigma X, K_{n+1}],$$

where the second equality is the adjointness of Σ and Ω. The key is that there is a map $\tau : \Sigma X \to \Sigma X \vee \Sigma X$ and that the sum $f + g$ of $f, g : \Sigma X \to K_{n+1}$ is given by $(f \vee g)\tau$.

Similarly, the abelian group structure on $[X, K_n]$ follows from $[X, K_n] = [\Sigma^2 X, K_{n+2}]$.

Step 2. Verify the reduced cohomology axioms: There are natural coboundary maps $\delta : h^n(A) \to h^{n+1}(X/A)$ such that

(1) (Homotopy axiom) If $f \simeq g : X \to Y$, then $f^* = g^* : h^n(Y) \to h^n(X)$.
(2) (Relative sequence) There is a long exact sequence

$$\to h^n(X/A) \to h^n(X) \to h^n(A) \to h^{n+1}(X/A) \to .$$

(3) (Wedge axiom) If $X = \vee_\alpha X_\alpha$, then $h^n(X) \to \Pi_\alpha h^n(X_\alpha)$ is an isomorphism.

Here "natural" means given $f : (X, A) \to (Y, B)$ the following diagram commutes:

$$\begin{array}{ccc} h^n(A) & \longrightarrow & h^n(X/A) \\ \uparrow & & \uparrow \\ h^n(B) & \longrightarrow & h^n(Y/B). \end{array}$$

(1) and (3) are clear. For (2), recall the Puppe sequence for (X, A):

$$A \xrightarrow{i} X \to C_i \to \Sigma A \to \Sigma X \to \Sigma C_i \to \dots,$$

where $C_i \simeq X/A$. Apply $[\cdot, K_n]$ to obtain:

$$[A, K_n] \leftarrow [X, K_n] \leftarrow [X/A, K_n] \leftarrow [\Sigma A, K_n] \leftarrow [\Sigma X, K_n] \leftarrow [\Sigma X/A, K_n],$$

where $[\Sigma A, K_n] = [A, \Omega K_n] = [A, K_{n-1}]$, $[\Sigma X, K_n] = [X, K_{n-1}]$, $[\Sigma X/A, K_n] = [X/A, K_{n-1}]$. This then becomes:

$$h^n(A) \leftarrow h^n(X) \leftarrow h^n(X/A) \leftarrow h^{n-1}(A) \leftarrow h^{n-1}(X) \leftarrow h^{n-1}(X/A).$$

□

Observe that $h^n(X) \simeq h^{n+1}(\Sigma X)$. This can be proved by using the relative sequence or by checking $[X, K_n] \simeq [\Sigma X, K_{n+1}]$.

Corollary 14.3.2. *For the Eilenberg-MacLane spectrum $K_n = K(G, n)$, $X \mapsto [X, K(G, n)]$ agrees with the (usual) reduced cohomology $\widetilde{H}^n(X; G)$ with G-coefficients.*

Theorem 14.3.1 implies that $X \mapsto [X, K(G, n)]$ gives a reduced cohomology theory. Note that the Hurewicz map

$$[S^n, K(G, i)] \to \widetilde{H}^i(S^n; G)$$

is an isomorphism for all i. To prove the corollary it remains to find natural isomorphisms

$$T : [X, K(G, n)] \to \widetilde{H}^n(X; G)$$

for all CW complexes X; we omit the proof and just remark that the isomorphisms can be constructed cell-by-cell.

14.4. **Brown representability.** Theorem 14.3.1 has the following amazing converse:

Theorem 14.4.1 (Brown representability theorem)**.** *Every reduced cohomology theory on* $\mathbf{CW}^\bullet$ *has the form* $h^n(X) = [X, K_n]$ *for some* Ω*-spectrum* $K = \{K_n\}_{n\in\mathbb{Z}}$*.*

We will not give full details of the proof (cf. Hatcher, Section 4.E). A functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if there exists K and a natural transformation $\eta : \mathrm{Hom}(X, K) \to F(X)$ which is a bijection for each X. (So, strictly speaking, we want to view h^n as a functor from the homotopy category $\mathbf{hCW}^\bullet$.)

