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STRONG REGULARITY OF TIME AND NORM OPTIMAL CONTROLS

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Abstract. Pontryagin's maximum principle in its infinite dimensional version provides (separate) necessary and sufficient conditions for both time and norm optimality for the system $y' = Ay + u$ (A the infinitesimal generator of a strongly continuous semigroup); in particular it provides a costate $z(t)$ for every time or norm optimal control $\bar{u}(t)$ hitting a target $\bar{y} \in D(A)$. This paper shows that for the right translation semigroup the same condition on $\bar{y}$ guarantees that $z(T) \in E^*$, which in turn implies continuity of optimal controls in the entire control interval $[0, T]$.

Keywords. Time optimality, norm optimality, Pontryagin's maximum principle, costate, smoothness of optimal controls.

AMS (MOS) subject classification: 93E20, 93E25.

1 Introduction

We study the control system

$$y'(t) = Ay(t) + u(t)\,, \quad y(0) = \zeta \tag{1.1}$$

with controls $u(\cdot) \in L^\infty(0, T; E)$, where A is the generator of a strongly continuous semigroup $S(t)$ in a Banach space E. We look at two optimal control problems for (1.1). One is the *norm optimal* problem, where we drive the initial point ζ to a point target,

$$y(T\,) = \bar{y} \tag{1.2}$$

in a fixed time interval $0 \leq t \leq T$ minimizing $\|u(\cdot)\|_{L^\infty(0,T;E)}$. The second is the *time optimal* problem, where we drive to the target with a bound on the norm of the control (say $\|u(\cdot)\|_{L^\infty(0,T;E)} \leq 1$) in optimal time T. The *solution* or *trajectory* of (1.1) is given by the variation-of-constants formula

$$y(t) = y(t, \zeta, u) = S(t)\zeta + \int_0^t S(t - \sigma)u(\sigma)d\sigma \tag{1.3}$$

and is continuous in $t \geq 0$. For the time optimal problem, controls in $L^\infty(0,T;E)$ with norm $\|u(\cdot)\|_{L^\infty(0,T;E)} \leq 1$ are called *admissible.*

Separate necessary and sufficient conditions for norm and time optimality are given in terms of the maximum principle (1.5) below, whose formulation requires the construction of spaces of multipliers (final values of costates). We summarize [2] or [4] Section 2.3 assuming (as we may in this paper) that the adjoint $S(t)^*$ is strongly continuous in E^*. When A has a bounded inverse, the space E^*_{-1} is the completion of E^* in the norm

$$\|y^*\|_{E^*_{-1}} = \|(A^{-1})^* y^*\|_{E^*} \,.$$

Each $S(t)^*$ can be extended to an operator $S(t)^* : E^*_{-1} \to E^*_{-1}$, and the space $Z(T) \subseteq E^*_{-1}$ consists of all $z \in E^*_{-1}$ such that $S(t)^* z \in E^*$ $(t > 0)$ and

$$\|z\|_{Z(T)} = \int_0^T \|S(t)^* z\| dt < \infty \,. \tag{1.4}$$

The space $Z(T)$ equipped with the norm $\|\cdot\|_{Z(T)}$ is a Banach space. All spaces $Z(T)$ coincide (that is, $Z(T) = Z(T')$ and the norms $\|\cdot\|_{Z(T)}, \|\cdot\|_{Z(T')}$ are equivalent for $T, T' > 0$). $Z(T)$ is an example of a *multiplier space,* an arbitrary linear space $\mathcal{Z} \supseteq E^*$ to which $S(t)^*$ can be extended in such a way that $S(t)^* \mathcal{Z} \subseteq E^*$. When A does not have a bounded inverse, a few changes are needed. Since A is a semigroup generator, $(\lambda I - A)^{-1}$ exists for $\lambda > \omega$ and E^*_{-1} is the completion of E^* in any of the equivalent norms

$$\|y^*\|_{E^*_{-1},\lambda} = \|((\lambda I - A)^{-1})^* y^*\|_{E^*} \quad (\lambda > \omega) \,.$$

The definition of $Z(T)$ (and of multiplier spaces) is the same. See [4] Section 2.3 for details.

A control $\bar{u}(\cdot) \in L^\infty(0,T;E)$ satisfies *Pontryagin's maximum principle* if

$$\langle S(T-t)^* z, \bar{u}(t) \rangle = \max_{\|u\| \leq \rho} \langle S(T-t)^* z, u \rangle \quad \text{a. e. in } 0 \leq t < T \,, \tag{1.5}$$

where $\langle \cdot\,, \cdot \rangle$ is the duality of E and the dual E^*, with $\rho = \|\bar{u}(\cdot)\|_{L^\infty(0,T;E)}$ and z in a multiplier space $\mathcal{Z}$. We call z the *multiplier* and $S(T-t)^* z$ the *costate* corresponding to (or associated with) the control $\bar{u}(t)$. We assume that (1.5) is nonempty, that is, that $S(T-t)^* z$ is not identically zero in the interval $0 \leq t < T$, although it may be zero in part of the interval (in which part (1.5) says nothing about $\bar{u}(t)$). The nonemptiness requirement implies that $z \neq 0$. When E is a Hilbert space the maximum principle reduces to

$$\bar{u}(t) = \rho \frac{S(T-t)^* z}{\|S(T-t)^* z\|} \qquad (0 \leq t \leq T) \,. \tag{1.6}$$

If $S(T-t)^* z = 0$ for some $t \in (0,T)$ then the interval in (1.6) is replaced by the maximal interval $(\delta, T]$ where $S(T-t)^* z \neq 0$.

A large part of the the theory of optimal controls for (1.1) deals with the relation between optimality and the maximum principle (1.5). There are separate necessary and sufficient conditions for optimality based on the maximum principle (Theorem 1.1 below). We call an optimal control $\bar{u}(t)$ *regular* if it satisfies (1.5) with $z \in Z(T)$, *strongly regular* if $z \in E^*$.

