Name:________________________ **PID:**________________________

TA:________________________ **Sec. No:**______ **Sec. Time:**____________

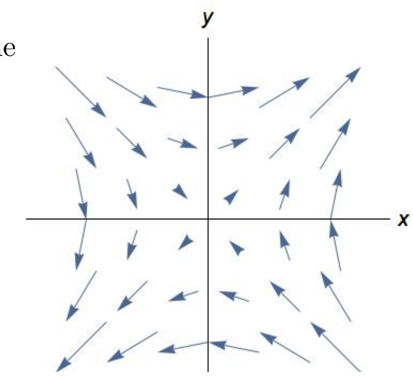
Q #:	Score:
1	/10
2	/10
3	/10
4	/10
5	/10
Total	/50

Math 20E Midterm 2

Summer Session II, 2015

1. [10 points total]: Circle your answer to each of the following true/false or multiple-choice questions.

(a) [3 points]: Which of the following formulas defines the vector field $\vec{\mathbf{F}} : \mathbb{R}^2 \to \mathbb{R}^2$ represented in the figure?

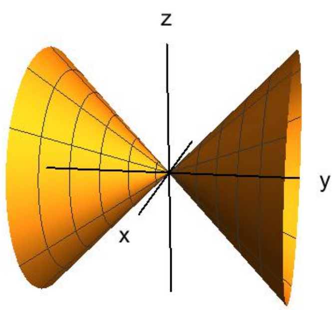


A. $\vec{\mathbf{F}}(x, y) = (x, -y)$

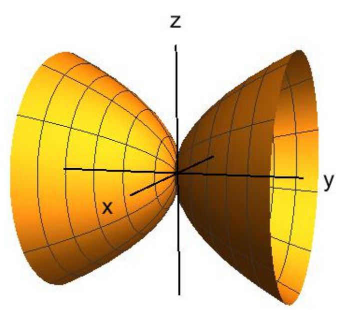
B. $\vec{\mathbf{F}}(x, y) = (y, x)$

C. $\vec{\mathbf{F}}(x, y) = (y, -x)$

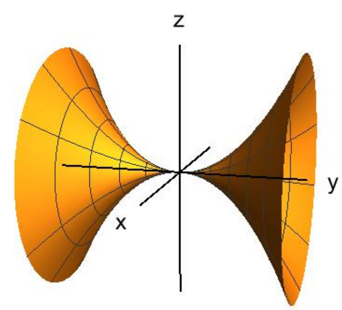
(b) [3 points]: Which of the following shows the surface $x^2 - y^2 + z^2 = 0$?



A.



B



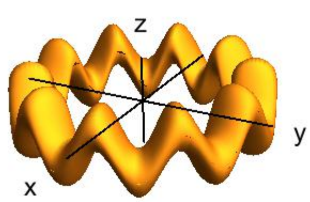
C

(c) [4 points]: Recall that a torus can be parametrized by

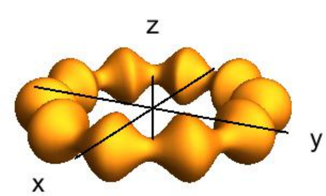
$$\Phi(\theta, \phi) = ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$$

for (θ, ϕ) in $D = [0, 2\pi] \times [0, 2\pi]$ where r is the minor radius and R is the major radius. Which of the following pictures shows $S = \Phi(D)$ for

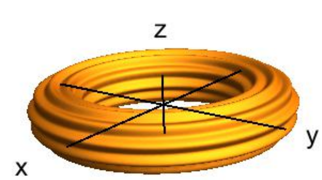
$$\Phi(\theta, \phi) = (\ \ (10 + (2 + \sin 10\phi)\cos\theta)\cos\phi, \quad (10 + (2 + \sin 10\phi)\cos\theta)\sin\phi, \quad (2 + \sin 10\phi)\sin\theta\ \ )?$$



A

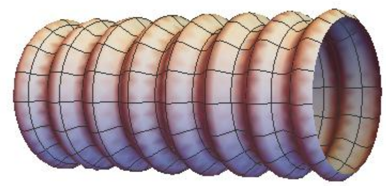


B.



