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TIME AND NORM OPTIMAL CONTROLS: A SURVEY OF RECENT RESULTS AND OPEN PROBLEMS*

Dedicated to Professor Peter D. Lax on the occasion of his 85th birthday

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Abstract We present in this paper a survey of recent results on the relation between time and norm optimality for linear systems and the infinite dimensional version of Pontryagin's maximum principle. In particular, we discuss optimality (or nonoptimality) of singular controls satisfying the maximum principle and smoothness of the costate in function of smoothness of the target.

Key words linear control systems in Banach spaces; time optimal problem; norm optimal problem

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1 Introduction

We study the control system

$$y'(t) = Ay(t) + u(t), \quad y(0) = \zeta \tag{1.1}$$

with controls $u(\cdot) \in L^\infty(0, T; E)$, where A is the generator of a strongly continuous semigroup $S(t)$ in a Banach space E. We look at two optimal control problems for (1.1). One is the norm optimal problem, where we drive the initial point ζ to a point target,

$$y(T) = \bar{y} \tag{1.2}$$

in a fixed time interval $0 \le t \le T$ minimizing the norm $\|u(\cdot)\|_{L^\infty(0,T;E)}$. The second is the time optimal problem, where we drive to the target with a fixed bound on the norm of the control (say $\|u(\cdot)\|_{L^\infty(0,T;E)} \le 1$) in optimal time T. The solution or trajectory of (1.1) is given by the variation-of-constants formula

$$y(t) = y(t, \zeta, u) = S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)\mathrm{d}\sigma \tag{1.3}$$

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and is continuous in $t \geq 0$. For the time optimal problem, controls in $L^\infty(0,T;E)$ with norm $\|u(\cdot)\|_{L^\infty(0,T;E)} \leq 1$ are called admissible. Separate necessary and sufficient conditions for norm and time optimality are given in terms of the maximum principle (1.5) below, whose formulation requires the construction of multiplier spaces. For simplicity, we assume that the adjoint $S(t)^*$ is strongly continuous in E^*. When A has a bounded inverse, the space E^*_{-1} is the completion of E^* in the norm

$$\|y^*\|_{E^*_{-1}} = \|(A^{-1})^* y^*\|_{E^*}.$$

Each $S(t)^*$ can be extended to an operator $S(t)^* : E^*_{-1} \to E^*_{-1}$, and $Z(T) \subseteq E^*_{-1}$ consists of all $z \in E^*_{-1}$ such that $S(t)^* z \in E^*$ $(t > 0)$ and

$$\|z\|_{Z(T)} = \int_0^T \|S(t)^* z\| \mathrm{d}t < \infty. \tag{1.4}$$

The space $Z(T)$ equipped with the norm $\|\cdot\|_{Z(T)}$ is a Banach space. All spaces $Z(T)$ coincide (that is, $Z(T) = Z(T')$ and the norms $\|\cdot\|_{Z(T)}, \|\cdot\|_{Z(T')}$ are equivalent for $T, T' > 0$). $Z(T)$ is an example of a multiplier space, an arbitrary linear space $\mathcal{Z} \supseteq E^*$ to which $S(t)^*$ can be extended in such a way that $S(t)^*\mathcal{Z} \subseteq E^*$ and $S(s)^*S(t)^*\psi = S(s+t)^*\psi$ $(s, t > 0)$; the largest space with these properties is called $\mathcal{M}$. When A does not have a bounded inverse, a few changes are needed. Since A is a semigroup generator, $(\lambda I - A)^{-1}$ exists for $\lambda > \omega$ and E^*_{-1} is the completion of E^* in any of the equivalent norms

$$\|y^*\|_{E^*_{-1},\lambda} = \|((\lambda I - A)^{-1})^* y^*\|_{E^*} \quad (\lambda > \omega).$$

The definition of $Z(T)$ (and of multiplier spaces) is the same. See [17, Section 2.3] for details.

A control $\bar{u}(\cdot) \in L^\infty(0,T;E)$ satisfies Pontryagin's maximum principle[1] if

$$\langle S(T-t)^*\psi, \bar{u}(t)\rangle = \max_{\|u\|\leq\rho} \langle S(T-t)^*\psi, u\rangle \quad \text{a.e. in } 0 \leq t < T, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ is the duality of the space E and the dual E^*, with $\rho = \|\bar{u}(\cdot)\|_{L^\infty(0,T;E)}$ and ψ in a multiplier space $\mathcal{Z}$. We call z the multiplier and $z(t) = S(T-t)^*z$ the costate corresponding to (or associated with) the control $\bar{u}(t)$. We assume that (1.5) is nontrivial, that is, that $S(T-t)^*z$ is not identically zero in $0 \leq t < T$, although it may be zero in part of the interval (in which part (1.5) gives no information on $\bar{u}(t)$). The nontriviality requirement reduces to $\psi \neq 0$. When E is a Hilbert space the maximum principle is equivalent to

$$\bar{u}(t) = \rho \frac{S(T-t)^*\psi}{\|S(T-t)^*\psi\|} \qquad (0 \leq t < T). \tag{1.6}$$

If $S(T-t)^*\psi = 0$ for some $t \in (0,T)$ then (1.6) is valid in $\delta < t < T$, (δ, T) the largest subinterval of $(0,T)$ where $S(T-t)^*\psi \neq 0$.

[1] The original "Pontryagin's maximum principle" was announced in [3] (superseding a linear version in [1]) with full proof in [24]. It applies to (generally nonlinear) ordinary differential systems, thus is finite dimensional in character. Equality (1.5) is one of its linear, infinite dimensional versions, corresponding to the "differential equations in Banach space" approach in this paper. There are other versions of the maximum principle, for instance for partial differential equations, not necessarily of evolution type.

A substantial part of the the theory of optimal control of the system (1.1) deals with the relation between optimality and the maximum principle (1.5). An exception is the result below, whose proof is in [9], [17, Theorem 2.1.3].[2]

Theorem 1.1 Let $\bar{u}(t)$ be a time optimal control. Then

$$\|\bar{u}(t)\| = 1 \quad (0 \le t \le T). \tag{1.7}$$

There are separate necessary and sufficient conditions for optimality based on the maximum principle (Theorems 1.2 and 1.3 below). We call an optimal control $\bar{u}(t)$ regular if it satisfies (1.5) with $z \in Z(T)$, strongly regular if $z \in E^*$.

Theorem 1.2 Assume $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ time or norm optimally in the interval $0 \le t \le T$ and that

$$\bar{y} - S(T)\zeta \in D(A). \tag{1.8}$$

Then $\bar{u}(t)$ satisfies (1.5) with $z \in Z(T)$, thus it is regular; if $\bar{u}(t)$ is norm optimal $\rho =$ minimum norm, if $\bar{u}(t)$ is time optimal $\rho = 1$.