Theorem 1.1 *Assume* $\bar{u}(t)$ *drives* $\zeta \in E$ *to* $\bar{y} = y(T, \zeta, \bar{u})$ *time or norm optimally in the interval* $0 \leq t \leq T$ *and that*

$$\bar{y} - S(T)\zeta \in D(A)\,. \tag{1.7}$$

Then $u(t)$ *is regular. Conversely, let* $\bar{u}(t)$ *be a regular control. Then* $\bar{u}(t)$ *drives* $\zeta \in E$ *to* $\bar{y} = y(T, \zeta, \bar{u})$ *norm optimally in the interval* $0 \leq t \leq T$; *if* $\rho = 1$ *the drive is time optimal.*

For the proof see [2] Theorem 5.1, [4] Theorem 2.5.1; in the sufficiency part of Theorem 1.1 no conditions like (1.7) are put on the initial value ζ or the target $\bar{y}$.[1] The notion of strongly regular control is of interest in applications. In fact, if E^* is a Hilbert space then (1.6) shows that a strongly regular control is (at least) continuous in $0 \leq t \leq T$, whereas a merely regular control may "chatter" at the endpoint T of the control interval. This makes a difference, for instance, in numerical approximations of the optimal control.

A natural question on regular controls is, *characterize the control systems* (1.1) *for which all* (*time, norm*) *optimal controls are strongly regular.* A sufficient condition for all optimal controls being strongly regular is

$$S(t)E = E \quad (t > 0)\,. \tag{1.8}$$

This condition is valid in any Banach space [4] Theorem 2.1. On the other hand, (1.8) is also a necessary condition when E is a Hilbert space or even under weaker assumptions [7] Corollary 4.8, thus we may consider this question answered.

The next natural question might be: in absence of (1.8), *does* (1.7) *guarantee that* $\bar{u}(t)$ *is strongly regular?* The answer is negative, as shown in [6]. We may then modify the question: *for what kind of space and semigroup does* (1.7) *guarantee that* $\bar{u}(t)$ *is strongly regular?* Of course, any interesting answer has to exclude (1.8). We don't know of any general result; all this paper presents is an example of a semigroup not satisfying (1.8) where condition (1.7) (with $\zeta = 0$) implies strong regularity of optimal controls. For special semigroups, there exist conditions on the target $\bar{y}$ (all stronger than $\bar{y} \in D(A)$) that guarantee that $z \in E^*$; for instance, if A is self adjoint and negative definite in Hilbert space, $\bar{y} \in D((-A)^\alpha)$ for $\alpha > 1$ guarantees that $z \in E$ (in fact, z is in a slightly smaller space, see [4] Theorem 6.1.2).

[1] The statement on time optimality needs additional assumptions on the initial condition ζ and the target $\bar{y}$. These conditions are satisfied if $\zeta = 0$ irrespective of the target ([4] Theorem 2.5.7), which is the only case of interest in this paper. We also need to assume that $S(t)^*z \neq 0$ in the entire interval $0 \leq t \leq T$.

However, this condition involves fractional powers and is not easy to verify, for instance, for a differential operator.

The punch line in our arguments (coming in Section 7) is a consequence of the following existence-uniqueness statements, where we assume E is a Hilbert space. The first is from [1] (see also [4] Theorems 3.1.2 and 2.1.7). For the second, we reproduce the argument in [4] since it is central to our arguments. A target $\bar{y}$ is called *r-reachable in time* T if we can drive from zero to $\bar{y}$ in time $T > 0$ with a control $u(\cdot)$ satisfying $\|u(\cdot)\|_{L^\infty(0,T;E)} \le r$. The target is *r-reachable* if it is r-reachable in some time T.

Theorem 1.2 (*a*) *Assume* $\bar{y}$ *is* 1*-reachable. Then there exists an admissible control* $\bar{u}(\cdot)$ *that does the drive in optimal time* T*. The optimal control is unique.* (*b*) *Assume* $\bar{y}$ *is r-reachable in time* T*. Then there exists a control that does the drive with minimum norm. If* $\bar{y} \in D(A)$ *and* $S(t)^* z \ne 0$ $(0 \le t < T)$ *this control is unique.*

Theorem 1.3 *Assume there exist* $z, \tilde{z} \in Z(T)$ *such that*

$$\int_0^T S(T-\sigma)\frac{S(T-\sigma)^* z}{\|S(T-\sigma)^* z\|}d\sigma = \bar{y} = \int_0^{\tilde{T}} S(\tilde{T}-\sigma)\frac{S(\tilde{T}-\sigma)^* \tilde{z}}{\|S(\tilde{T}-\sigma)^* \tilde{z}\|}d\sigma \quad (1.9)$$

(*we assume the denominators in each integral nonzero in the whole interval of integration*)*. Then*

$$T = \tilde{T} \quad \textit{and} \quad \tilde{z} = \mu z\,,\ \mu \ne 0. \quad (1.10)$$

Proof. Using the sufficiency part of Theorem 1.1 (which does not require any condition on $\bar{y}$) the first equality (1.9) says that the first control drives 0 to $\bar{y}$ in optimal time T, the second equality that the second control drives 0 to $\bar{y}$ in optimal time $\tilde{T}$. The optimal time is unique, thus $\tilde{T} = T$ and the first equality (1.10) ensues. Uniqueness of optimal controls (Theorem 1.2) means

$$\frac{S(T-\sigma)^* z}{\|S(T-\sigma)^* z\|} = \frac{S(T-\sigma)^* \tilde{z}}{\|S(T-\sigma)^* \tilde{z}\|}\,. \quad (1.11)$$

Applying $(\lambda I - A^*)^{-1}$ (λ large enough) on both sides and multiplying by the denominator of the second fraction,

$$\mu(t) S(T-\sigma)^*(\lambda I - A^*)^{-1} z = S(T-\sigma)^*(\lambda I - A^*)^{-1}\tilde{z} \quad (1.12)$$

Since $Z(T) \subseteq E^*_{-1}$ and $(\lambda I - A^*)^{-1} : E^*_{-1} \to E^*$, $(\lambda I - A^*)^{-1} z \in E^*$ and $(\lambda I - A^*)^{-1}\tilde{z} \in E^*$. Equality (1.12) and strong continuity of $S(t)^*$ imply that $\mu(t)$ is continuous with $\mu(T) = \mu \ne 0$. Accordingly, $(\lambda I - A^*)^{-1} z = (\lambda I - A^*)^{-1}\tilde{z}$ which implies the second equality (1.10). Note that "uniqueness modulo multiplication by a nonzero constant" is all that can be proved for multipliers z; the control (1.6) is the same for z or μz.