C

2. [10 points total]: Consider the 15π-meter long storm drain pipe pictured and parametrized by



$$\Phi(y,\theta) = (\,(10+\sin y)\cos\theta,\;\; y,\;\; (10+\sin y)\sin\theta\,)$$

for (y,θ) in the domain $D = [0, 15\pi] \times [0, 2\pi]$.

(a) [4 points]: Calculate $\|\vec{\mathbf{T}}_y \times \vec{\mathbf{T}}_\theta\|$.

$$\vec{\mathbf{T}}_y \times \vec{\mathbf{T}}_\theta = \begin{vmatrix} \cos y\cos\theta & -(10+\sin y)\sin\theta & \vec{\mathbf{i}} \\ 1 & 0 & \vec{\mathbf{j}} \\ \cos y\sin\theta & (10+\sin y)\cos\theta & \vec{\mathbf{k}} \end{vmatrix}$$

$$= (10+\sin y)\cos\theta\,\vec{\mathbf{i}} + \cos y(10+\sin y)(\cos^2\theta+\sin^2\theta)(-\vec{\mathbf{j}}\,) + (10+\sin y)\sin\theta\,\vec{\mathbf{k}}$$

$$\implies \|\vec{\mathbf{T}}_y \times \vec{\mathbf{T}}_\theta\| = \sqrt{(10+\sin y)^2(\cos^2\theta+\cos^2 y+\sin^2\theta)} = (10+\sin y)\sqrt{1+\cos^2 y}$$

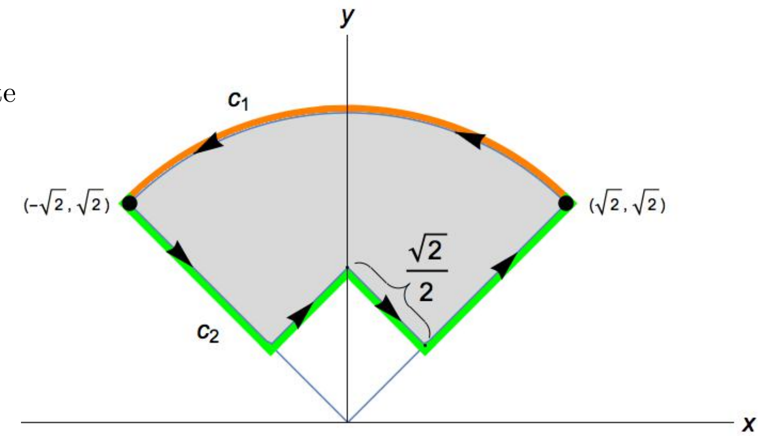
where $\sqrt{(10+\sin y)^2} = 10+\sin y$, because $10+\sin y$ is always positive.

(b) [6 points]: Over the years, dirt accumulates on the surface S of the pipe. If the density of dirt at a point $(x,y,z) \in S$ is given by $f(x,y,z) = \sqrt{1+\cos^2 y}$ kg/m^2, what is the total mass of dirt on the surface of the pipe in kilograms (kg)?

