Name:__________________________ **PID:**__________________________

TA:__________________________ **Sec. No:**______ **Sec. Time:**____________

Q #:	**Score:**
1	/10
2	/10
3	/10
4	/10
5	/10
Total	/50

Math 20E Midterm 1

Summer Session II, 2015

1. [10 points total]: Circle your answer to each of the following true/false or multiple-choice questions. [2 points each]:

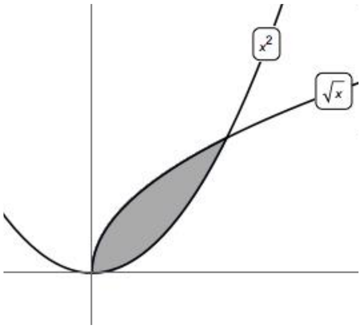
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\vec{x_0} \in \mathbb{R}^n$.

(a) [✓ **True**] **or False:** If f is differentiable at $\vec{x_0}$, then f must be continuous at $\vec{x_0}$.

(b) [✓ **True**] **or False:** If f is differentiable at $\vec{x_0}$, then all of the partial derivatives of f must exist at $\vec{x_0}$.

(c) **True or** [✓ **False**]**:** If all of the partial derivatives of f exist at $\vec{x_0}$, then f must be differentiable at $\vec{x_0}$.

(d) Which of the following double integrals are evaluated over this shaded domain?

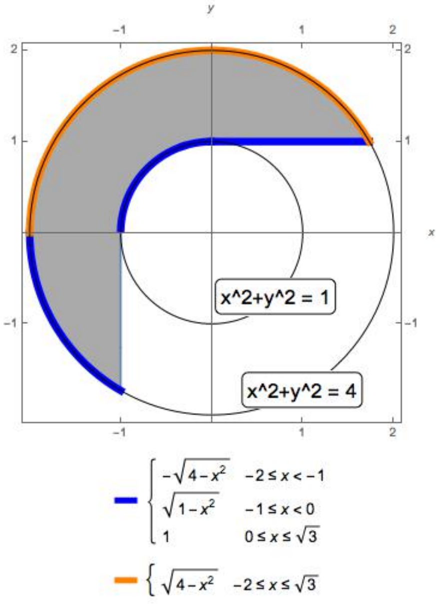


(i) $\int_0^1 \int_{x^2}^{\sqrt{x}} dydx$ (ii) $\int_0^1 \int_{\sqrt{y}}^{y^2} dxdy$ (iii) $\int_0^1 \int_{y^2}^{\sqrt{y}} dxdy$

A. (i) only
B. (ii) only
C. (iii) only
D. (i) and (ii)
[✓ **E.**] (i) and (iii)

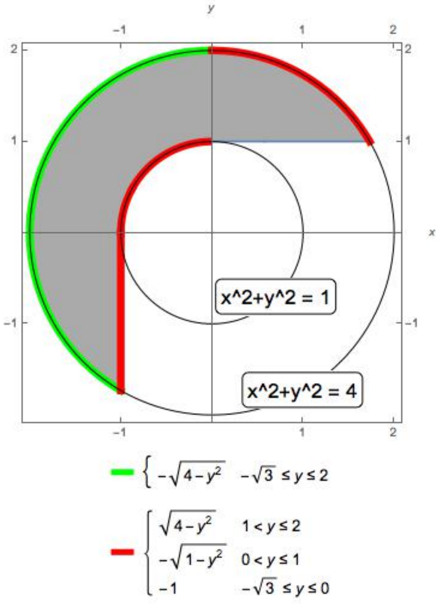
(e) The shaded region below is

A. x-simple
B. y-simple
[✓ **C.**] both x-simple and y-simple
D. neither x-simple nor y-simple



🟦 $\begin{cases} -\sqrt{4-x^2} & -2 \le x < -1 \\ \sqrt{1-x^2} & -1 \le x < 0 \\ 1 & 0 \le x \le \sqrt{3} \end{cases}$

🟧 $\left\{ \sqrt{4-x^2} \quad -2 \le x \le \sqrt{3} \right.$



🟩 $\left\{ -\sqrt{4-y^2} \quad -\sqrt{3} \le y \le 2 \right.$

🟥 $\begin{cases} \sqrt{4-y^2} & 1 < y \le 2 \\ -\sqrt{1-y^2} & 0 < y \le 1 \\ -1 & -\sqrt{3} \le y \le 0 \end{cases}$

2. [10 points total]:

(a) [2 points]: Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(x, y, z) = zx^2 + \cos y$. Find the matrix of partial derivatives $\mathrm{D}g(x, y, z)$.