The main results in this paper (for the right translation semigroup) are

Theorem 1.4 *Let $\bar{y} \in D(A)$, and assume $\bar{y}$ is 1-reachable. Then the (unique) time optimal control in Theorem 1.2 is of the form*

$$\bar{u}(t) = \frac{S(T-t)^* z}{\|S(T-t)^* z\|} \quad (0 \le t \le T = \textit{optimal time}\,, \ z \in E^*)\,. \tag{1.13}$$

Theorem 1.5 *Let $\bar{y} \in D(A)$ and assume $\bar{y} \in E$ is r-reachable in time T. Then the (unique) norm optimal control in Theorem 1.3 is of the form*

$$\bar{u}(t) = \rho \frac{S(T-t)^* z}{\|S(T-t)^* z\|} \quad (0 \le t \le T, \quad \rho = \textit{optimal norm}\,, \ z \in E^*)\,. \tag{1.14}$$

For a general semigroup, all that can be said is that $z \in Z(T)$.[2] The arguments reveal that, for the right translation semigroup, the key condition is not $\bar{y} \in D(A)$ but $\bar{y} \in \mathcal{R}^2 \supset D(A)$ (see Section 4) where the class $\mathcal{R}^2$ is characterized by the growth of $\bar{y}(x)$ at 0, not by the smoothness involved in the definition of $D(A)$. The approach is constructive; it allows explicit computation of costates (thus of optimal controls) using ODE software. Examples in Section 8.

2 The right translation semigroup

The space is $E = L^2(0,\infty)$. The *right translation* semigroup $S(t)$ in E is

$$S(t)y(x) = \begin{cases} y(x-t) & (x \ge t) \\ 0 & (x < t) \end{cases} \tag{2.1}$$

This semigroup is strongly continuous and isometric with $\|S(t)\| \le 1$. The adjoint semigroup is the *left translation* (and chop-off) semigroup

$$S(t)^* z(x) = z(x+t) \quad (x \ge 0)\,. \tag{2.2}$$

We have

$$S(t)^* S(t) = I\,, \quad S(t)S(t)^* y(x) = \chi(t,x) y(x) \tag{2.3}$$

where $\chi(t,x) = 1$ if $x \ge t$, $\chi(t,x) = 0$ elsewhere. The infinitesimal generator A of $S(t)$ is

$$Ay(x) = -y'(x)\,, \tag{2.4}$$

with domain $D(A) = \{y(\cdot) \in L^2(0,\infty);\ y'(\cdot) \in L^2(0,\infty),\ y(0) = 0\}$, the derivative understood in the sense of distributions. The semigroup $S(t)$ is associated with the control system

$$\begin{aligned} \frac{\partial y(t,x)}{\partial t} &= -\frac{\partial y(t,x)}{\partial x} + u(t,x) \quad (0 \le t,\ x < \infty) \\ y(0,x) &= \zeta(x)\,, \qquad y(t,0) = 0 \end{aligned} \tag{2.5}$$

[2] For a space and semigroup where the costate z in (1.13) or (1.14) does *not* belong to E^* for certain targets $\bar{y} \in D(A)$ see [6].

in the sense that $S(t)$ is the propagation semigroup of the homogeneous equation ($u(t,x)=0$). Formula (1.3)for the control $u(t)(x)=u(t,x)$ is

$$\begin{aligned} y(t,x,\zeta,u) = y(t,\zeta,u)(x) &= \left(S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)d\sigma\right)(x) \\ &= \zeta(x-t) + \int_0^t u(\sigma, x-(t-\sigma))d\sigma\,, \end{aligned} \tag{2.6}$$

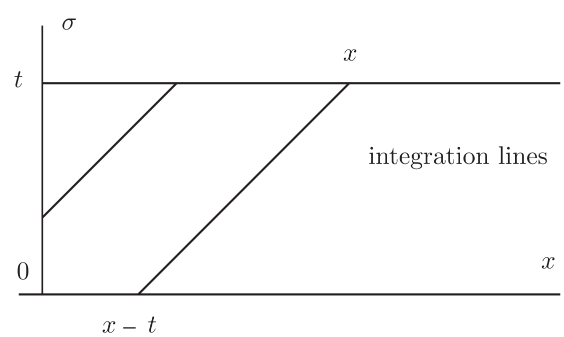

Figure 1

We name $\mathcal{Z}$ the space of all measurable $z(x)$ defined in $x>0$ and such that

$$\kappa(\sigma,z) = \|S(\sigma)^* z(\cdot)\| = \sqrt{\int_0^\infty z(x+\sigma)^2dx} = \sqrt{\int_\sigma^\infty z(x)^2dx} < \infty \tag{2.7}$$

for $\sigma > 0$. The space $\mathcal{Z}$ is the largest multiplier space. The space $Z(T)$ consists of all $z(\cdot) \in \mathcal{Z}$ with

$$\int_0^T \|S(\sigma)^* z(\cdot)\| d\sigma = \int_0^T \kappa(\sigma,z)d\sigma < \infty\,. \tag{2.8}$$