$$\begin{aligned}
\text{Total Mass } &= \iint_S f\,dS = \iint_D f(\Phi(y,\theta))\|\vec{\mathbf{T}}_y \times \vec{\mathbf{T}}_\theta\|\,dA \\
&= \int_0^{2\pi}\int_0^{15\pi} \sqrt{1+\cos^2 y}(10+\sin y)\sqrt{1+\cos^2 y}\,dy\,d\theta \\
&= \left[\int_0^{2\pi} d\theta\right]\cdot\left[\int_0^{15\pi}(10+\sin y+10\cos^2 y+\sin y\cos^2 y)\,dy\right] \\
&(\text{Next using } \cos 2y = 2\cos^2 y - 1 \implies \cos^2 y = \frac{\cos 2y+1}{2}) \\
&= 2\pi\left[\int_0^{15\pi}(10+\sin y+5\cos 2y+5+\sin y\cos^2 y)\,dy\right] \\
&= 2\pi\left[15y - \cos y + \frac{5}{2}\sin 2y + \frac{-1}{3}\cos^3 y\right]_0^{15\pi} \\
&= 2\pi\left[\left[15\cdot 15\pi - (-1) + \frac{-1}{3}(-1)^3\right] - \left[-1 + \frac{-1}{3}(1)^3\right]\right] \\
&= 2\pi\left[(15)^2\pi + 2 + \frac{2}{3}\right]
\end{aligned}$$

3. [10 points total]: Let D be the shaded portion of the baseball field pictured, which consists of square of side length $\frac{\sqrt{2}}{2}$ inside one quarter of the disc $x^2 + y^2 \leq 4$. Let $P(x, y) = x^2 - y$ and $Q(x, y) = x$.

(a) [1 point]: Use geometric formulas to calculate the area of D.



$$\text{Area}(D) = \frac{1}{4} \cdot \pi(2)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \pi - \frac{1}{2}$$

(b) [4 points]: The outer arc $\mathbf{c_1}$ of the field is parametrized by $\mathbf{c_1}(t) = (2\cos t, 2\sin t)$ for $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$. Evaluate $\int_{\mathbf{c_1}} Pdx + Qdy$.

$c_1'(t) = (-2\sin t, 2\cos t) = (-y, x)$.

$$\int_{c_1} Pdx + Qdy = \int_{\pi/4}^{3\pi/4} \left[((2\cos t)^2 - 2\sin t, 2\cos t) \cdot (-2\sin t, 2\cos t)\right] dt$$

$$= \int_{\pi/4}^{3\pi/4} \left[-8\cos^2 t \sin t + 4\sin^2 t + 4\cos^2 t\right] dt = \int_{\pi/4}^{3\pi/4} \left[-8\cos^2 t \sin t + 4\right] dt$$

$$= \left[\frac{8}{3}\cos^3 t + 4t\right]_{\pi/4}^{3\pi/4} = \left[\frac{8}{3}\left(-\frac{1}{\sqrt{2}}\right)^3 + 3\pi\right] - \left[\frac{8}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \pi\right]$$

$$= -2 \cdot \frac{8}{3}\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + 2\pi = -\frac{8}{3\sqrt{2}} + 2\pi$$

(c) [5 points]: Use Green's Theorem and your answers for parts **(a)** and **(b)** to evaluate $\int_{\mathbf{c_2}} Pdx + Qdy$ where $\mathbf{c_2}$ is the part of ∂D not covered by $\mathbf{c_1}$, oriented as shown.
[If you did not solve **(a)** and/or **(b)**, you may use constants a and b in place of those answers]

By Green's Theorem,

$$\iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \int_{\partial D} Pdx + Qdy = \left[\int_{c_1} Pdx + Qdy\right] + \left[\int_{c_2} Pdx + Qdy\right]$$

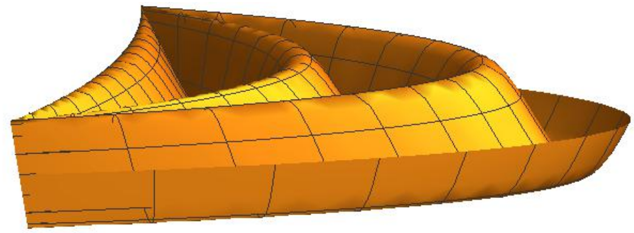
and so

$$\int_{c_2} Pdx + Qdy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA - \left[\int_{c_1} Pdx + Qdy\right]$$

$$= \iint_D [1 - (-1)] dA - \left[-\frac{8}{3\sqrt{2}} + 2\pi\right] = 2\iint_D 1 \cdot dA + \frac{8}{3\sqrt{2}} - 2\pi$$

$$= 2\text{Area}(D) + \frac{8}{3\sqrt{2}} - 2\pi = 2\left[\pi - \frac{1}{2}\right] + \frac{8}{3\sqrt{2}} - 2\pi = \frac{8}{3\sqrt{2}} - 1$$