Solution:

$$\mathrm{D}g(x, y, z) = \begin{bmatrix} 2zx & -\sin y & x^2 \end{bmatrix}$$

(b) [3 points]: Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(u, v) = (u \sin v, e^{u-v}, v)$. Find the matrix of partial derivatives $\mathrm{D}f(u, v)$.

Solution:

$$\mathrm{D}f(u, v) = \begin{bmatrix} \sin v & u \cos v \\ e^{u-v} & -e^{u-v} \\ 0 & 1 \end{bmatrix}$$

(c) [2 points]: Find $f(1,0)$ and $\mathrm{D}f(1,0)$.

Solution:

$$f(1,0) = (1 \cdot \sin 0, e^{1-0}, 0) = (0, e, 0)$$

$$\mathrm{D}f(1,0) = \begin{bmatrix} \sin 0 & (1)\cos 0 \\ e^{1-0} & -e^{1-0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ e & -e \\ 0 & 1 \end{bmatrix}$$

(d) [3 points]: Let $h : \mathbb{R}^3 \to \mathbb{R}$ be defined by $h = g \circ f$. Use the chain rule to find $\mathrm{D}h(1,0)$.

Solution: By the chain rule,

$$\mathrm{D}h(1,0) = [\mathrm{D}g(f(1,0))][\mathrm{D}f(1,0)] = [\mathrm{D}g(0,e,0)] \begin{bmatrix} 0 & 1 \\ e & -e \\ 0 & 1 \end{bmatrix}$$

We have

$$[\mathrm{D}g(0,e,0)] = \begin{bmatrix} 2(0)(0) & -\sin e & 0^2 \end{bmatrix} = \begin{bmatrix} 0 & -\sin e & 0 \end{bmatrix}$$

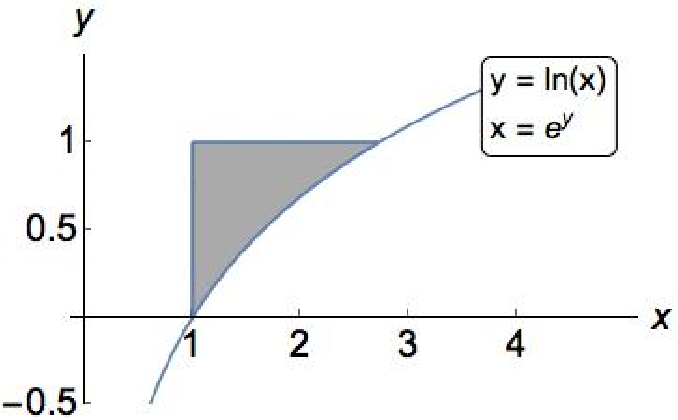
and hence

$$\begin{aligned} \mathrm{D}h(1,0) &= \begin{bmatrix} 0 & -\sin e & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ e & -e \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (0)(0) + (-\sin e)(e) + (0)(0) & (0)(1) + (-\sin e)(-e) + (0)(1) \end{bmatrix} \\ &= \begin{bmatrix} -e\sin e & e\sin e \end{bmatrix} \end{aligned}$$

3. [10 points]: Evaluate

$$\int_{1}^{e}\int_{\ln x}^{1} \frac{\sin(y^2)}{x}\, dydx$$

Solution: Since we don't know an antiderivative (in y) for $\frac{\sin(y^2)}{x}$, we want to switch the order of integration. To do so, we draw the region of integration:



Then

$$\begin{aligned}
\int_{1}^{e}\int_{\ln x}^{1} \frac{\sin(y^2)}{x}\, dydx &= \int_0^1 \int_1^{e^y} \frac{\sin(y^2)}{x}\, dxdy \\
&= \int_0^1 \sin(y^2)\,[\ln x]_1^{e^y}\, dy \\
&= \int_0^1 \sin(y^2)\,[\ln(e^y) - \ln 1]\, dy \\
&= \int_0^1 \sin(y^2)\,[y - 0]\, dy \\
&= \int_0^1 y\sin(y^2) dy \\
&= \left[-\frac{\cos y^2}{2}\right]_0^1 \\
&= \left[-\frac{\cos(1)}{2}\right] - \left[-\frac{\cos 0}{2}\right] \\
&= \frac{-\cos(1) + 1}{2}
\end{aligned}$$