Since we are in a Hilbert space (1.6) applies. Setting $\rho=1$ for simplicity any control that satisfies the maximum principle (1.5) is given a. e. by

$$\bar{u}(\sigma,x) = \frac{S(T-\sigma)^* z(x)}{\|S(T-\sigma)^* z(\cdot)\|} = \chi(0,x)\frac{z(x+(T-\sigma))}{\kappa(T-\sigma,z)} \quad (0 \le \sigma \le T)\,, \tag{2.9}$$

the interval $(0,T]$ replaced by the maximal interval where $S(t)^*z \neq 0$ if necessary. Using the second equality (2.3) we obtain

$$S(T-\sigma)\bar{u}(\sigma)(x) = \frac{S(T-\sigma)S(T-\sigma)^* z(x)}{\|S(T-\sigma)^* z(\cdot)\|} = \frac{\chi(T-\sigma,x)z(x)}{\kappa(T-\sigma,z)} \tag{2.10}$$

in $0 < \sigma \le T$. Formula (1.3) for $t = T$ becomes

$$
\begin{aligned}
&y(T, x, \zeta, \bar{u})(x) = S(T)\zeta(x) + \int_0^T S(T-\sigma)\bar{u}(\sigma, x)d\sigma \\
&= \zeta(x-T) + \int_0^T \frac{\chi(T-\sigma, x)z(x)}{\kappa(T-\sigma, z)}d\sigma = \zeta(x-T) + z(x)\int_0^T \frac{\chi(T-\sigma, x)}{\kappa(T-\sigma, z)}d\sigma \\
&= \zeta(x-T) + z(x)\int_0^T \frac{\chi(\sigma, x)}{\kappa(\sigma, z)}d\sigma = \zeta(x-T) + z(x)\omega(T, x, z)\,, \qquad (2.11)
\end{aligned}
$$

where

$$
\omega(T, x, z) = \int_0^T \frac{\chi(\sigma, x)}{\kappa(\sigma, z)}d\sigma = \int_0^{\min(x,T)} \frac{d\sigma}{\kappa(\sigma, z)}\,. \qquad (2.12)
$$

If we drive from 0 to $\bar{y}(x)$ in time T, the target $\bar{y}(x)$ and the costate $z(x)$ are related by

$$
\bar{y}(x) = y(T, x, 0, \bar{u}) = z(x)\omega(T, x, z)\,. \qquad (2.13)
$$

Formula (2.12) requires clarification if $z(x)$ has compact support. Let $a(z)$ be the least a such that $z(x)$ has support contained in $[0, a]$. ($a(z) = \infty$ if no such a exists). When $a(z) < \infty$ we have

$$
\kappa(\sigma, z)\begin{cases} > 0 & (0 < x < a(z)) \\ = 0 & (x \ge a(z)) \end{cases}
$$

so that $\omega(x)$ is only defined in $0 \le x < a(z)$; if $T > a(z)$ this leaves $\omega(T, x, z)$ undefined in the interval $[a(z), T]$. We show below that this interval, if nonempty, consists of a single point.

Lemma 2.1 *We have $a(z) = a(\bar{y})$. If $\bar{y}$ is 1-reachable then the optimal time satisfies $T \le a(\bar{y})$.*

Proof. The fact that $a(z) = a(\bar{y})$ follows directly from

$$
z(x) = 0 \quad \Longleftrightarrow \quad \bar{y}(x) = 0\,, \qquad (2.14)
$$

a consequence of formula (2.13). For the second statement, we may assume $a(\bar{y}) < \infty$. If $T > a(\bar{y})$ we have the configuration in Figure 2.

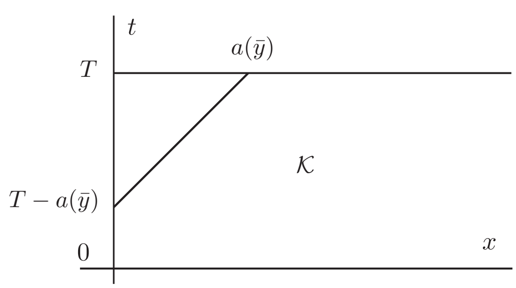

Figure 2

If the time optimal control $\bar{u}(t,x)$ is not zero in $\mathcal{K}$ we may modify its definition to $\bar{u}(t,x) = 0$ there. This doesn't affect the target and, if anything, improves each norm $\|\bar{u}(t,\cdot)\|_{L^2(0,\infty)}$. The modified control is admissible, thus time optimal; by uniqueness it is equal to $\bar{u}(t,x)$. If $T > a(\bar{y})$ then $\|\bar{u}(t,\cdot)\|_{L^2(0,\infty)} = 0$ in $0 \le t \le T - a(\bar{y})$. However, the bang-bang Theorem 2.1.2 in [4] says that a time optimal control must satisfy $\|u(t)\| = 1$ for all t, thus we have a contradiction.

3 Target and costate, I

Formula (2.12) says that $\omega(T,x,z) = \omega(T,T,x) = C =$ constant for $x \ge T$. Multiplying the costate $z(x)$ by C multiplies $\omega(T,x,z)$ by $1/C$, thus we may assume that

$$\omega(T,x,z) = 1 \quad (x \ge T)\,. \tag{3.1}$$

Formula (2.13) then says that the costate and the target are equal for $x \ge T$.