4. [10 points total]: Let S be the surface parametrized by



$$\Phi(x,\theta) = (x, \theta(1-x^2), \sin\theta)$$

for (x,θ) in the domain $D = [-1,1]\times[0,6\pi]$.

(a) [3 points]: Find $\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_\theta$.

$$\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_\theta = \begin{vmatrix} 1 & 0 & \vec{\mathbf{i}} \\ -2x\theta & (1-x^2) & \vec{\mathbf{j}} \\ 0 & \cos\theta & \vec{\mathbf{k}} \end{vmatrix}$$

$$= -2x\theta\cos\theta\,\vec{\mathbf{i}} + \cos\theta(-\vec{\mathbf{j}}) + (1-x^2)\,\vec{\mathbf{k}} = -2x\theta\cos\theta\,\vec{\mathbf{i}} - \cos\theta\,\vec{\mathbf{j}} + (1-x^2)\,\vec{\mathbf{k}}$$

(b) [7 points]: Let $\vec{\mathbf{F}}(x,y,z) = (x,\ x^2,\ z)$. Evaluate $\iint\limits_{\Phi} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}$.

$$\iint\limits_{\Phi} \vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} = \iint\limits_{D} \left[\vec{\mathbf{F}}(\Phi(x,\theta))\cdot(\vec{\mathbf{T}}_x\times\vec{\mathbf{T}}_\theta)\right] dA$$

$$= \iint\limits_{D} \left[(x,x^2,\sin\theta)\cdot(-2x\theta\cos\theta, -\cos\theta, 1-x^2)\right] dA$$

$$= \iint\limits_{D} \left[-2x^2\theta\cos\theta - x^2\cos\theta + (1-x^2)\sin\theta\right] dA$$

$$= \int_0^{6\pi}\int_{-1}^{1} \left[-x^2(2\theta\cos\theta+\cos\theta+\sin\theta)+\sin\theta\right] dx\,d\theta$$

$$= \int_0^{6\pi} \left[-\frac{x^3}{3}(2\theta\cos\theta+\cos\theta+\sin\theta)+x\sin\theta\right]_{-1}^{1} d\theta$$

$$= \int_0^{6\pi} \left[\left[-\frac{1}{3}(2\theta\cos\theta+\cos\theta+\sin\theta)+\sin\theta\right] - \left[-\frac{(-1)^3}{3}(2\theta\cos\theta+\cos\theta+\sin\theta)+(-1)\sin\theta\right]\right] d\theta$$

$$= \int_0^{6\pi} \left[-\frac{2}{3}(2\theta\cos\theta+\cos\theta+\sin\theta)+2\sin\theta\right] d\theta = \int_0^{6\pi}\left[-\frac{2}{3}\cos\theta+\frac{4}{3}\sin\theta\right] d\theta - \frac{4}{3}\int_0^{6\pi}\theta\cos\theta\,d\theta$$