For a proof see [13, Theorem 5.1], [17, Theorem 2.5.1]. The converse holds without any smoothness condition like (1.8) on the target ([13, Theorem 5.2], [17, Theorems 2.5.5 and 2.5.7]).

Theorem 1.3 Let $\bar{u}(t)$ be a regular control. Then $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ norm optimally in the interval $0 \le t \le T$; if $\rho = 1$ the drive is time optimal.[3]

In the linear case existence of optimal controls is a simple consequence of Alaoglu's theorem on weak convergence of (a subsequence of) bounded sequences of controls in $L^2(0, T; E)$; the proof needs some regularity of the space E, as well as assumptions on the possibility of driving to the target by appropriate controls. To precise this we define: $\bar{y}$ is r-reachable from ζ in time T if there exists a control $u(t)$ with $\|u(\cdot)\|_{L^\infty(0,T;E)} \le r$ driving from ζ to $\bar{y}$ in time T.

Theorem 1.4 Let E be reflexive. (a) Assume the target $\bar{y}$ is r-reachable from ζ in time T for some $r \ge 0$. Then a norm optimal control exists. (b) Assume the target $\bar{y}$ is 1-reachable from ζ for some $T \ge 0$. Then a time optimal control exists.

The proof is in [9, Lemma 1.1] for the time optimal problem and in general in [17, Theorem 3.1.2]. Uniqueness requires (a somewhat different) regularity of E, namely strict convexity; if $u, v \in E, \|u\| = \|v\| = 1, u \ne v \Longrightarrow \|u + v\| < 2$. If E is strictly convex, uniqueness of time optimal controls follows directly from Theorem 1.1; in fact, if two different time optimal controls $\bar{u}(t)$, $\bar{v}(t)$ drive ζ to $\bar{y}$ in optimal time T (obviously, the "optimal times" for each control must be the same) then $(\bar{u}(t) + \bar{v}(t))/2$ does the same drive optimally as well and does not satisfy (1.7). See [9, Corollary 2.3] and [17, Theorem 3.1.4].

Uniqueness for norm optimal controls requires additional conditions. If $\bar{u}(t)$ is a norm optimal control with $\bar{y} - S(T)\zeta \in D(A)$ Theorem 1.2 applies and the maximum principle (1.5) holds. If

$$S(T-t)^* z \ne 0 \quad (0 \le t \le T),$$

[2] I am grateful to Peter Lax for his belief (at the time where he was my dissertation advisor) that the results in [9] were worth publishing and for encouraging me to do so. Note that (1.7) is a consequence of the maximum principle (1.5) (for $\rho = 1$) if $S(T-t)^*\psi \ne 0$; however (1.7) holds independently of the maximum principle, and there are time optimal controls that do not satisfy the maximum principle (Section 7).

[3] The statement on time optimality needs additional assumptions on the initial condition ζ and the target $\bar{y}$. These conditions are satisfied if $\zeta = 0$ irrespective of the target [15], [17, Theorem 2.5.7]. We also need to assume that $S(t)^* z \ne 0$ in the entire interval $0 < t \le T$.

then (1.7) follows, and uniqueness in strictly convex spaces is proved in the same way as as for time optimal controls [17, Theorem 3.1.5].

2 Strongly Regular Controls, I

"Strongly regular" is of interest in applications. In fact, if E is a Hilbert space then (1.6) shows that a strongly regular control is (at least) continuous in $0 \le t \le T$, whereas a merely regular control may "chatter" (that is, it may be discontinuous) at the endpoint T of the control interval. This introduces complications, for instance, in numerical approximations of the optimal control.

A natural question on regular controls is, characterize the control systems (1.1) for which all (time, norm) optimal controls are strongly regular. A sufficient condition (which includes groups $\{S(t); -\infty < t < \infty\}$ is

Theorem 2.1 Assume

$$S(t)E = E \quad (t > 0). \tag{2.1}$$

Then every (time, norm) optimal control is strongly regular.

This result is valid in any Banach space [17, Theorem 2.1] with no conditions on the target $\bar{y}$. To gain perspective, we summarize the way the theorem is proved (as well as how the argument is modified in absence of (2.1) to obtain Theorem 1.2). Given $T > 0$ the reachable space $R^\infty(T)$ is the set of all elements of the form

$$y = \int_0^T S(T-\sigma)u(\sigma)\mathrm{d}\sigma \quad (u(\cdot) \in L^\infty(0,T;E)) \tag{2.2}$$

equipped with the norm

$$\|y\|_{R^\infty(T)} = \inf \|u(\cdot)\|_{L^\infty(0,T;E)}, \tag{2.3}$$

the infimum taken over all $u(\cdot)$ that satisfy (2.2). Equivalently, $R^\infty(T)$ is the quotient space $L^\infty(0,T;E)/\mathcal{N}^\infty$ with the standard quotient norm, where

$$\mathcal{N}^\infty(T) = \left\{u(\cdot) \in L^\infty(0,T;E)\,;\, \int_0^T S(T-\sigma)u(\sigma)\mathrm{d}\sigma = 0\right\}.$$

It is easily proved that if the control $\bar{u}(t)$ drives ζ to $\bar{y}$ in optimal time T then $\bar{y} - S(T)\zeta$ is a boundary point of $B^\infty(T)$, the unit ball of $R^\infty(T)$. Assumption 2.1 implies that

$$R^\infty(T) = E \tag{2.4}$$

and that the norm (2.3) is equivalent to the norm of E thus the interior of $B^\infty(T)$ (in E) is nonempty and $\bar{y} - S(T)\zeta$ can be separated from $B^\infty(T)$ by a functional $z \in E^*$. This gives the maximum principle (1.5). For the norm optimal principle the argument is the same with $B^\infty(T)$ replaced by $\rho B^\infty(T)$ ($\rho =$ optimal norm).

The converse of Theorem 2.1 is almost true (Theorem 2.2 below) but there are some caveats regarding the uniqueness of the multiplier ψ associated with a given control $\bar{u}(t)$ in the maximum principle (1.5). Of course ψ is never unique (ψ can be replaced by $\lambda\psi$, $\lambda \ne 0$). We say then that ψ is essentially unique for a given control $\bar{u}(\cdot) \in L^\infty(0,T;E)$ if any other multiplier giving (1.5) is of the form $\lambda\psi$, $\lambda \neq 0$. In a general multiplier space $\mathcal{Z}$ equality of multipliers has the meaning $\psi = \eta$ if and only if $S(T-t)^*\psi = S(T-t)^*\eta$ $(0 \leq t < T)$; equivalently, $S(t)^*\psi = S(t)^*\eta$ $(0 < t \leq T)$. The particular T is irrelevant; if the equality is satisfied for any $T > 0$ the semigroup equation implies that it is satisfied in $0 < t < \infty$.

