Stokes' Theorem Example

The following is an example of the time-saving power of Stokes' Theorem.

Ex: Let $\vec{F}(x,y,z) = \arctan(xyz)\ \vec{\mathbf{i}}\ + (x + xy + \sin(z^2))\ \vec{\mathbf{j}}\ + z\sin(x^2)\ \vec{\mathbf{k}}$. Evaluate $\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for each of the following oriented surfaces S.

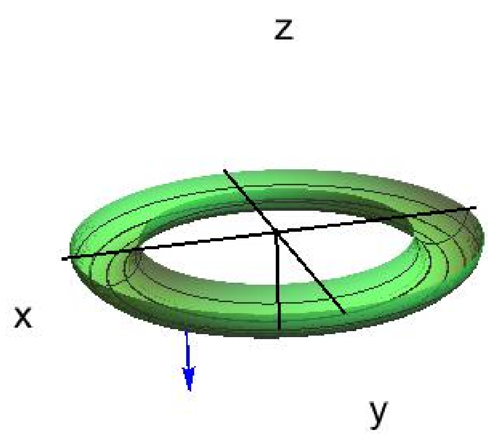
(a) S is the unit sphere oriented by the outward pointing normal.

(b) S is the unit sphere oriented by the inward pointing normal.

(c) S is a torus with $r = 1$, $R = 5$, oriented by the outward pointing normal.

(d) S is a torus with $r = 1$, $R = 5$, oriented by the inward pointing normal.

(e) S is the half of the torus below the x-y plane with $r = 1$, $R = 5$, oriented by the outward pointing normal, as shown.



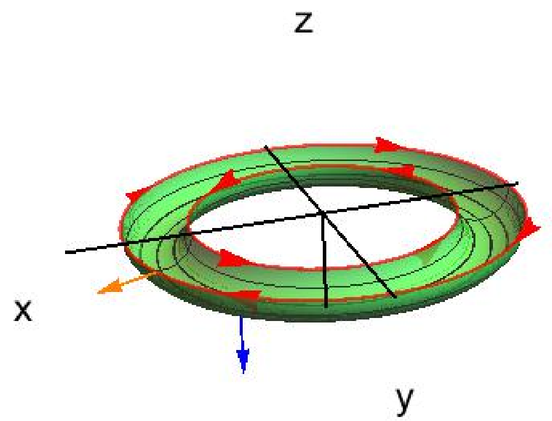
Solution: At first glance, this looks like it's going to be a ton of work to do this. Just computing $\nabla \times \vec{\mathbf{F}}$ takes a while, much less evaluating $\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$ for each of the above surfaces. Thank goodness for Stokes' Theorem:

As discussed in lecture, for **(a)-(d)**, S has no boundary ∂S, so by Stokes' Theorem,

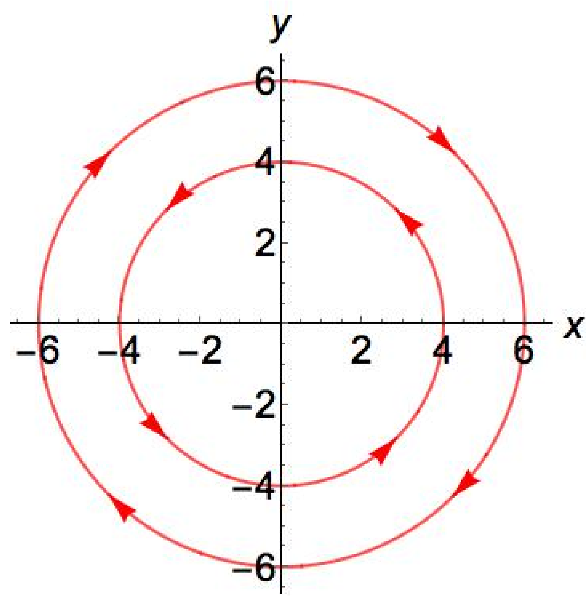
$$\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 0$$

For **(e)**, Stokes' Theorem will allow us to compute the surface integral without ever having to parametrize the surface! The boundary ∂S consists of two circles in the x-y plane, one of radius $5 - 1 = 4$ and the other of radius $5 + 1 = 6$. We can parametrize them by $c_4(t) = (4\cos t, 4\sin t, 0)$ for $0 \le t \le 2\pi$ and $c_6(t) = (6\cos t, 6\sin t, 0)$ for $0 \le t \le 2\pi$.

Because these circles come from the boundary of a surface, we can't just orient them the way we would when using Green's Theorem on region between them. Instead, we have to imagine walking on ∂S with our head pointing in the same direction as the orienting normal vector. In this problem, that means walking with our head pointing with the outward pointing normal. So in the picture below, we are represented by the orange vector as we walk around the boundary.



In order to keep S on our left, we have to walk in the direction as indicated in the figure. As in lecture, if we draw ∂S in the x-y plane, this means it gets the following orientation:



This is exactly the opposite of the orientation we would have if we were simply considering those circles as the boundary of the region in the x-y plane between them. This happens because we had to walk with our head pointing with the given normal.

Now let's compute the integral. We see that our parametrization for c_6 is goes the wrong way, so we get minus sign in front of the corresponding integral here:

$$\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = -\int_0^{2\pi} [\vec{\mathbf{F}}(c_6(t)) \cdot c_6'(t)]dt + \int_0^{2\pi} [\vec{\mathbf{F}}(c_4(t)) \cdot c_4'(t)]dt$$

Since $z = 0$ on both circles, so we get a much simpler integrals:

$$\begin{aligned}
\int_0^{2\pi} [\vec{\mathbf{F}}(c_6(t)) \cdot c_6'(t)] dt &= \int_0^{2\pi} [(0, 6\cos t + 6^2 \cos t \sin t, 0) \cdot (-6\sin t, 6\cos t, 0)] dt \\
&= \int_0^{2\pi} [6^2 \cos^2 t + 6^3 \cos^2 t \sin t] dt \\
&\text{(Now using } \cos 2t = 2\cos^2 t - 1 \implies \cos^2 t = \frac{\cos 2t + 1}{2} \implies \int \cos^2 t dt = \frac{\sin 2t}{4} + \frac{t}{2} + C), \\
&= 6^2 \left[\frac{\sin 2t}{4} + \frac{t}{2} - \frac{6}{3} \cos^3 t \right]_0^{2\pi} \\
&= 6^2 \left[\frac{2\pi}{2} - \frac{6}{3} + \frac{6}{3} \right] = 36\pi
\end{aligned}$$

and similarly,

$$\begin{aligned}
\int_0^{2\pi} [\vec{\mathbf{F}}(c_4(t)) \cdot c_4'(t)] dt &= \int_0^{2\pi} [(0, 4\cos t + 4^2 \cos t \sin t, 0) \cdot (-4\sin t, 4\cos t, 0)] dt \\
&= \int_0^{2\pi} [4^2 \cos^2 t + 4^3 \cos^2 t \sin t] dt \\
&= 4^2 \left[\frac{\sin 2t}{4} + \frac{t}{2} - \frac{4}{3} \cos^3 t \right]_0^{2\pi} \\
&= 4^2 \left[\frac{2\pi}{2} - \frac{4}{3} + \frac{4}{3} \right] = 16\pi
\end{aligned}$$

Plugging these values in, we have

$$\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = -36\pi + 16\pi = -20\pi$$