(now integrating $\int_0^{6\pi}\theta\cos\theta\,d\theta$ by parts)

$$= \left[-\frac{2}{3}\sin\theta+\frac{4}{3}(-\cos\theta)\right]_0^{6\pi} - \frac{4}{3}\left[[\theta\sin\theta]_0^{6\pi} - \int_0^{6\pi}\sin\theta\,d\theta\right]$$

$$= \left[0+\frac{4}{3}(-1)-0-\frac{4}{3}(-1)\right] - \frac{4}{3}\left[[6\pi\cdot 0 - 0] + [\cos\theta]_0^{6\pi}\right]$$

$$= 0 - \frac{4}{3}\,[[0]+[1-1]] = 0$$

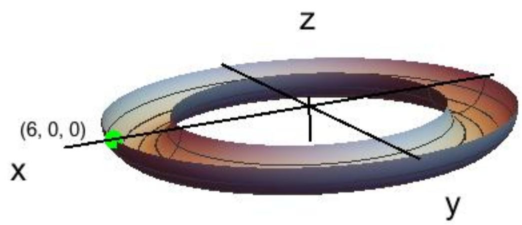
5. [10 points total]:

(a) [3 points]: Recall that a torus can be parametrized by

$$\Phi(\theta, \phi) = ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$$

for (θ, ϕ) in $D = [0, 2\pi] \times [0, 2\pi]$ where r is the minor radius and R is the major radius.

The pictured tray S consists of the half of a torus below the x-y plane for $r = 1$ and $R = 5$. Find a domain $D \subset \mathbb{R}^2$ and a parametrization $\Phi : D \to \mathbb{R}^3$ such that $S = \Phi(D)$.



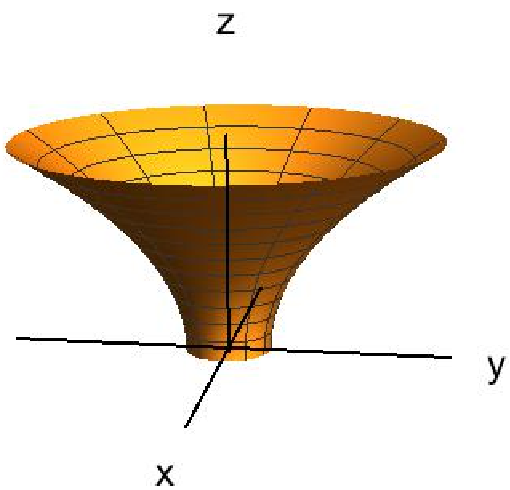
We need only restrict the domain D for the torus parametrization. Since θ is the angle going around the small circles, we can restrict $\pi \le \theta \le 2\pi$ (or $-\pi \le \theta \le 0$, or $-47\pi \le \theta \le -46\pi$...), so plugging in $R = 5$ and $r = 1$, we get

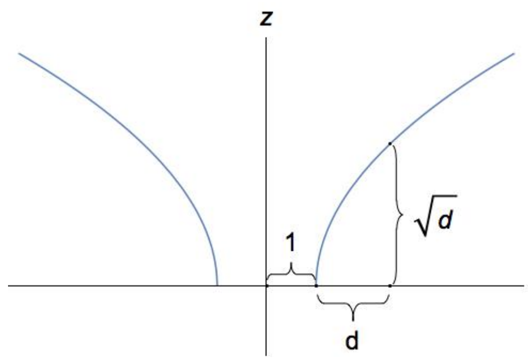
$$\Phi(\theta, \phi) = ((5 + \cos\theta)\cos\phi, (5 + \cos\theta)\sin\phi, \sin\theta)$$

for (θ, ϕ) in $D = [\pi, 2\pi] \times [0, 2\pi]$.

(b) [7 points]: Let S be the surface of the pictured birdbath for $0 \leq z \leq 4$. Horizontal cross-sections of the S are circles, the base of the birdbath being a circle of radius 1 in the x-y plane. At a distance d from the base of the birdbath, the birdbath has height $\sqrt{d}$ as shown below.

Find a domain $D \subset \mathbb{R}^2$ and a parametrization $\Phi : \ D \ \rightarrow \ \mathbb{R}^3$ such that $S = \Phi(D)$.





Just like on the review guide, we can use modified cylindrical coordinates where the radius changes with z, so we just need to find the radius r at each height z. The radius is $r = 1 + d$, so that $z = \sqrt{d} = \sqrt{r-1} \implies r = z^2 + 1$. So our parametrization is

$$\Phi(z, \theta) = ((z^2+1)\cos\theta, (z^2+1)\sin\theta, z)$$

for (z, θ) in $D = [0, 4] \times [0, 2\pi]$.