Theorem 2.2 Let E be reflexive and separable. Assume all optimal controls for (1.1) are strongly regular and that z in the maximum principle (1.5) is essentially unique for any control $\bar{u}(t)$. Then $S(t)$ satisfies (2.1).

For a proof see [19, Corollary 4.8]. The essential uniqueness property holds for a class of spaces slightly smaller than that in Theorem 2.2 (for the precise definition see Remark 2.4 below). We begin with a simple result that generalizes [19, Section 3]:

Lemma 2.3 Let E be a Hilbert space. Then the multiplier ψ in the maximum principle (1.5) is essentially unique for any control $\bar{u}(t)$.

Proof Assume that the control $\bar{u}(t)$ satisfies (1.5) with two (possibly different) multipliers ψ, η. Due to the requirement that (1.5) be nontrivial there exists $\alpha > 0$ such that $S(T-t)^*\psi \neq 0$, $S(T-t)^*\eta \neq 0$ for $\alpha \leq t < T$. We have

$$\bar{u}(t) = \frac{S(T-t)^*\psi}{\|S(T-t)^*\psi\|} = \frac{S(T-t)^*\eta}{\|S(T-t)^*\eta\|} \quad (\alpha \leq t \leq T) \tag{2.5}$$

or

$$S(T-t)^*\eta = \beta(t)S(T-t)^*\psi \quad (\alpha \leq t \leq T) \tag{2.6}$$

with $\beta(t) = \|S(T-t)^*\eta\|/\|S(T-t)^*\psi\|$. Writing (2.6) for $t = \alpha$ and then applying $S(\alpha - t)^*$ $(0 < t \leq \alpha)$ to both sides we obtain

$$S(T-t)^*\eta = \beta(\alpha)S(T-t)^*\psi = S(T-t)^*\beta(\alpha)\psi \quad (0 < t \leq \alpha)$$

thus $\eta = \beta(\alpha)\psi$ and we are all done.

Remark 2.4 The result in Lemma 2.3 can be easily generalized to a reflexive and separable E (the setting of Theorem 2.2) under the additional assumptions that both E and E^* are strictly convex. To see this, let $y^* \in E^* \setminus \{0\}$ and denote by $\Phi(y^*) \subset E$ the set of all $y \in E$ satisfying

$$\|\Phi(y^*)\|_E = 1, \quad \langle y^*, \Phi(y^*)\rangle = \|y^*\|_{E^*}.$$

Strict convexity of E^* implies that $\Phi(y^*)$ is single valued [17, Remark 3.1.8] thus the maximum principle (1.5) implies the following generalization of (1.6):

$$\bar{u}(t) = \Phi(S(T-t)^*\psi) \quad (0 \leq t < T). \tag{2.7}$$

Obviously, we have $\Phi(\lambda y^*) = \Phi(y^*)$ and strict convexity of the dual E^* implies the converse,

$$\Phi(y_1^*) = \Phi(y_2^*) \implies y_2^* = \lambda y_2^* \quad (\lambda \neq 0)$$

([17, Lemma 3.2.5]). Equality (2.5) then becomes

$$\bar{u}(t) = \Phi(S(T-t)^*\psi) = \Phi(S(T-t)^*\eta)$$

which implies (2.6). The rest of the proof is the same.

The following example [18, Lemma 6.1] shows that Theorem 2.2 may fail without essential uniqueness of multipliers.

Example 2.5 Consider the space $E = \ell^0$ consisting of all numerical sequences $y = \{y_n\} = \{y_1, y_2, \cdots\}$ with $\lim\limits_{n\to\infty} y_n = 0$, equipped with the norm $\|y\|_0 = \max\limits_{n\geq 1} |y_n|$. The dual is $E^* = \ell^1$, the space of all numerical sequences $y^* = \{y_n^*\}$ with $\|y^*\|_1 = \sum\limits_{n=1}^{\infty} |y_n^*| < \infty$, the duality of both spaces given by $\langle y^*, y\rangle = \sum\limits_{n=1}^{\infty} y_n^* y_n$. The semigroup and generator are

$$S(t)\{y_n\} = \{e^{-nt} y_n\}, \quad A\{y_n\} = \{-ny_n\}, \tag{2.8}$$

A with maximal domain ($\lim\limits_{n\to\infty} n|y_n| = 0$). The adjoint semigroup is

$$S(t)^*\{y_n^*\} = \{e^{-nt} y_n^*\}, \tag{2.9}$$

and the space E_{-1}^* consists of all sequences $\{y_n^*\}$ with

$$\|(A^{-1})^*\{y_n^*\}\| = \sum_{n=1}^{\infty} \frac{|y_n^*|}{n} < \infty.$$

If $\{y_n^*\} \in E_{-1}^*$ we have

$$\int_0^T \|S(t)^* z\| \mathrm{d}t = \left\| \left\{ \int_0^T e^{-nt} y_n^* \mathrm{d}t \right\} \right\| = \sum_{n=1}^{\infty} |y_n^*| \frac{1 - e^{-nT}}{n} \leq \|(A^{-1})^*\{y_n^*\}\|,$$

thus $E_{-1}^* = Z(T)$. Due to existence requirements for optimal controls for (1.1), controls are taken in $L^\infty(0, T; \ell^\infty)$ rather than[4] in $L^\infty(0, T; \ell^0)$, where ℓ^∞ is the space of all bounded numerical sequences $y = \{y_n\}$ equipped with the norm $\|y\|_\infty = \sup\limits_{n\geq 1} |y_n|$. This means u in the maximum principle (1.5) belongs to ℓ^∞ rather than to ℓ^0. The space $\mathcal{M}$ of all multipliers consists of all sequences $\{\psi_n\}$ with $\|S(t)^*\psi\|_1 = \sum\limits_{n=1}^{\infty} e^{-nt}|\psi_n| < \infty$ $(t > 0)$. We take this result from [18].

Lemma 2.6 An admissible control $\bar{u}(t) = \{\bar{u}_n(t)\}$ satisfies the maximum principle (1.5) with $\psi = \{\psi_n\}$ in any multiplier space if and only if $\bar{u}_m(t) = 1$ $(0 \leq t \leq T)$ or $u_m(t) = -1$ $(0 \leq t \leq T)$ for at least one $m \geq 1$.