If we know $h^n(S^i)$ for all i, then we can try to construct K_n by the requirement $\pi_i(K_n) = h^n(S^i)$. One case where this works is if h^n is reduced singular cohomology with G-coefficients. Then K_n must be a $K(G, n)$. This implies Corollary 14.3.2.

Theorem 14.4.1 follows from the following slightly more general theorem, together with a step relating the K_n's (which will not be explained here).

Theorem 14.4.2. *Let* $F : \mathbf{CW}^\bullet \to \mathbf{Set}^\bullet$ *be a contravariant functor such that:*

(1) *(Homotopy axiom) If* $f, g : X \to Y$ *are homotopic, then* $Ff = Fg$*. (Equivalently view* F *as a functor from the homotopy category* $\mathbf{hCW}^\bullet$*.)*
(2) *(Sheaf axiom) If* $X \in \mathbf{CW}^\bullet$ *and* $X = A \cup B$*, where* $A, B, A \cap B \in \mathbf{CW}^\bullet$*, then if* $a \in F(A)$ *and* $b \in F(B)$ *that restrict to the same element in* $F(A \cap B)$*, there exists* $x \in F(X)$ *that restricts to* a *and* b*.*
(3) *(Wedge axiom) If* $X = \vee_\alpha X_\alpha$*, then* $F(X) = \Pi_\alpha F(X_\alpha)$*.*

Then there exists $K \in \mathbf{CW}^\bullet$ *and* $u \in F(K)$ *such that*

$$T_u : [X, K] \to F(X), \quad T_u(f) = Ff(u)$$

is a bijection for all X*. (Note that* T_u *is a natural transformation.)*

Remark 14.4.3. Axiom (3) implies that $F(pt)$ is a one-element set. (HW: Verify this using $X \wedge \{*\} = X$.)

Remark 14.4.4. Axioms (1)–(3) for $F = h^n$ together with the existence of natural isomorphisms $h^n(X) \simeq h^{n+1}(\Sigma X)$ for all $X \in \mathbf{CW}^\bullet$ is equivalent to the reduced cohomology axioms. (Proof omitted here.)

Example. If T is a pointed connected topological space, then apply Theorem 14.4.2 to the functor $[\cdot, T]$. Then there exists $C \in \mathbf{CW}^\bullet$ such that $[X, C] = [X, T]$ for all $X \in \mathbf{CW}^\bullet$. In other words, the Brown representability theorem implies the CW approximation theorem.

14.5. **Proof of Theorem 14.4.2.**

Step 1. We construct K inductively cell-by-cell as usual so that *the theorem holds for all spheres* $X = S^n$ *with* $n \geq 1$. Recall that if we know $F(S^n)$ for all $n > 0$, then we can try to construct K by the requirement $\pi_n(K) = F(S^n)$ (just like for $K(G, i)$). The complication in the proof comes from keeping track of the element $u \in K$.

Start with $K_0 = pt$. Let u_0 be the unique element of $F(K_0)$, recalling Remark 14.4.3. Arguing by induction, assume that there exist K_n and $u_n \in F(K_n)$ such that

$$T_{u_n} : \pi_i(K_n) \to F(S^i), \quad f \mapsto Ff(u_n),$$

is surjective for $i \leq n$ and has trivial kernel for $i < n$.

We will construct $K_{n+1} \supset K_n$ and u_{n+1} that restricts to u_n: Denote the representatives of the kernel of $T_{u_n} : \pi_n(K_n) \to F(S^n)$ by $f_\alpha : S^n_\alpha \to K_n$ and set $f = \vee_\alpha f_\alpha$. Let M_f and C_f be the mapping cylinder and mapping cone of f. We then set

$$K_{n+1} = C_f \vee (\vee_\beta S^{n+1}_\beta),$$

where β ranges over $F(S^{n+1})$.

Fact/HW: Axioms (1)–(3) in Theorem 14.4.2 imply the exactness of

$$F(X/A) \to F(X) \to F(A)$$

for each inclusion $A \hookrightarrow X$.

