Changing the Order of Integration in the Wild

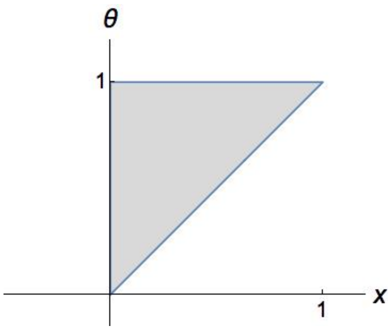
Below is an example of how changing the order of integration can appear "in the wild," that is, when trying to evaluate a surface integral. The surface is the same as in question 4 of Midterm 2 but with the domain D restricted. You will find that this problem is actually easier!

1. Let S be the surface parametrized by

$$\Phi(x, \theta) = (x, \theta(1 - x^2), \sin\theta)$$

for (x, θ) in the domain $D = \{(x, \theta) \in \mathbb{R}^2 : 0 \leq x \leq 1, x \leq \theta \leq 1\}$.

(a) Sketch the region D.



(b) Evaluate $\iint\limits_{\Phi} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ where $\vec{\mathbf{F}}(x, y, z) = \vec{\mathbf{i}} - 3x^2\, \vec{\mathbf{j}} + \vec{\mathbf{k}}$.

Solution: Just as on Midterm 2,

$$\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_\theta = \begin{vmatrix} 1 & 0 & \vec{\mathbf{i}} \\ -2x\theta & (1-x^2) & \vec{\mathbf{j}} \\ 0 & \cos\theta & \vec{\mathbf{k}} \end{vmatrix} = -2x\theta\cos\theta\,\vec{\mathbf{i}} + \cos\theta(-\vec{\mathbf{j}}) + (1-x^2)\,\vec{\mathbf{k}} = -2x\theta\cos\theta\,\vec{\mathbf{i}} - \cos\theta\,\vec{\mathbf{j}} + (1-x^2)\,\vec{\mathbf{k}}$$

Then for $\vec{\mathbf{F}}(x, y, z) = \vec{\mathbf{i}} - 3x^2\, \vec{\mathbf{j}} + \vec{\mathbf{k}}$,

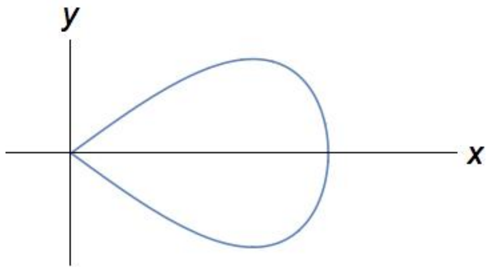
$$\iint\limits_{\Phi} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint\limits_{D} \left[\vec{\mathbf{F}}(\Phi(x,\theta)) \cdot (\vec{\mathbf{T}}_x \times \vec{\mathbf{T}}_\theta)\right] dA = \iint\limits_{D} \left[(1, -3x^2, 1) \cdot (-2x\theta\cos\theta, -\cos\theta, 1 - x^2)\right] dA$$
$$= \int_0^1 \int_x^1 \left[-2x\theta\cos\theta + 3x^2\cos\theta + 1 - x^2\right] d\theta dx$$

Now, we could integrate in θ, but we would have to do integration by parts. Instead, we can change the order of integration!

$$\iint\limits_{\Phi} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_0^1 \int_0^\theta \left[-2x\theta\cos\theta + 3x^2\cos\theta + 1 - x^2\right] dx d\theta$$
$$= \int_0^1 \left[-x^2\theta\cos\theta + x^3\cos\theta + x - \frac{1}{3}x^3\right]_0^\theta d\theta = \int_0^1 \left[-\theta^3\cos\theta + \theta^3\cos\theta + \theta - \frac{1}{3}\theta^3\right] d\theta$$
$$= \int_0^1 \left[\theta - \frac{1}{3}\theta^3\right] d\theta = \left[\frac{1}{2}\theta^2 - \frac{1}{12}\theta^4\right]_0^1 = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

Green's Theorem Example from Class

2. **(a)** Let $\boldsymbol{c}(t) = (2\cos t, 5\sin(2t))$ for $-\pi/2 \le t \le \pi/2$. Compute the area of the region D enclosed by $\boldsymbol{c}$:



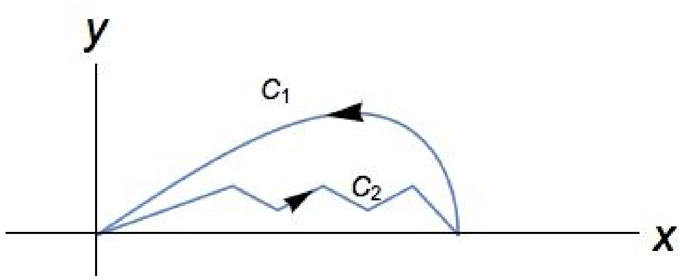
We have $c'(t) = (-2\sin t, 10\cos(2t))$. Using the area formula:

$$\text{Area}(D) = \iint_D 1dA = \frac{1}{2}\int_{\partial D} -ydx + xdy = \frac{1}{2}\int_{-\pi/2}^{\pi/2} [(-5\sin(2t), 2\cos t)\cdot(-2\sin t, 10\cos(2t))]\,dt$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} [10\sin(2t)\sin t + 20\cos t\cos(2t)]\,dt$$

which you can now integrate by using the double angle formulas $\sin(2t) = 2\sin t\cos t$ and $\cos 2t = 1 - 2\sin^2 t$. The answer will be $\frac{40}{3}$.

(b) Let $\boldsymbol{c}_1$ be the top half of the curve from part **(a)**, i.e. $\boldsymbol{c}_1(t) = (2\cos t, 5\sin(2t))$ for $0 \le t \le \pi/2$, and let $\boldsymbol{c}_2$ be the jagged curve pictured:



Given that $\frac{1}{2}\int_{\boldsymbol{c}_2} -ydx + xdy = M$, find the area of the region $\tilde{D}$ enclosed by $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ in terms of M.

$$\text{Area}(\tilde{D}) = \iint_{\tilde{D}} 1dA = \left[\frac{1}{2}\int_{\boldsymbol{c}_1} -ydx + xdy\right] + \left[\frac{1}{2}\int_{\boldsymbol{c}_2} -ydx + xdy\right]$$

$$= \left[\frac{1}{2}\int_{\boldsymbol{c}_1} -ydx + xdy\right] + M$$

$$= M + \frac{1}{2}\int_0^{\pi/2} [10\sin(2t)\sin t + 20\cos t\cos(2t)]\,dt$$

$$= M + \frac{20}{3}$$

where you can get to the last line using double angle formulas as above.