Proof We take $\rho = 1$. The maximum principle is

$$\begin{aligned}
\langle S(T-t)^*\{\psi_n\}, \{\bar{u}_n(t)\}\rangle &= \sum_{n=1}^{\infty} e^{-n(T-t)} \psi_n \bar{u}_n(t) \\
&= \max_{\|\{u_n\}\|_{\ell^\infty} \leq 1} \langle S(T-t)^*\{\psi_n\}, \{u_n\}\rangle \\
&= \max_{|u_n| \leq 1} \sum_{n=1}^{\infty} e^{-n(T-t)} \psi_n u_n \\
&= \sum_{n=1}^{\infty} e^{-n(T-t)} |\psi_n|,
\end{aligned} \tag{2.10}$$

[4]The "right" space is actually $L_w^\infty(0, T; \ell^\infty)$, where the subindex indicates that the strong measurability assumption on the elements of $L^\infty(0, T; \ell^\infty)$ is replaced by E-weak measurability. This "weak measurability setup" is also natural in the treatment of systems modelled by parabolic equations in L^1 and L^∞ spaces (see [17, Chapters 4 and 5]).

so that we must have $\bar{u}_m(t) = \text{sign } \psi_m$ whenever $\psi_m \neq 0$. Conversely, if the assumptions of Theorem 2.6 are satisfied for $\{\bar{u}_n(t)\}$ we obtain the maximum principle (1.5) with $\{\psi_n\} = \{\pm\delta_{mn}\}$ (δ_{mn} the Kronecker delta).

Lemma 2.6 obviously implies that every control $\bar{u}(t)$ satisfying (1.5) (with any multiplier) qualifies as strongly regular; this means $\psi \in \mathcal{M}$ in the maximum principle (1.5) can be replaced by a multiplier $\{\pm\delta_{mn}\} \in E^*$ without any change in $\bar{u}(t)$. The semigroup (2.8) does not satisfy (2.1), thus Theorem 2.2 does not hold.

3 Strongly Regular Controls, II

Condition (2.1) is (under some restrictions) necessary and sufficient for validity of (2.4). Accordingly, in absence of (2.1) we have $R^\infty(T) \hookrightarrow E$ (strictly); the unit ball $B^\infty(T)$ has empty interior in E and the separation of $\bar{y} - S(T)\zeta$ from $B^\infty(T)$ must be performed by means of a functional ξ in the dual $R^\infty(T)^*$.

An element $z \in Z(T)$ defines a functional $\xi_z \in R^\infty(T)^*$ by

$$\langle \xi_z, y\rangle = \left\langle \xi_z, \int_0^T S(T-\sigma)u(\sigma)\mathrm{d}\sigma \right\rangle = \int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle \mathrm{d}\sigma. \tag{3.1}$$

It can be easily seen that (3.1) respects the equivalence relation in $R^\infty(T) = L^\infty(0,T;E)/\mathcal{N}^\infty(T)$ [13], [17, Lemma 2.3.5] and that ξ_z is a bounded functional in $R^\infty(T)$, precisely

$$\|\xi_z\|_{R^\infty(T)^*} = \int_0^T \|S(T-\sigma)^* z\|\mathrm{d}\sigma = \int_0^T \|S(\sigma)^* z\|\mathrm{d}\sigma. \tag{3.2}$$

The inequality $\leq$ is obvious; for the equality, see [13] or [17, Section 2.3]. Functionals of the form (3.1) are called regular and $\mathcal{R}(T) \subseteq R^\infty(T)^*$ is the space of all regular functionals.

The semigroup equality

$$\bar{y} = \int_0^T S(T-\sigma)\frac{1}{T}(y - \sigma A\bar{y})\mathrm{d}\sigma$$

gives $D(A) \subseteq R^\infty(T)$ and $\|y\|_{R^\infty(T)} \leq C(T)(\|y\| + \|Ay\|) = C(T)\|y\|_{D(A)}$ (in other words, $D(A) \hookrightarrow R^\infty(T)$). Functionals $\xi_s \in R^\infty(T)^*$ that vanish in $D(A)$ are called singular; the space of all such functionals is $\mathcal{S}(T) \subseteq R^\infty(T)^*$.

Theorem 3.1 [17, Theorem 2.4.1] [5]

$$R^\infty(T)^* = \mathcal{R}(T) \oplus \mathcal{S}(T) \quad \text{(Banach direct sum).} \tag{3.3}$$

Separation of $\bar{y} - S(T)\zeta$ from the unit ball $B^\infty(T)$ by a regular functional ξ_z (or by a functional having a nonzero "regular part" in (3.3)) produces the maximum principle (1.5) with z as multiplier. On the other hand, separation with a singular functional does not imply the maximum principle. The condition $\bar{y} - S(T)\zeta \in D(A)$ (or, more generally, $\bar{y} - S(T)\zeta \in \overline{D(A)}$ = closure of $D(A)$ in $R^\infty(T)$ in the norm of $R^\infty(T)$) guarantees separation with a regular functional. Arguing along the lines of the previous section we may ask whether $\bar{y} - S(T)\zeta \in D(A)$

[5] "Banach direct sum" means algebraic direct sum plus bounded projections from the space into each of the two subspaces.

actually implies that $z \in E^*$, that is, that the control is strongly regular. The following example [18, Theorem 3.6] shows that (without restrictions on the semigroup and the space) the answer is "no".

Example 3.2 The space is ℓ^1 (all sequences $y = \{y_1, y_2, \cdots\} = \{y_n\}$ with norm $\|y\|_1 = \|\{y_n\}\|_1 = \sum\limits_{n=1}^{\infty} |y_n| < \infty$). The infinitesimal generator and the semigroup are

$$A\{y_n\} = \{-ny_n\}, \qquad S(t)\{y_n\} = \{e^{-nt}y_n\}, \tag{3.4}$$

$D(A)$ the set of all $\{y_n\} \in \ell^1$ with $\{ny_n\} \in \ell^1$. We have $\ell_1^* = \ell^\infty$, the space of all sequences $y^* = \{y_1^*, y_2^*, \cdots\} = \{y_n^*\}$ with $\|y^*\|_\infty = \|\{y_n^*\}\|_\infty = \sup\limits_{n \ge 1} |y_n^*| < \infty$ equipped with $\|\cdot\|_\infty$, the duality of ℓ^1 and ℓ^∞ given by $\langle y^*, y\rangle = \langle \{y_n^*\}, \{y_n\}\rangle = \sum\limits_{n=1}^{\infty} y_n^* y_n$. The adjoint semigroup is

