Math 20E Homework 1

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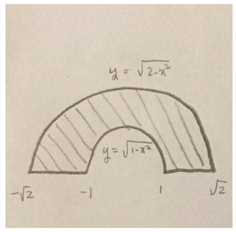
Problem 5.3 #11: Find

$$\iint_D 1 + xy \; dA$$

for the region D described in the picture.

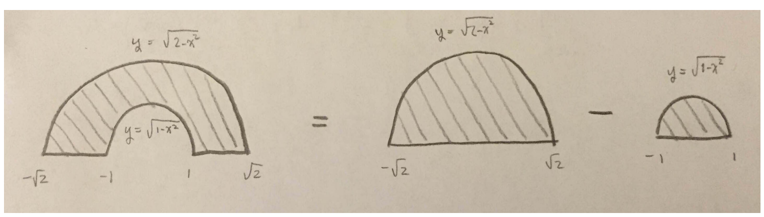


Method 1:

$$\iint_D 1 + xy \; dA = \iint_D 1 \; dA + \iint_D xy \; dA$$

Being the area of the region D, the first integral of RHS equals $\frac{1}{2}(\pi(\sqrt{2})^2 - \pi) = \frac{1}{2}\pi$. The second integral is split into integrals over two regions

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xy \; dydx - \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} xy \; dydx.$$



Integrating against y, the first integral becomes

$$\int_{-\sqrt{2}}^{\sqrt{2}} x(\frac{2 - x^2}{2}) \; dx$$

The above integral is indeed 0 because the integrand is an odd function and the domain of integration is symmetric. By the same reasoning, $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} xy \; dydx = 0$. Hence,

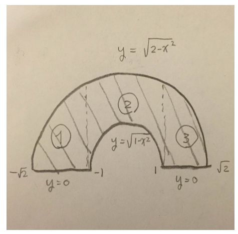
$$\iint_D 1 + xy \; dA = \frac{\pi}{2}.$$

Method 2:

$$\iint_D 1 + xy\ dA = \int_0^{\sqrt{2}} \int_0^{2\pi} (1 + r^2 \cos\theta \sin\theta)\ r d\theta dr = \text{...easy...} = \pi/2.$$

Method 3 (highly NOT recommended):

Split the original integral into integrals over three regions



$$\begin{aligned}
\iint_D xy\ dA &= \int_{-\sqrt{2}}^{-1} \int_0^{\sqrt{2-x^2}} xy\ dydx + \int_{-1}^{-1} \int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} xy\ dydx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xy\ dydx \\
&= \int_{-\sqrt{2}}^{-1} \frac{x(2-x^2)}{2}\ dx + \int_{-1}^{1} \frac{x((2-x^2)-(1-x^2))}{2}\ dx + \int_1^{\sqrt{2}} \frac{x(2-x^2)}{2}\ dx \\
&= -\frac{1}{8} + 0 + \frac{1}{8} \ = 0.
\end{aligned}$$

$$\begin{aligned}
\iint_D 1\ dA &= \int_{-\sqrt{2}}^{-1} \int_0^{\sqrt{2-x^2}} 1\ dydx + \int_{-1}^{-1} \int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} 1\ dydx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 1\ dydx \\
&= \int_{-\sqrt{2}}^{-1} \sqrt{2-x^2}\ dx + \int_{-1}^{1} \sqrt{2-x^2} - \sqrt{1-x^2}\ dx + \int_1^{\sqrt{2}} \sqrt{1-x^2}\ dx \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\ dx - \int_{-1}^{1} \sqrt{1-x^2}\ dx
\end{aligned}$$

Substituting x in the first integral with $\sqrt{2}\sin\theta$ and the second with $\sin\theta$, we finally get

$$\iint_D 1\ dA = \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

Problem 5.3 #15: Find the volume V of the region inside the surface $z = x^2 + y^2$ and between $z = 0$ and $z = 10$.

Method 1:

The volume can be written as

$$\int_{-\sqrt{10}}^{\sqrt{10}} \int_{-\sqrt{10-x^2}}^{\sqrt{10-x^2}} 10 - (x^2 + y^2)\ dydx$$

Use polar coordinate to convert the above integral to

$$\int_0^{\sqrt{10}} \int_0^{2\pi} (10 - r^2) r d\theta dr = 2\pi \int_0^{\sqrt{10}} (r - r^3) dr = 50\pi.$$

Method 2:

By the Slice Method-Cavalieri's Principle in Section 5.1, the volume V is

$$\int_0^{10} \pi(\sqrt{z})^2\ dz = 50\pi.$$