We turn (2.13) into a differential equation for $\omega(T,x,z)$. To lighten up the notation we set $\kappa(x,z) = \kappa(x)$, $\omega(T,x,z) = \omega(x)$. We have

$$\omega(x) = \int_0^x \frac{d\sigma}{\kappa(\sigma)}\,, \qquad \omega'(x) = \frac{1}{\kappa(x)}\,, \tag{3.2}$$

and $\omega(x)$ is well defined in $0 \le x < a(\bar{y})$, where the arguments below apply. On the other hand, we have $\kappa(a) = 0 \iff \bar{y}(x) = 0$ $(x \ge a)$. In view of its definition (2.7) we have

$$(\kappa(x)^2)'(x) = -z(x)^2 \quad \Longrightarrow \quad \kappa'(x) = -\frac{z(x)^2}{2\kappa(x)}\,. \tag{3.3}$$

It follows from (3.2) that $\omega(x)$ is twice differentiable in the sense of distributions with

$$\omega''(x) = \Big(\frac{1}{\kappa(x)}\Big)' = -\frac{\kappa'(x)}{\kappa(x)^2} = \frac{z(x)^2}{2\kappa(x)^3}\,.$$

Hence, $\bar{y}(x)^2 = z(x)^2\omega(x)^2 = 2\kappa(x)^3\omega''(x)\omega(x)^2 = 2\omega(x)^2\omega''(x)/\omega'(x)^3$ or

$$\omega''(x) = \frac{\bar{y}(x)^2\omega'(x)^3}{2\omega(x)^2}\,. \tag{3.4}$$

In case $z(\cdot) \in L^2(0,\infty)$ the initial conditions are

$$\omega(0) = 0\,, \quad \omega'(0) = \frac{1}{\kappa(0)} = \frac{1}{\|z(\cdot)\|} > 0 \tag{3.5}$$

while if $z(\cdot) \in \mathcal{Z} \setminus L^2(0,\infty)$ the initial conditions are

$$\omega(0) = 0\,, \quad \omega'(0) = \frac{1}{\kappa(0)} = \frac{1}{\infty} = 0\,. \tag{3.6}$$

When $z(\cdot) \in Z(T) \setminus L^2(0,\infty)$, since $1/\omega(x) = \kappa(x)$ the initial conditions are complemented with

$$\int_0^\epsilon \frac{dx}{\omega'(x)} < \infty \tag{3.7}$$

where $\epsilon > 0$ is arbitrary. The costate is recovered from the target via (2.13),

$$z(x) = \frac{\bar{y}(x)}{\omega(x)}\,. \tag{3.8}$$

Equation (3.4) does not take into consideration the sign of $\bar{y}(x)$; (3.8) restores the sign.

4 The initial value problem: existence

The space $\mathcal{R}^p$ $(p > 1)$ consists of all $y(\cdot) \in L^p(0,\infty)$ such that

$$\int_0^\infty \Big(\frac{y(x)}{x}\Big)^p dx < \infty\,. \tag{4.1}$$

Since $y(x)$ is assumed to be in $L^p(0,\infty)$, the definition of $\mathcal{R}^p$ only affects the behavior of $y(\cdot)$ near 0; we may integrate in (4.1) in any finite interval $[0,a]$ as well.

Lemma 4.1 *Let*

$$\mathcal{B}y(x) = \frac{1}{x}\int_0^x y(\xi)d\xi\,.$$

Then $\mathcal{B} : L^p(0,1) \to L^p(0,1)$ *and* $\mathcal{B}$ *is bounded for* $1 < p \le \infty$.

Proof. Clearly $\mathcal{B} : L^\infty(0,1) \to L^\infty(0,1)$ is bounded. On the other hand, if

$$|\mathcal{B}y(x)| = \left|\frac{1}{x}\int_0^x y(x)dx\right| \ge c \tag{4.2}$$

we have

$$\|y(\cdot)\|_1 = \int_0^x |y(\xi)|d\xi \ge \left|\int_0^x y(\xi)d\xi\right| \ge cx\,,$$

so that the set $\mathcal{L}(c)$ where (4.2) holds is contained in $0 \le x \le \|y\|_1/c$ and

$$|\mathcal{L}(c)| \le \frac{\|y\|_1}{c}$$

thus showing that $\mathcal{B}$ is of weak type (1, 1). Applying the Marcinkiewicz interpolation theorem [10] p. 183, the result for $p > 1$ follows. It doesn't hold for $p = 1$; $y(x) = 1/x(\log x)^2 \in L^1(0,1)$ but

$$\mathcal{B}y(x) = \frac{1}{x}\int_0^x \frac{d\xi}{\xi(\log\xi)^2} = -\frac{1}{x\log x} \notin L^1(0,1)\,.$$

Corollary 4.2 *We have $D(A) \subset \mathcal{R}^2$; precisely, if $y(\cdot) \in D(A)$ then there exists C_2 such that*

$$\int_0^\infty \Big(\frac{y(x)}{x}\Big)^2 dx \le C_2 \int_0^\infty \big(y'(x)^2 + y(x)^2\big) dx\,. \tag{4.3}$$

Proof: Since

$$\frac{y(x)}{x} = \frac{y(x) - y(0)}{x} = \frac{1}{x}\int_0^x y'(\xi)d\xi = \mathcal{B}y'(x)$$

Lemma 4.1 applies. We have

$$\int_0^\infty \Big(\frac{y(x)}{x}\Big)^2 dx \le \int_0^1 \Big(\frac{y(x)}{x}\Big)^2 + \int_1^\infty y(x)^2 dx \le C\int_0^1 y'(x)^2 dx + \int_1^\infty y(x)^2 dx$$

which ends the proof.

Equation (3.4) is *homogeneous* in the following sense: if $\omega(x)$ is a solution, so is $\mu\omega(x)$ for any $\mu \neq 0$. Consequently, to find a solution of (3.4)-(3.5) we only have to solve

$$\theta''(x) = \frac{\bar{y}(x)^2\theta'(x)^3}{2\theta(x)^2}\,, \quad \theta(0) = 0\,, \quad \theta'(0) = 1 \tag{4.4}$$

and set $\omega(x) = \theta(x)/\kappa(0)$. When the boundary conditions are (3.6), any $\mu \neq 0$ will do. Existence (or uniqueness) of solutions of (4.4) does not follow from standard theorems, since the initial conditions are given at the "bad" point where the right hand side is $= 0$.

Theorem 4.3 *Assume $\bar{y}(x)/x \in L^2(0, a)$ for some $a > 0$. Then the initial value problem* (4.4) *has a solution in some interval* $0 \le x \le \delta \le a$.