$$S(t)^*\{y_n^*\} = \{e^{-nt}y_n^*\}, \tag{3.5}$$

analytic in $t > 0$ but not strongly continuous at $t = 0$; for instance, if $\{y_n^*\} = \{1, 1, \cdots\}$ then $\|S(t)^*\{y_n^*\} - \{y_n^*\}\|_\infty \to 1 \quad$ as $t \to 0$. The space $E_{-1}^* = (\ell^1)_{-1}^* = \ell_{-1}^\infty$ consists of all sequences $z = \{z_1, z_2, \cdots\} = \{z_n\}$ with

$$\|z\|_{\infty,-1} = \|\{z_n\}\|_{\infty,-1} = \sup_{n \ge 1} \left|\frac{z_n}{n}\right| < \infty \tag{3.6}$$

and $Z(T)$ is the subspace of ℓ_{-1}^∞ determined by the condition

$$\int_0^T \|\{z_n e^{-n\sigma}\}\|_\infty \mathrm{d}\sigma < \infty.$$

The maximum principle for this system is

$$\sum_{n=1}^{\infty} z_n e^{-n(T-t)} \bar{u}_n(t) = \max_{\|\{u_n\}\|_1 \le \rho} \sum_{n=1}^{\infty} z_n e^{-n(T-t)} u_n. \tag{3.7}$$

Theorem 3.3 There exists a control $\{\bar{u}_n(t)\}$ driving optimally from the origin to a target $\{y_n\} \in D(A)$ such that $\{\bar{u}_n(t)\}$ satisfies the maximum principle (1.5) with $\{z_n\} \in Z(T)$ but not with $\{z_n\} \in E^* = \ell^\infty$.

In other words, there exist controls driving optimally 0 to $\bar{y} \in D(A)$ which are regular (this is guaranteed by Theorem 1.2) but are not strongly regular.

On the other hand, there exist particular semigroups and spaces where the condition $\bar{y} \in D(A)$ guarantees that an optimal control driving from 0 to $\bar{y}$ is strongly regular.[6] The example below (from [21]) exhibits this phenomenon.

Example 3.4 The right translation semigroup $S(t)$ in $E = L^2(0, \infty)$ is

$$S(t)y(x) = \begin{cases} y(x-t) & (x \ge t), \\ 0 & (x < t). \end{cases} \tag{3.8}$$

[6] Obviously any interesting example must not satisfy (2.1), for under this assumption every optimal control is strongly regular (without conditions on the target $\bar{y}$).

This semigroup is strongly continuous and isometric. The adjoint semigroup is the left translation (and chop-off) semigroup

$$S(t)^* z(x) = z(x+t) \quad (x \ge 0). \tag{3.9}$$

The infinitesimal generator A of $S(t)$ is

$$Ay(x) = -y'(x), \tag{3.10}$$

with domain $D(A) = \{\text{all } y(\cdot) \in L^2(0,\infty) \text{ with } y'(\cdot) \text{ in } L^2(0,\infty) \text{ and } y(0) = 0\}$. The semigroup $S(t)$ is associated with the control system

$$\begin{aligned} &\frac{\partial y(t,x)}{\partial t} = -\frac{\partial y(t,x)}{\partial x} + u(t,x) \quad (0 \le t,\ x < \infty), \\ &y(0,x) = \zeta(x), \qquad y(t,0) = 0, \end{aligned} \tag{3.11}$$

in the sense that $S(t)$ is the propagation semigroup of the homogeneous equation ($u(t,x) = 0$). The space $\mathcal{M}$ of all multipliers for this system is the set of all $\psi(x)$ defined in $x \ge 0$ and such that

$$\|S(t)^*\psi\| = \sqrt{\int_0^\infty \psi(x+t)^2 \mathrm{d}x} = \sqrt{\int_t^\infty \psi(x)^2 \mathrm{d}x} = \kappa(t,\psi) < \infty \quad (t > 0), \tag{3.12}$$

a condition that allows arbitrary growth of $\psi(x)$ as $x \to 0$. The space $Z(T)$ is characterized by

$$\int_0^T \|S(t)^* z\| \mathrm{d}t = \int_0^T \kappa(t,z) \mathrm{d}t < \infty.$$

It follows from Theorem 1.2 that if $\bar{u}(t,x)$ is a control that drives 0 optimally to $\bar{y}(\cdot) \in D(A)$ then $\bar{u}(t,x)$ satisfies the maximum principle (1.5), here of the form

$$\begin{aligned} \langle S(T-t)^* z, \bar{u}(t) \rangle &= \int_0^\infty z(x+T-t)\bar{u}(t,x)\mathrm{d}x \\ &= \max_{\|u\|_{L^2(0,\infty)} \le 1} \int_0^\infty z(x+T-t)u(x)\mathrm{d}x \\ &= \max_{\|u\|_{L^2(0,\infty)} \le 1} \langle S(T-t)^* z, u \rangle \end{aligned}$$

with $z(\cdot) \in Z(T)$, that is, the control is regular. However, a more precise result holds: it is proved in [21, Section 5] that $\bar{u}(t,x)$ is strongly regular, that is, that $z(\cdot) \in L^2(0,\infty)$. Interestingly, the method used in [21] allows explicit computation of the costate $z(x)$ in terms of the target $\bar{y}(x)$ (solving a singular nonlinear ordinary differential equation). Numerical examples are given in [21, Section 8]. Solutions of the differential equation have in general finite blowup time, which is an upper bound for the optimal time (for related results see [23]).

4 Smoothness of the Costate from Smoothness of the Target and Back

Whether or not the implication "$\bar{y} \in D(A) \implies z \in E^*$" holds was the subject of last section. We may generalize this line of inquiry asking whether smoothness of the target implies smoothness of the costate and viceversa, "smoothness" of $\bar{y}$ (resp. of z) defined in terms of membership in the domain of a (possibly fractional) power of A (resp. of A^*). There is a general result, although limited to self adjoint operators. If A is self adjoint in a separable Hilbert space, A generates a strongly continuous semigroup if and only it is bounded below, $A \le \omega I$. If $\mu > \omega$ and $\alpha \ge 0$ we define

$$H_\alpha = D((\mu I - A)^\alpha).$$

For negative indices $\alpha = -\beta < 0$ the definition is

$$H_\alpha = H_{-\beta} = \mathcal{C}_{-\beta}(H),$$

where $\mathcal{C}_{-\beta}(H)$ is the completion of H in the norm $\|y\|_{H_{-\beta}} = \|(\mu I - A)^{-\beta} y\|_H$. The definition is independent of μ.

The following result relates smoothness of the target and the multiplier in a simple way [16], [17, Theorems 6.1.2 and 6.1.4].

