Name:______________________ PID:______________________

TA:______________________ Sec. No:_____ Sec. Time:___________

Q #:	Score:
1	/10
2	/10
3	/10
4	/10
5	/10
Total	/50

Math 20E Midterm 1

Summer Session II, 2015

1. [10 points total]: Circle your answer to each of the following true/false or multiple-choice questions. [2 points each]:

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\vec{x_0} \in \mathbb{R}^n$.

(a) **True** **or False:** If f is differentiable at $\vec{x_0}$, then all of the partial derivatives of f must exist at $\vec{x_0}$.

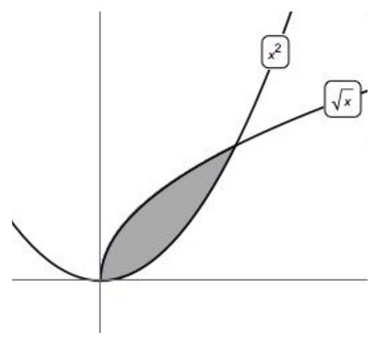
(b) **True or** **False** **:** If all of the partial derivatives of f exist at $\vec{x_0}$, then f must be differentiable at $\vec{x_0}$.

(c) **True** **or False:** If f is differentiable at $\vec{x_0}$, then f must be continuous at $\vec{x_0}$.

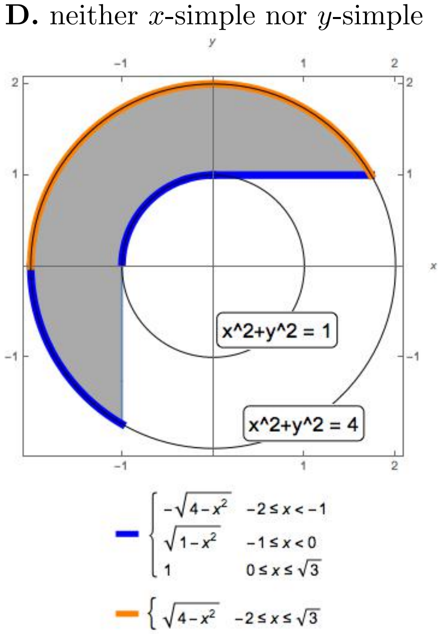
(d) Which of the following double integrals are evaluated over this shaded domain?

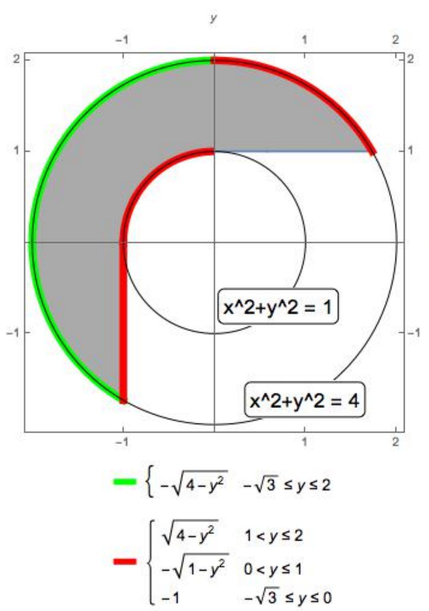
(i) $\int_0^1 \int_{x^2}^{\sqrt{x}} dydx$ (ii) $\int_0^1 \int_{y^2}^{\sqrt{y}} dxdy$ (iii) $\int_0^1 \int_{\sqrt{y}}^{y^2} dxdy$

A. (i) only
B. (ii) only
C. (iii) only
D. (i) and (ii)
E. (i) and (iii)



(e) The shaded region below is
A. x-simple
B. y-simple
C. both x-simple and y-simple
D. neither x-simple nor y-simple





2. [10 points total]:

(a) [2 points]: Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(x, y, z) = xy^2 + \cos z$. Find the matrix of partial derivatives $\mathrm{D}g(x, y, z)$.

Solution:

$$\mathrm{D}g(x, y, z) = \begin{bmatrix} y^2 & 2xy & -\sin z \end{bmatrix}$$

(b) [3 points]: Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(u, v) = (u, v \sin u, e^{v-u})$. Find the matrix of partial derivatives $\mathrm{D}f(u, v)$.