4. [10 points]: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(u,v) = (u - e^v, u + e^v)$. Let $D^* = [0,1] \times [0,1]$ in the u-v plane. Calculate

$$\iint\limits_{T(D^*)} [x + y]\, dA$$

Solution: Since $\mathrm{D}T(u,v) = \begin{bmatrix} 1 & -e^v \\ 1 & e^v \end{bmatrix}$ we see that the partial derivatives of T all exist and are continuous. Moreover, T is one-to-one on D^* since if $x = u - e^v$ and $y = u + e^v$ for $(u,v) \in D^*$, then $u = \frac{x+y}{2}$ and $v = \ln \frac{y-x}{2}$. Thus T is one-to-one and continuously differentiable, so by the change of variables theorem,

$$\iint\limits_{T(D^*)} f(x,y)\, dA = \iint\limits_{D^*} f(u - e^v, u + e^v) \cdot |\det \mathrm{D}T(u,v)| dA$$

for any integrable function $f : \mathbb{R}^2 \to \mathbb{R}$. We have

$$\det \mathrm{D}T(u,v) = \begin{vmatrix} 1 & -e^v \\ 1 & e^v \end{vmatrix} = (1)(e^v) - (1)(-e^v) = 2e^v$$

and so taking $f(x,y) = x + y$ we have

$$\begin{aligned} \iint\limits_{T(D^*)} x + y\, dA &= \iint\limits_{D^*} (u - e^v + u + e^v)|2e^v|\, dA \\ &= \int_0^1 \int_0^1 (2u)(2e^v) dv du \\ &= \int_0^1 4u \left[e^v\right]_0^1 du \\ &= \int_0^1 4u \left[e^1 - e^0\right] du \\ &= 4(e-1) \left[\frac{u^2}{2}\right]_0^1 \\ &= 2(e-1) \end{aligned}$$

5. [10 points]: One day, Bill the baker spills an entire bag of sugar on his (2 meter)-by-(2 meter) baking table, represented by $D = [-1, 1] \times [-1, 1]$. Assume the *planar* density of sugar over a point (x, y) is given by

$$f(x, y) = x^2 + y^2 \quad \text{mg/m}^2$$

Luke the very lucky ant then eats his way across the table in path given by

$$c(t) = (e^{-t}\cos t, e^{-t}\sin t) \quad \text{from} \quad t = 0 \quad \text{to} \quad t = 10.$$

If the path Luke eats is 1 millimeter ($= \frac{1}{1000}$ meters) wide, then

$$[f(c(t)) \text{ mg/m}^2][\frac{1}{1000} \text{ m}] = \frac{f(c(t))}{1000} \text{ mg/m}$$

approximates the *linear* density of sugar along Luke's path. In mg, about how much sugar did Luke eat?

Solution: Since $\frac{f(c(t))}{1000}$ approximates the linear density of sugar along the path $\mathbf{c}$, the total mass of sugar eaten (in mg) is given by $\int_{\mathbf{c}} \frac{f(x,y)}{1000} ds = \frac{1}{1000}\int_0^{10} f(c(t))\|c'(t)\|dt$.
For $0 \le t \le 10$ we have

$$f(c(t)) = e^{-2t}\cos^2 t + e^{-2t}\sin^2 t = e^{-2t} \qquad (\text{using } \sin^2 t + \cos^2 t = 1)$$

and

$$\begin{aligned}
\|c'(t)\| &= \|(-e^{-t}\cos t + e^{-t}(-\sin t), -e^{-t}\sin t + e^{-t}\cos t)\| \\
&= \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2} \\
&= \sqrt{e^{-2t}\cos^2 t + 2(e^{-t}\cos t)(e^{-t}\sin t) + e^{-2t}\sin^2 t + e^{-2t}\sin^2 t - 2(e^{-t}\sin t)(e^{-t}\cos t) + e^{-2t}\cos^2 t} \\
&= \sqrt{2e^{-2t}} \qquad (\text{using } \sin^2 t + \cos^2 t = 1) \\
&= \sqrt{2}\, e^{-t}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{1000}\int_0^{10} f(c(t))\|c'(t)\|dt &= \frac{1}{1000}\int_0^{10} (e^{-2t})(\sqrt{2}\, e^{-t})dt \\
&= \frac{\sqrt{2}}{1000}\int_0^{10} e^{-3t}dt \\
&= \frac{\sqrt{2}}{1000}\left[\frac{-e^{-3t}}{3}\right]_0^{10} \\
&= \frac{\sqrt{2}}{1000}\left[\frac{-e^{-30}}{3} - \frac{-e^{0}}{3}\right] \\
&= \frac{\sqrt{2}}{3000}(1 - e^{-30})
\end{aligned}$$

so Luke ate $\frac{\sqrt{2}}{3000}(1 - e^{-30})$ mg of sugar. That must have been a small bag of sugar!