Theorem 4.1 (a) Let $\alpha > 1$. Then $\bar{y} \in H_\alpha$ and $z \in Z(T) \Longrightarrow z \in H_{\alpha-1}$. (b) Let $-\infty < \alpha < \infty$. Then $z \in H_\alpha \Longrightarrow \bar{y} \in H_{\alpha-1}$.

Significantly, the proof of Part (a) of Theorem 4.1 excludes the case $\alpha = 1$, where the implication would be $\bar{y} \in D(A) \Longrightarrow z \in E$ or, equivalently, regular $\Longrightarrow$ strongly regular. Whether this implication holds in the present self adjoint setting seems to be an open problem.

It is natural to ask whether something like Theorem 4.1 could be extended to general (not necessarily self adjoint) infinitesimal generators. The right translation semigroup (3.8) in Example 3.4 shows this is not possible. In fact, it is shown in [16], [17, Example 6.2.1] that there exists a target $\bar{y} \in D(A^\infty) = \bigcap_{n\ge 1} D(A^n)$ such that $z \notin D((A^*)^2)$ and, conversely, there exists a multiplier $z \in D((A^*)^\infty) = \bigcap_{n\ge 1} D((A^*)^n)$ such that $\bar{y} \notin D(A^2)$.

5 Singular Controls, I

An admissible control $\bar{u}(t)$ is called singular if (a) it satisfies the maximum principle (1.5) with a multiplier ψ not in $Z(T)$, that is, such that

$$\int_0^T \|S(t)^*\psi\| \mathrm{d}t = \infty, \tag{5.1}$$

(b) it is not regular (that is, is doesn't satisfy (1.5) for any $z \in Z(T)$). Part (b) of the definition addresses the possibility of a control satisfying (1.5) with essentially different multipliers; for instance, in Example 2.5 a control satisfying (1.5) with a totally arbitrary multiplier $\psi \in \mathcal{M}$ satisfies (1.5) with $z \in E^*$; for this control system there are no singular controls whatsoever.

Obviously, a control $\bar{u}(t)$ can be singular only if $\bar{y} \notin D(A)$; if $\bar{y} \in D(A)$ Theorem 1.2 applies and any control driving optimally from 0 to $\bar{y}$ must be regular.

Condition (1.4) on the costate is sufficient for the corresponding control $\bar{u}(t)$ to be optimal (Theorem 1.3) The following two examples show that (1.4) is not a necessary condition; there exist singular controls $\bar{u}(t)$ driving 0 to $\bar{y}$ optimally (of course, Theorem 1.2 says that $\bar{y} \notin D(A)$).

Example 5.1 The space is $E = L^2(0, \infty)$ and the infinitesimal generator

$$Au(\lambda) = -(\lambda + c)u(\lambda) \tag{5.2}$$

($c > 0$ arbitrary) with maximal domain. The operator A is self adjoint with $A \leq -cI$ and generates the self adjoint analytic semigroup

$$S_c(t)u(\lambda) = e^{-(\lambda+c)t}u(\lambda) \quad (t > 0). \tag{5.3}$$

The space $\mathcal{M}$ of all multipliers consists of all measurable functions $\psi(\lambda)$ defined in $\lambda \geq 0$ with

$$S_c(t)^*\psi(\cdot) = S_c(t)\psi(\cdot) = e^{-(\cdot+c)t}\psi(\cdot) \in L^2(0,\infty) \quad (t > 0).$$

In particular,

$$\psi_\alpha(\lambda) = \lambda^\alpha \tag{5.4}$$

belongs to $\mathcal{M}$ for $\alpha > -1/2$. We have

$$\|S_c(t)\psi_\alpha\|^2 = e^{-2ct}\int_0^\infty e^{-2\lambda t}\lambda^{2\alpha}\mathrm{d}\lambda = \frac{\Gamma(1+2\alpha)}{e^{2ct}(2t)^{1+2\alpha}},$$

so that

$$\|S_c(t)^*\psi_\alpha\| = \|S_c(t)\psi_\alpha\| = \frac{\sqrt{\Gamma(1+2\alpha)}}{2^{1/2+\alpha}}\cdot\frac{e^{-ct}}{t^{1/2+\alpha}}$$

and $\psi_\alpha(\cdot) \in Z(T)$ if and only if $\alpha < 1/2$. By Lemma 2.3 there is essential uniqueness of multipliers, hence the control

$$\bar{u}_\alpha(t,\lambda) = \frac{S_c(T-t)^*\psi_\alpha(\lambda)}{\|S_c(T-t)^*\psi_\alpha(\lambda)\|} = \frac{S_c(T-t)\psi_\alpha(\lambda)}{\|S_c(T-t)\psi_\alpha(\lambda)\|} \tag{5.5}$$

is regular for $-1/2 < \alpha < 1/2$, singular for $\alpha \geq 1/2$. Using an argument entirely independent of the maximum principle, it is shown in [11], [17, Theorems 3.4.2 and 5.1] that

Theorem 5.2 The control $\bar{u}_\alpha(t,\lambda)$ drives 0 to $y(T,0,\bar{u}_\alpha)$ time and norm optimally if $\alpha \leq 1/2$. The drive is neither time or norm optimal if $\alpha > 1/2$.

Theorem 5.2 says something new only in the range $\alpha \geq 1/2$ (for $\alpha < 1/2$ optimality is covered by Theorem 1.3). For $\alpha = 1/2$ the control is singular but still time and norm optimal (thus showing that condition (1.4) is not necessary for optimality). For $\alpha > 1/2$ the control is singular and no longer optimal.

Example 5.3 The semigroup is (3.8) in Example 3.4. We use the family of multipliers

$$z_\alpha(x) = \frac{1}{x^\alpha} \quad \Big(\alpha > \frac{1}{2}\Big) \tag{5.6}$$

associated with the controls

$$\bar{u}_\alpha(t,x) = \frac{\chi_0(x)}{\kappa(T-t,z_\alpha)(x+(T-t))^\alpha} \quad (0 \leq t \leq T) \tag{5.7}$$

($\kappa(t,z)$ defined in (3.12), $\chi_0(x)$ the characteristic function of $x \geq 0$). We have

$$\kappa(t,z_\alpha)^2 = \int_t^\infty \frac{\mathrm{d}x}{x^{2\alpha}} = \frac{t^{1-2\alpha}}{2\alpha-1} = \frac{1}{(2\alpha-1)t^{2\alpha-1}},$$

thus

$$\|S(t)^*z_\alpha\| = \kappa(t,z_\alpha) = \frac{1}{\sqrt{2\alpha-1}\,t^{\alpha-1/2}}. \tag{5.8}$$

By Lemma 2.3 there is essential uniqueness of multipliers, thus the the control $\bar{u}_\alpha(t,x)$ is regular (that is, $z_\alpha(\cdot) \in Z(T)$) if and only if

$$\alpha - \frac{1}{2} < 1 \quad \Longleftrightarrow \quad \alpha < \frac{3}{2}\,;$$

on the other hand $z_\alpha(\cdot) \in \mathcal{M}$ for $\alpha \geq 3/2$. Combining (5.7) and (5.8)

$$\bar{u}_\alpha(t,x) = \frac{\sqrt{2\alpha-1}\chi_0(x)(T-t)^{\alpha-1/2}}{(x+(T-t))^\alpha} \quad (0 \leq t \leq T).$$

When $\alpha < 3/2$, $\bar{u}_\alpha(t,x)$ is regular and its optimality follows from Theorem 1.3. Thus the following result in [20, Section 3] (again proved bypassing the maximum principle) gives new information only in the range $\alpha \geq 3/2$.