Solution:

$$\mathrm{D}f(u, v) = \begin{bmatrix} 1 & 0 \\ v \cos u & \sin u \\ -e^{v-u} & e^{v-u} \end{bmatrix}$$

(c) [2 points]: Find $f(0,1)$ and $\mathrm{D}f(0,1)$.

Solution:

$$f(0,1) = (0, 1\cdot\sin 0, e^{1-0}) = (0,0,e)$$

$$\mathrm{D}f(0,1) = \begin{bmatrix} 1 & 0 \\ (1)\cos 0 & \sin 0 \\ -e^{1-0} & e^{1-0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -e & e \end{bmatrix}$$

(d) [3 points]: Let $h : \mathbb{R}^3 \to \mathbb{R}$ be defined by $h = g \circ f$. Use the chain rule to find $\mathrm{D}h(0,1)$.

Solution: By the chain rule,

$$\mathrm{D}h(0,1) = [\mathrm{D}g(f(0,1))][\mathrm{D}f(0,1)] = [\mathrm{D}g(0,0,e)] \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -e & e \end{bmatrix}$$

We have

$$[\mathrm{D}g(0,0,e)] = \begin{bmatrix} 0^2 & 2(0)(0) & -\sin e \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin e \end{bmatrix}$$

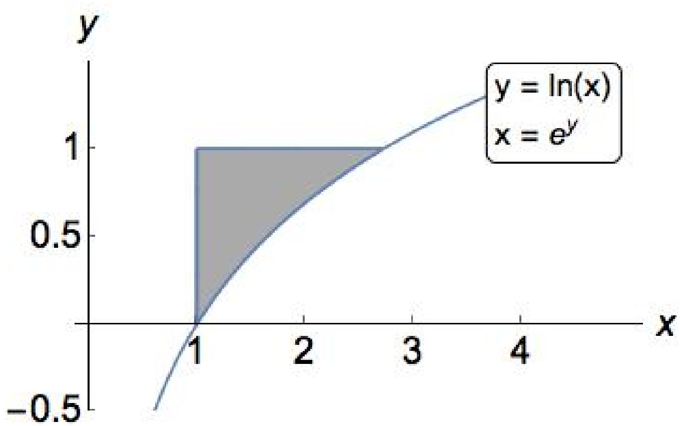
and hence

$$\begin{aligned} \mathrm{D}h(0,1) &= \begin{bmatrix} 0 & 0 & -\sin e \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -e & e \end{bmatrix} \\ &= \begin{bmatrix} (0)(1) + (0)(1) + (-\sin e)(-e) & (0)(0) + (0)(0) + (-\sin e)(e) \end{bmatrix} \\ &= \begin{bmatrix} e\sin e & -e\sin e \end{bmatrix} \end{aligned}$$

3. [10 points]: Evaluate

$$\int_1^e \int_{\ln x}^1 \frac{\cos(y^2)}{x}\, dydx$$

Solution: Since we don't know an easy y-antiderivative for $\frac{\cos(y^2)}{x}$, we want to switch the order of integration. To do so, we draw the region of integration:



Then

$$\begin{aligned}
\int_1^e \int_{\ln x}^1 \frac{\cos(y^2)}{x}\, dydx &= \int_0^1 \int_1^{e^y} \frac{\cos(y^2)}{x}\, dxdy \\
&= \int_0^1 \cos(y^2)\, [\ln x]_1^{e^y}\, dy \\
&= \int_0^1 \cos(y^2)\, [\ln(e^y) - \ln 1]\, dy \\
&= \int_0^1 \cos(y^2)\, [y - 0]\, dy \\
&= \int_0^1 y\cos(y^2) dy \\
&= \left[\frac{\sin y^2}{2}\right]_0^1 \\
&= \left[\frac{\sin(1)}{2}\right] - \left[\frac{\sin 0}{2}\right] \\
&= \frac{\sin(1)}{2}
\end{aligned}$$