Apply the fact to $F(C_f) \to F(K_n) = F(M_f) \to F(\vee_\alpha S^n_\alpha)$. Viewing $u_n \in F(M_f)$, it is mapped to $0 \in F(\vee_\alpha S^n_\alpha)$, and hence comes from $w \in F(C_f)$. We then define $u_{n+1} \in F(K_{n+1})$ to restrict to $u_n \in F(K_n)$ and $\beta \in F(S^{n+1}_\beta)$; this exists by (3) and the fact that u_n and β restrict to the same point. For HW, verify that $T_{u_{n+1}}$ is surjective for $i \leq n+1$ and has trivial kernel for $i < n+1$.

We then set $K = \cup_n K_n$. For technical reasons of defining u, we use the *mapping telescope* of $K_0 \hookrightarrow K_1 \hookrightarrow \dots$ given by

$$T = \cup_i (K_i \times [i, i+1]) \subset K \times [0, \infty),$$

with the appropriate quotienting in the pointed category. The mapping telescope T is homotopy equivalent to K. Cut up $T = A \cup B$ where A (resp. B) is the union of the $K_i \times [i, i+1]$ for i even (resp. odd). Since A and B are wedges, we can define $\vee_{i \text{ even}} u_i$ and $\vee_{i \text{ odd}} u_i$ on A and B. They can be glued into u using (2).

Step 2. Show that the theorem holds for any $X \in \mathbf{CW}^\bullet$.

Surjectivity of T_u: we show that given any $x \in F(X)$ there exists a map $f : X \to K$ such that $Ff(u) = x$. Let $Z = X \vee K$ and $z = x \vee u \in F(X \vee K)$. By the methods of Step 1, there exists $(Z, z) \hookrightarrow (K', u')$ where (K', u') also satisfies the conditions of the theorem for spheres. Here $(K', u') \supset (K, u)$ and is a weak homotopy equivalence. Hence K' deformation retracts onto K and we obtain a map $X \to K'$ with the desired property.

The injectivity of T_u is analogous and is omitted.

Example. Let G be a topological group and let $F : \mathbf{CW}^\bullet \to \mathbf{Set}^\bullet$ be the functor such that $F(X)$ is the set of isomorphism classes of principal G-bundles $P \to X$. Then F is represented by $K = BG$, the *classifying space of* G. While BG admits an explicit construction, the Brown representability theorem proves the existence of BG.

Example. (Complex topological K-theory) Let $\mathrm{Vect}(X)$ be the group of isomorphism classes of complex vector finite rank complex vector bundles over X. $\mathrm{Vect}(X)$ is a monoid under direct sum $\oplus$, and the *Grothendieck group* $K^0(X)$ is the *group completion* of the monoid $\mathrm{Vect}(X)$. [The *group completion* of a monoid M is $M \times M/\sim$, where $(m_1, m_2) \sim (m_1', m_2')$ if there exists $m \in M$ such that $m_1 + m_2' + m = m_1' + m_2 + m$. Think of (m_1, m_2) as $m_1 - m_2$.] There is a reduced version of $K^0(X)$, denoted by $\widetilde{K}^0(X)$. The functor $\widetilde{K}^0$ is represented by $BU \times \mathbb{Z}$, where $BU = \lim_{n\to\infty} BU(n)$.

Bott periodicity: Apply Ω iteratively to $BU \times \mathbb{Z}$. It's not hard to see that $\Omega(BU \times \mathbb{Z}) \simeq U$, where $\simeq$ denotes weak homotopy equivalence. Bott periodicity states that $\Omega U \simeq BU \times \mathbb{Z}$ and hence $\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}$. This implies that the $\Omega^k(BU \times \mathbb{Z})$ are 2-periodic. The Ω-spectrum $K_n = U$ for n odd and $BU \times \mathbb{Z}$ for n even represents the "higher" K-groups $\widetilde{K}^n$.

This would be a good starting point for the study of vector bundles and K-theory, but alas we are out of time....