Theorem 5.4 The control $\bar{u}_\alpha(t,x)$ drives 0 to $y(T,0,\bar{u}_\alpha)$ both time and norm optimally if $\alpha \leq 3/2$. The drive is neither time or norm optimal if $\alpha > 3/2$.

The formal similarity in the statements of Theorem 5.2 and 5.4 is obvious, and it is remarkable that similar results (proved in different ways) work for semigroups as different as (5.3) (analytic, with (1.1) an abstract parabolic equation) and (3.8) (isometric, with equation (1.1) having a finite velocity of propagation, thus qualifying as "abstract hyperbolic").

6 Singular Controls, II

A general way of constructing singular controls which are not optimal is the following. Restricting ourselves to Hilbert spaces, we take

$$\tilde{u}(t) = \frac{S(T-t)^*\psi}{\|S(T-t)^*\psi\|} \tag{6.1}$$

with $\psi \in \mathcal{M} \setminus Z(T)$ (we assume the denominator is not zero in $0 \leq t \leq T$). If we can show that

$$\bar{y} = y(T,0,\tilde{u}) = \int_0^T S(T-\sigma)\tilde{u}(\sigma)\mathrm{d}\sigma \in D(A), \tag{6.2}$$

then we can use Existence Theorem 1.4 to construct a control $\bar{u}(t)$ that drives 0 to $\bar{y}$ norm optimally in $0 \leq t \leq T$. Since $\bar{u} \in D(A)$ Theorem 1.2 applies and $\bar{u}(t)$ satisfies the maximum principle (1.5) with a multiplier $z \in Z(T)$; this means

$$\bar{u}(t) = \rho\,\frac{S(T-t)^*z}{\|S(T-t)^*z\|}, \tag{6.3}$$

where $\rho = \|\bar{u}(\cdot)\|_{L^\infty(0,T;E)}$. Since $\|\bar{u}(\cdot)\|_{L^\infty(0,T;E)} = 1$ we have $\rho \leq 1$. If $\rho < 1$ then (6.1) is not norm optimal, thus not time optimal (time optimality implies norm optimality [9], [17, Theorem 2.1.2]). On the other hand, if $\rho = 1$ then both (6.1) and (6.3) are norm optimal; by uniqueness of norm optimal controls (closing comments in Section 1)

$$\frac{S(T-t)^*z}{\|S(T-t)^*z\|} = \frac{S(T-t)^*\psi}{\|S(T-t)^*\psi\|}$$

and it follows from Lemma 2.3 that $\psi = \lambda z$, a contradiction. This argument requires

$$S(T-t)^*\psi \neq 0, \quad S(T-t)^*z \neq 0 \quad (0 \leq t \leq T) \tag{6.4}$$

To our knowledge the only general result on controls of the form (6.1) with $y(T,0,\tilde{u}) \in D(A)$ (that is, satisfying (6.2)) is proved in [12, Lemma 8.3], [17, Lemma 3.5.9].

Lemma 6.1 Let E be a Hilbert space and let A be self adjoint, $A \le \omega I$. Assume that, for some r, $1 < r < 2$ we have

$$\int_0^1 \frac{\|S(r\sigma)\psi\|}{\sigma\|S(\sigma)\psi\|}\mathrm{d}\sigma < \infty, \tag{6.5}$$

and that $\psi \notin Z(T)$. Then (6.2) holds. In particular, the control (6.1) is not time or norm optimal.

This result is used for construction of nonoptimal controls satisfying the maximum principle in [17, Example 3.5.11] for the semigroup (5.3) in Example 5.1 and the function

$$\psi(\lambda) = \sqrt{\frac{2I_1(4\sqrt{\lambda})}{\sqrt{\lambda}}}, \tag{6.6}$$

where $I_1(x)$ is the Bessel function $I_1(x) = \sum\limits_{n=0}^{\infty} x^{2n+1}/2^{2n+1}n!(n+1)!$. It is proved there that

$$\|S_c(t)^*\psi\| = \|S_c(t)\psi\| = e^{-ct}(e^{2/t} - 1),$$

which shows that (6.5) holds. Condition (6.4) is automatic for an analytic semigroup, thus another example of a nonoptimal singular control ensues.

Lemma 6.1 can be informally stated as “if $\|S(t)^*\psi\|$ increases very fast as $t \to 0$ then the control (6.1) drives 0 to a target $\bar{y} \in D(A)$” (and thus is an example of a nonoptimal singular control, since “increasing very fast” destroys the regularity condition (1.4)). It is natural to ask whether the same sort of result exist for other semigroups. This is in fact the case for the semigroup (3.8) in Example 3.4, as shown in [21]. For the multiplier

$$\psi(x) = \frac{e^{1/2x}}{x} \quad (x > 0), \tag{6.7}$$

we have the “very fast increase” estimate

$$\|S(t)^*\psi\| = e^{1/2t}(1 + O(t)),$$

and it is proved in [21, Lemma 5.1] that the control $\bar{u}(t,x)$ satisfying the maximum principle with multiplier (6.7) drives 0 to the target $\bar{y} = y(T,0,\bar{u}) \in D(A)$. Since $\psi \notin Z(T)$ the drive is not norm or time optimal. Again, it is remarkable that results of similar type hold for semigroups as different as (3.8) and (5.3). The proofs (as can be expected) are totally different.