Proof. "Solution" means a function $\theta(x)$ with second derivative $\theta''(x)$ (in the sense of distributions) in $L^1(0, \delta)$ satisfying (4.4) a. e., which means that the right side must belong to $L^1(0, \delta)$. Obviously, this will be the case if a solution exists, but we can't assume this *a priori;* we show that the present assumptions on $\theta(x)$ and the initial conditions alone guarantee that the right side of the equation belongs to $L^1(0, \delta)$ for δ sufficiently small. We have

$$\theta'(x) = 1 + \int_0^x \theta''(\xi)d\xi = 1 + o(1)$$

thus $\theta'(x) > 0$ for $0 \le x \le \delta$ sufficiently small and $\theta(x)$ is strictly increasing. Accordingly, we only have to check what happens at $x = 0$. We have

$$\theta(x) = x + \int_0^x (x - \xi)\theta''(\xi)d\xi = x + xo(1) = x(1 + o(1))\,,$$

so that near $x = 0$

$$\frac{\bar{y}(x)^2\theta'(x)^3}{2\theta(x)^2} = \frac{\bar{y}(x)}{2x^2} \cdot \frac{x^2}{x^2(1+o(1))^2}(1+o(1))^3 = \frac{1}{2}\left(\frac{\bar{y}(x)}{x}\right)^2(1+o(1))$$

which belongs to $L^1(0,\delta)$ in view of the assumptions on $\bar{y}(x)$. This argument shows that the initial value problem (4.4) is equivalent to the system of integral equations

$$\theta(x) = x + \int_0^x (x-\xi)\frac{\bar{y}(\xi)^2\theta'(\xi)^3}{\theta(\xi)^2}d\xi\,, \tag{4.5}$$

$$\theta'(x) = 1 + \int_0^x \frac{\bar{y}(\xi)^2\theta'(\xi)^3}{2\theta(\xi)^2}d\xi\,. \tag{4.6}$$

We solve this system by (a variation of) a well known approximation procedure whose punch line is the Arzelà - Ascoli theorem. We work in an interval $[0,\delta]$ *a priori* arbitrary (restrictions on δ come later). We denote by $\{\delta_n\}$ a strictly decreasing sequence in $(0,\delta)$ with $\delta_n \to 0$ and define

$$\theta_n(x) = \begin{cases} x & (0 \le x \le \delta_n) \\ \rho_n(x) & (\delta_n < x) \end{cases} \tag{4.7}$$

where $\rho_n(x)$ is the solution of the initial value problem

$$\rho_n''(x) = \frac{\bar{y}(x)^2\rho'(x)^3}{2\rho(x)^2} \quad (x \ge \delta_n)\,, \quad \rho_n(\delta_n) = \delta_n\,, \quad \rho_n'(\delta_n) = 1 \tag{4.8}$$

equivalent to the system of integral equations

$$\rho_n(x) = x + \int_{\delta_n}^x (x-\xi)\frac{\bar{y}(\xi)^2\rho_n'(\xi)^3}{\rho_n(\xi)^2}d\xi\,, \tag{4.9}$$

$$\rho_n'(x) = 1 + \int_{\delta_n}^x \frac{\bar{y}(\xi)^2\rho_n'(\xi)^3}{2\rho_n(\xi)^2}d\xi\,. \tag{4.10}$$

Local existence and uniqueness of $\rho_n(x)$ can be proved on the basis of the standard (Lipschitz continuous) existence theorem, provided we can show the iterative approximations $\{\rho_{nm}(x); m \ge 1\}$ to the solution $\rho_n(x)$ stay out of the bad region "ρ small" of the function $f(x,\rho,\rho') = \bar{y}(x)\rho'^3/2\rho^2$. The iterations are

$$\rho_{n,m+1}(x) = x + \int_{\delta_n}^x (x-\xi)\frac{\bar{y}(\xi)^2\rho_{nm}'(\xi)^3}{2\rho_{nm}(\xi)^2}d\xi\,,$$

$$\rho_{n,m+1}'(x) = 1 + \int_{\delta_n}^x \frac{\bar{y}(\xi)^2\rho_{nm}'(\xi)^3}{2\rho_{nm}(\xi)^2}d\xi\,.$$

Starting with $\rho_{n0}(x) = x$ it is easily proved by induction that

$$x \le \rho_{nm}(x)\,, \quad 1 \le \rho_{nm}'(x)\,, \tag{4.11}$$

both inequalities assuring that each approximation $(\rho_{nm}(x), \rho'_{nm}(x))$ is bounded below and thus stays out of trouble.

The second line of (4.7) is supposed to be used for $\delta_n \leq x \leq \delta$ if $\rho_n(x)$ exists that far, otherwise in $\delta_n \leq x \leq$ blowup point of $\rho_n(x)$. The approximate solution $\theta_n(x)$ satisfies

$$\theta''_n(x) = 0 = \frac{\bar{y}(x)^2}{2x^2} - \frac{\bar{y}(x)^2}{2x^2} = \frac{\bar{y}(x)\theta'_n(x)^3}{2\theta_n(x)^2} - \frac{\bar{y}(x)^2}{2x^2} \quad (0 \leq x \leq \delta_n) \tag{4.12}$$

and satisfies the exact equation for $x \geq \delta_n$. Integrating and using the initial conditions,

$$\theta_n(x) = x + \int_0^x (x-\xi)\frac{\bar{y}(\xi)^2\theta'_n(\xi)^3}{2\theta_n(\xi)^2}d\xi - \int_0^{\min(x,\delta_n)} (x-\xi)\frac{\bar{y}(\xi)^2}{2\xi^2}d\xi\,, \tag{4.13}$$

$$\theta'_n(x) = 1 + \int_0^x \frac{\bar{y}(\xi)^2\theta'_n(\xi)^3}{2\theta_n(\xi)^2}d\xi - \int_0^{\min(x,\delta_n)} \frac{\bar{y}(\xi)^2}{2\xi^2}d\xi\,. \tag{4.14}$$