It is of course possible to parametrize in many other ways, for example using d as a parameter:

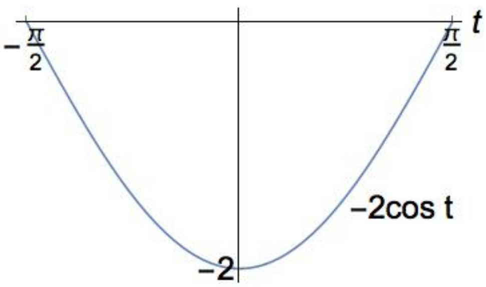
$$\Phi(d, \theta) = ((1+d)\cos\theta, (1+d)\sin\theta, \sqrt{d})$$

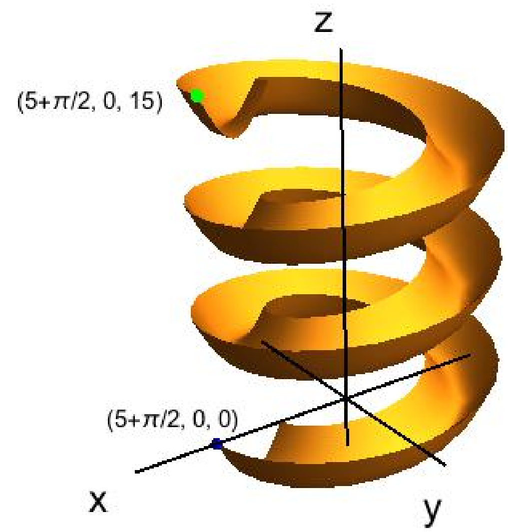
for (d, θ) in $D = [0, 16] \times [0, 2\pi]$, or using r as a parameter:

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{r-1})$$

for (r, θ) in $D = [1, 17] \times [0, 2\pi]$.

(c) [Extra Credit, 5 points]: The pictured slide S consists of a chute whose vertical cross-sections have the shape of the function $-2\cos t$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.





The bottom of the chute is always a distant $R = 5$ from the z-axis, and the slide spirals around the z-axis at a constant incline, making 3 full revolutions as shown. Find a domain $D \subset \mathbb{R}^2$ and a parametrization $\Phi : D \to \mathbb{R}^3$ such that $S = \Phi(D)$.

One idea here is to adapt the parametrization of a torus with $R = 5$. We keep the parameter ϕ to revolve the slide around the vertical axis, but since the vertical cross sections are not circular, we replace θ. The role of $R + r\cos\theta$ (the distance from the z-axis in the torus parametrization) is instead played by $R + t$, so that our parametrization takes the form

$$\Phi(t, \phi) = ((R + t)\cos\phi, (R + t)\sin\phi, \underline{\hspace{3cm}})$$

for $(t, \phi) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 6\pi]$, where we use 6π since three full revolutions are made. To figure out the z-coordinate, we add the contribution from the $-2\cos t$ shape to the contribution from the slide spiraling upward. Since the rate of incline is constant and a total vertical change of 15 is achieved over 6π radians of revolution, the latter is given by $\frac{15}{6\pi}\phi = \frac{5}{2\pi}\phi$. Hence our answer is:

$$\Phi(t, \phi) = ((R + t)\cos\phi, (R + t)\sin\phi, -2\cos t + \frac{5}{2\pi}\phi)$$

for $(t, \phi) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 6\pi]$. Plugging in $R = 5$, this is

$$\Phi(t, \phi) = ((5 + t)\cos\phi, (5 + t)\sin\phi, -2\cos t + \frac{5}{2\pi}\phi)$$

for $(t, \phi) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 6\pi]$. There are, of course, other possible correct answers.