7 Strongly Singular and Hypersingular Controls

An admissible control $\bar{u}(t)$ is strongly singular if it does not satisfy the maximum principle (1.5) for any multiplier $z \in \mathcal{M}$. Examples of strongly singular norm optimal controls were the first to be constructed [13], [17, Theorem 2.8.6] for a class of semigroups including analytic semigroups. (they were first constructed for self adjoint semigroups in [10]). For time optimal controls no such general result can exist as the following result [17, Theorem 3.2.3] shows:

Theorem 7.1 Assume that $S(t)$ is an analytic semigroup in a Hilbert space E. Then there exists a multiplier space $\mathcal{Z}$ such that, if $\bar{u}(t)$ is time optimal in the interval $0 \le t \le T$ then $\bar{u}(t)$ satisfies the maximum principle (1.5) with $\psi \in \mathcal{Z}$ and $S(t)^*\psi \neq 0$ $(t > 0)$.

(This result actually holds for the larger class of ζ-convex Banach spaces; see the comments in [17] after Theorem 3.2.3).

Theorems 7.1 and 2.1 force us to look for examples of strongly singular time optimal controls through a narrow window (Theorem 7.1 excludes many analytic semigroups, while groups satisfy (2.1) and are thus also excluded). Examples of strongly singular time optimal controls were first discovered for the semigroup (3.8) in Example 3.4 in [14], [17, Theorem 2.8.4]. The strongly singular time optimal controls $\bar{u}(t)$ constructed there are discontinuous at $t = T_n$, where $\{T_n\}$ is a sequence in the control interval $[0, T]$ with $T = T_0 < T_1 < T_2 < \cdots$, $T_n \to T$. These controls cannot satisfy (1.5) in the interval $0 \le t \le T$; if they did, formula (1.6) would be in action. Since $S(T-t)^*\psi \neq 0$ for $T - \delta < t < T$ (this is a standing assumption in the maximum principle) (1.6) would imply that $\bar{u}(t)$ is continuous in $T-\delta < t < T$, a contradiction. The examples are enhanced by the fact that, in the Hilbert space setting $E = L^2(0, \infty)$ there is uniqueness of time optimal controls, thus there exist targets that can be hit time optimally only by a strongly singular control.

The last frontier in singularity is hypersingularity. Let $\bar{u}(t)$ be a time optimal control defined in the interval $0 \le t \le T$. We say that $\bar{u}(t)$ is hypersingular if it is strongly singular in any subinterval $[\alpha, \beta]$, $0 \le \alpha < \beta \le T$. The definition is only significant for time optimal controls, since time optimality is inherited by subintervals from the interval $[0, T]$; this is not the case for the norm optimal problem and makes the concept of hypersingularity uninteresting in this case.

The strongly singular controls constructed in [14] for the semigroup (3.8) are not hypersingular, in fact, they are regular in each interval $[T_{n-1}, T_n]$; that they have a jump at each T_n is due to the fact that they "switch costates" as t moves across T_n. The only examples of hypersingular controls we know of are assembled in [18] for the semigroup (2.8) in Example 2.5. Since the construction is elementary, we include it below in full.

Lemma 7.2 Let $\{T_n\}$ be a sequence $0 = T_0 < T_1 < \cdots < T_n \to T$ and let $\bar{y} = \{\bar{y}_n\}$ be a sequence satisfying

$$1 > \frac{n|\bar{y}_n|}{1 - e^{-nT}} > \rho_n = \frac{1 - e^{-nT_n}}{1 - e^{-nT}}. \tag{7.1}$$

Then $\{\bar{y}_n\} \in \ell^0 = E$ and the constant admissible control

$$\bar{u}(t) = \{\bar{u}_n(t)\} = \left\{ \frac{n\bar{y}_n}{1 - e^{-nT}} \right\} \tag{7.2}$$

drives 0 to the target $\{\bar{y}_n\}$ time optimally.

Proof The first inequality (7.1) implies $|\bar{y}_n| = O(1/n)$ thus the first statement follows. Assume we have an admissible control $\{u_n(t)\}$ driving 0 to $\{\bar{y}_n\}$ in time $T' < T$. Then, if $T_m > T'$ the control $\{v_n(t)\}$ with

$$v_n(t) = \begin{cases} 0 & (0 \le t \le T_m - T'), \\ u_n(t - (T_m - T')) & (T_m - T' \le t \le T_m) \end{cases}$$

drives from 0 to $\{\bar{y}_n\}$ in time T_m, which means

$$\bar{y}_n = \int_0^{T_m} e^{-n(T-\sigma)} v_n(\sigma)\mathrm{d}\sigma \Longrightarrow |\bar{y}_n| \le \int_0^{T_m} e^{-n(T-\sigma)}\mathrm{d}\sigma = \frac{1-e^{-nT_m}}{n}$$

thus, combining with (7.1),

$$\frac{1-e^{-nT_n}}{n} = \rho_n \frac{1-e^{-nT}}{n} < |\bar{y}_n| \le \frac{1-e^{-nT_m}}{n},$$

which is a contradiction for $n = m$. This ends the proof.

The control (7.2) is hypersingular; we have $0 < u_n(t) < 1$ $(0 \le t \le T)$ for all n, thus Lemma 2.6 precludes it from satisfying the maximum principle (1.5) in any subinterval $[\alpha, \beta]$.

The value of this example is diminished by the extreme lack of smoothness of the space ℓ^0; in this space there is no uniqueness of time optimal controls. In fact, it is shown in [18] that there exists an admissible control $v(t)$ which is strongly singular (but not hypersingular) such that $y(T, 0, \bar{u}) = y(T, 0, \bar{v})$, thus $\bar{v}(t)$ does the same drive as $\bar{u}(t)$ time optimally. It would obviously be better to have an example of an hypersingular control in a space where uniqueness of time optimal controls holds (e.g. a Hilbert space) but we don't know of any.

8 Other Approaches to Optimality

The value function of $y \in E$ in the time optimal problem is the time $T(y)$ needed to drive y to 0 optimally by means of an admissible control; if the drive is not possible, $T(y) = \infty$. A second approach to the time optimal problem (the dynamic programming method) was outlined in [2], and it is based on an equation of Hamilton - Jacobi type for the value function. As given in [2] this method in purely heuristic, although it produces explicit solutions of some problems; optimal controls $u(t)$ are given by a feedback law $\bar{u}(t) = \mathcal{F}(y(t))$ from the state $y(t)$. In contrast, the maximum principle requires only the initial and final point of the trajectory, but explicit computation of the control from the maximum principle (except for some examples, mostly finite dimensional) is in general not possible.

The dynamic programming method was put on a firm mathematical footing in [4] by differential geometric methods, although only in the finite dimensional case (that is, for systems described by ordinary differential equations) and under stringent smoothness conditions. Justification of dynamic programming for infinite dimensional systems like (1.1) (and nonlinear versions) had to wait for new tools in nonlinear nonsmooth analysis [5], [6], [8], [22]. For a comparison of time optimality and norm (energy) optimality in this context see [7], [8].

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