We prove that, given an arbitrary constant C there exists $\delta > 0$ such that all $\theta_n(x)$ exist in $0 \leq x \leq \delta$ and

$$x \leq \theta_n(x) \leq x + C\,, \quad 1 \leq \theta'_n(x) \leq 1 + C \quad (0 \leq x \leq \delta,\ n = 1, 2, \dots)\,. \tag{4.15}$$

The two equalities are obvious in $0 \leq x \leq \delta_n$. Both left inequalities $x \leq \theta_n$ and $1 \leq \theta'_n(x)$ follow from (4.11). For the right inequalities, select δ with

$$\int_0^\delta \left(\frac{\bar{y}(\xi)}{\xi}\right)^2 d\xi < \min\left(\frac{2C}{(C+1)^3}, \frac{2C}{\delta(C+1)^3}\right), \tag{4.16}$$

and let $[0, \mu_n]$ be the maximal interval where both inequalities (4.15) hold: obviously we must have

$$\theta_n(\mu_n) = \mu_n + C \quad \text{or} \quad \theta'(\mu_n) = 1 + C\,, \tag{4.17}$$

otherwise a standard continuation argument would refute maximality. We show below that, under the hypotheses on δ there exists n_0 such that

$$\delta \leq \mu_n \quad (n \geq n_0)\,, \tag{4.18}$$

thus showing that all $\theta_n(x)$ exist in the interval $[0, \delta]$ for $n \geq n_0$. To this end, we estimate both integral equations (4.13)-(4.14). Since both inequalities (4.15) are in force in $[0, \mu_n]$ we obtain, using the first estimation from below and the second estimation from above in (4.15) that

$$\frac{\bar{y}(x)^2\theta'_n(x)^3}{2\theta_n(x)^2} \leq \Big(\frac{\bar{y}(x)}{x}\Big)^2 \frac{\theta'_n(x)^3}{2} \leq \frac{(C+1)^3}{2}\Big(\frac{\bar{y}(x)}{x}\Big)^2 \tag{4.19}$$

in $0 \le x \le \mu_n$. It is obvious that $\mu_n > \delta_n$. If (4.18) fails (that is, if $\mu_n < \delta$) we have

$$\begin{aligned}
&\theta_{n+1}(\mu_n) - \mu_n \\
&\quad \le \frac{(C+1)^3}{2} \int_0^{\mu_n} (\mu_n - \xi)\Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi + \frac{1}{2}\int_0^{\delta_n} (\mu_n - \xi)\Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi \\
&\quad < \frac{\delta(C+1)^3}{2} \int_0^{\delta} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi + \frac{\delta}{2}\int_0^{\delta_n} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi < C \qquad (4.20)
\end{aligned}$$

for $n \ge n_0$ large enough; this follows from (4.16) and the fact that $\delta_n \to 0$. The corresponding estimation for (4.14) is

$$\begin{aligned}
\theta'_{n+1}(\mu_n) - 1 &\le \frac{(C+1)^3}{2} \int_0^{\mu_n} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi + \frac{1}{2}\int_0^{\delta_n} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi \\
&< \frac{(C+1)^3}{2} \int_0^{\delta} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi + \frac{1}{2}\int_0^{\delta_n} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi < C \qquad (4.21)
\end{aligned}$$

for n_0 large enough. In both estimations we use the fact that

$$\int_0^{\delta_n} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi \to 0 \quad \text{as } n \to \infty\,.$$

Obviously, the conjunction of inequalities (4.20) and (4.21) contradicts (4.17) and thus shows that $\mu_n \ge \delta$; all approximate solutions $\{\theta_n(x); n \ge n_0\}$ exist in the common interval $[0, \delta]$. With this information in hand, we finish the proof by an application of the Arzelà - Ascoli theorem to the sequence $\{\theta_n(x)\}$. Uniform boundedness has been shown, thus we prove that the sequence is equi(uniformly)continuous. It follows from the differential equation (4.8) for $\rho_n(x) = \theta_n(x)$ in $\delta_n \le x \le \delta$, the estimation (4.19) and the fact that $\theta''_n(x) = 0$ in $0 \le x \le \delta_n$ that, if $0 \le x < \tilde{x} \le \delta$ we have

$$\begin{aligned}
|\theta'_n(\tilde{x}) - \theta'_n(x)| &\le \Big|\int_x^{\tilde{x}} \theta''_n(\xi) d\xi\Big| \\
&\le \int_x^{\tilde{x}} |\theta''_n(\xi)| d\xi \le \frac{(C+1)^2}{2} \int_x^{\tilde{x}} \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 d\xi
\end{aligned}$$

independently of n, thus the required equicontinuity property follows from equicontinuity of the Lebesgue integral [9] p. 148. Arzelà - Ascoli implies that (if necessary passing to a subsequence) we may assume $\{\theta'_n(x)\}$ uniformly convergent to a function $\theta^1(x)$ continuous in $0 \le x \le \delta$ satisfying $\theta^1(0) = 1$. On the other hand, since

$$\theta_n(x) = \int_0^x \theta'_n(\xi) d\xi$$

$\{\theta_n(x)\}$ is uniformly convergent to a function $\theta(x)$ continuous in $0 \le x \le \delta$ and satisfying $\theta(0) = 0$. Taking limits,

$$\theta(x) = \int_0^x \theta^1(\xi)d\xi$$

so that $\theta^1(x) = \theta'(x)$. It only remains to show that $\theta(x)$ is the claimed solution; to do this, we take limits in the integral equations (4.13)-(4.14) for $x > 0$. The limit in the second integral of each line is taken using equicontinuity of the integral. The limit in the first integral of each line uses (4.19)and the dominated convergence theorem. This ends the proof.