4. [10 points]: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(u, v) = (u + e^v, -2u + e^v)$. Let $D^* = [0, 1] \times [0, 1]$ in the u-v plane. Calculate

$$\iint_{T(D^*)} [x - y] \, dA$$

Solution: Since $\mathrm{D}T(u, v) = \begin{bmatrix} 1 & e^v \\ -2 & e^v \end{bmatrix}$ we see that the partial derivatives of T all exist and are continuous. Moreover, T is one-to-one on D^* since if $x = u + e^v$ and $y = -2u + e^v$ for $(u, v) \in D^*$, then $u = \frac{x-y}{3}$ and $v = \ln \frac{2x+y}{3}$. Thus T is one-to-one and continuously differentiable, so by the change of variables theorem,

$$\iint_{T(D^*)} f(x, y) \, dA = \iint_{D^*} f(u + e^v, -2u + e^v) \cdot |\det \mathrm{D}T(u, v)| dA$$

for any integrable function $f : \mathbb{R}^2 \to \mathbb{R}$. We have

$$\det \mathrm{D}T(u, v) = \begin{vmatrix} 1 & e^v \\ -2 & e^v \end{vmatrix} = (1)(e^v) - (-2)(e^v) = 3e^v$$

and so taking $f(x, y) = x - y$ we have

$$\begin{aligned} \iint_{T(D^*)} x - y \, dA &= \iint_{D^*} ((u + e^v) - (-2u + e^v))|3e^v| \, dA \\ &= \int_0^1 \int_0^1 (3u)(3e^v) dv du \\ &= \int_0^1 9u \left[e^v\right]_0^1 du \\ &= \int_0^1 9u \left[e^1 - e^0\right] du \\ &= 9(e - 1) \left[\frac{u^2}{2}\right]_0^1 \\ &= \frac{9}{2}(e - 1) \end{aligned}$$

5. [10 points]: One day, Bill the baker spills an entire bag of sugar on his (2 meter)-by-(2 meter) baking table, represented by $D = [-1,1] \times [-1,1]$. Assume the *planar* density of sugar over a point (x, y) is given by

$$f(x,y) = x^2 + y^2 \quad \text{mg/m}^2$$

Luke the very lucky ant then eats his way across the table in path given by

$$c(t) = (e^{-t}\cos t, e^{-t}\sin t) \quad \text{from} \quad t=0 \quad \text{to} \quad t=10.$$

If the path Luke eats is 1 millimeter ($= \frac{1}{1000}$ meters) wide, then

$$[f(c(t)) \text{ mg/m}^2][\frac{1}{1000} \text{ m}] = \frac{f(c(t))}{1000} \text{ mg/m}$$

approximates the *linear* density of sugar along Luke's path. In mg, about how much sugar did Luke eat?

Solution: Since $\frac{f(c(t))}{1000}$ approximates the linear density of sugar along the path $\mathbf{c}$, the total mass of sugar eaten (in mg) is given by $\int_{\mathbf{c}} \frac{f(x,y)}{1000} ds = \frac{1}{1000}\int_0^{10} f(c(t))\|c'(t)\|dt$.
For $0 \le t \le 10$ we have

$$f(c(t)) = e^{-2t}\cos^2 t + e^{-2t}\sin^2 t = e^{-2t} \qquad (\text{using } \sin^2 t + \cos^2 t = 1)$$

and

$$\begin{aligned}
\|c'(t)\| &= \|(-e^{-t}\cos t + e^{-t}(-\sin t), -e^{-t}\sin t + e^{-t}\cos t)\| \\
&= \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2} \\
&= \sqrt{e^{-2t}\cos^2 t + 2(e^{-t}\cos t)(e^{-t}\sin t) + e^{-2t}\sin^2 t + e^{-2t}\sin^2 t - 2(e^{-t}\sin t)(e^{-t}\cos t) + e^{-2t}\cos^2 t} \\
&= \sqrt{2e^{-2t}} \qquad (\text{using } \sin^2 t + \cos^2 t = 1) \\
&= \sqrt{2}\, e^{-t}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{1000}\int_0^{10} f(c(t))\|c'(t)\|dt &= \frac{1}{1000}\int_0^{10} (e^{-2t})(\sqrt{2}\, e^{-t})dt \\
&= \frac{\sqrt{2}}{1000}\int_0^{10} e^{-3t}dt \\
&= \frac{\sqrt{2}}{1000}\left[\frac{-e^{-3t}}{3}\right]_0^{10} \\
&= \frac{\sqrt{2}}{1000}\left[\frac{-e^{-30}}{3} - \frac{-e^{0}}{3}\right] \\
&= \frac{\sqrt{2}}{3000}(1 - e^{-30})
\end{aligned}$$

so Luke ate $\frac{\sqrt{2}}{3000}(1-e^{-30})$ mg of sugar. That must have been a small bag of sugar!