Remark 4.4 In view of (4.16), the "best C" (that is, the one yielding the largest δ) is the one that maximizes the right side of (4.16). The maximum of $C/(C+1)^3$ is attained at $C = 1/2$, thus inequality (4.16) becomes

$$\int_0^\delta \Big(\frac{\bar{y}(\xi)}{\xi}\Big)^2 \le \frac{4}{27} \min\Big(1, \frac{1}{\delta}\Big) . \tag{4.22}$$

5 Target and costate, II

Going in a direction opposite to that Section 3 we construct an unknown costate from the solution $\omega(x)$ of the differential equation (3.4). We look first at initial conditions (3.5) thus the pertinent initial value problem is (4.4). Let $\theta(x)$ be a solution of (4.4). The equation implies that $\theta'(x) > 0$ near zero, thus $\theta''(x)$ near zero as well, and $\theta(x)$ is increasing and convex near zero. This makes the derivative increasing, thus $\theta(x)$ is increasing and convex in its entire interval of definition $[0, \mu)$ and

$$\theta(x) \ge x\,, \quad \theta'(x) \ge 1 \quad (0 \le x < \mu)\,. \tag{5.1}$$

We obtain a solution of (3.4)-(3.5) setting $\omega(x) = \lambda\theta(x)$, $\theta(x)$ the solution of (4.4). We define

$$\kappa(x) = \frac{1}{\omega'(x)} = \frac{1}{\lambda\theta'(x)} \tag{5.2}$$

(λ to be fixed later). The costate $z(x)$ under construction must satisfy

$$z(x)^2 = -(\kappa(x)^2)'\,, \tag{5.3}$$

an equation that can only make sense if

$$(\kappa(x)^2)' \le 0\,.$$

Using the differential equation (3.4) we have

$$
\begin{aligned}
(\kappa(x)^2)' = \Big(\frac{1}{\omega'(x)^2}\Big)' &= -\frac{2\omega'(x)}{\omega'(x)^4}\omega''(x) \\
= -\frac{2\omega'(x)}{\omega'(x)^4} \cdot \frac{\bar{y}(x)^2\omega'(x)^3}{2\omega(x)^2} &= -\frac{\bar{y}(x)^2}{\omega(x)^2}
\end{aligned}
$$

so that $z(x)$ can be defined by (5.3) and satisfies

$$
z(x) = \frac{\bar{y}(x)}{\omega(x)} \,. \tag{5.4}
$$

It follows from (5.2) that $\kappa(0) = 1/\lambda$, thus $z(\cdot) \in L^2(0, \infty)$ and it only remains to determine λ. We obtain from (3.1) that

$$
\omega(T) = \lambda\theta(T) = 1 \tag{5.5}
$$

if $T < \mu =$ blowup point of $(\theta(x), \theta'(x))$. The second initial condition (3.5) is

$$
\frac{1}{\lambda^2\theta'(0)^2} = \frac{1}{\omega'(0)^2} = \kappa(0)^2 = \|z(\cdot)\|^2 = \int_0^\infty z(x)^2 dx \,. \tag{5.6}
$$

Since $z(x) = \bar{y}(x)$ for $x \geq T$ (consequence of (3.1)) we have

$$
\begin{aligned}
\frac{1}{\omega'(0)^2} &= \kappa(0)^2 = \int_0^T z(x)^2 dx + \int_T^\infty z(x)^2 dx \\
&= \kappa(0)^2 - \kappa(T)^2 + \kappa(T)^2 \\
&= \frac{1}{\omega'(0)^2} - \frac{1}{\omega'(T)^2} + \int_T^\infty \bar{y}(x)^2 dx \,,
\end{aligned} \tag{5.7}
$$

so that (5.6) becomes

$$
\int_T^\infty \bar{y}(x)^2 dx = \frac{1}{\omega'(T)^2} = \frac{1}{\lambda^2\theta'(T)^2} \,. \tag{5.8}
$$

Replacing $\lambda = 1/\theta(T)$ from (5.5) we obtain

$$
F(T) = \int_T^\infty \bar{y}(x)^2 dx = \frac{\theta(T)^2}{\theta'(T)^2} \tag{5.9}
$$

which is an equation for the optimal time T alone. Once T has been determined the costate is given by

$$
z(x) = \begin{cases} \dfrac{\theta(T)}{\theta(x)}\bar{y}(x) & (0 \leq x \leq T) \\[2ex] \bar{y}(x) & (x \geq T) \,. \end{cases} \tag{5.10}
$$

If a solution T of (5.9) can be found then we have a costate that drives 0 to $\bar{y}$ with a control of the form (2.9) thus this control is time optimal by the sufficiency part of Theorem 1.1.

For boundary conditions (3.6) and condition (3.7) the pertinent initial value problem is

$$\theta''(x) = \frac{\bar{y}(x)^2\theta'(x)^3}{2\theta(x)^2}\,, \quad \theta(0) = 0\,, \quad \theta'(0) = 0\,, \tag{5.11}$$

with the added condition

$$\int_0^\epsilon \frac{dx}{\theta'(x)} < \infty \tag{5.12}$$

for some $\epsilon > 0$. Assume a solution $\theta(x)$ of (5.11)-(5.12) exists. This means in particular that $\theta(x) \neq 0$ for $x > 0$ thus (if necessary multiplying by -1) we may assume that $\theta(x) > 0$ for $x > 0$. If δ is arbitrarily small we must obviously have $x_0 \in [0, \delta]$ such that $\omega'(x_0) > 0$. By translation invariance $\theta(x)$ satisfies the initial value problem

$$\theta''(x) = \frac{\bar{y}(x)^2\theta'(x)^3}{2\theta(x)^2}\,, \quad \theta(x_0) > 0\,, \quad \theta'(x_0) > 0\,,$$

and we show as in the proof of Theorem 4.3 that $\theta(x)$ is increasing and convex in $x \geq x_0$ with

$$\theta(x) \geq x - x_0 \quad \theta'(x) \geq 1\,.$$

Since δ is arbitrary, $\theta(x)$ is increasing and convex for $x > 0$. Thus, the costate $z(x)$ can be constructed in $x > 0$ as before. As to its behavior near zero, (5.2) and (5.12) guarantee that $z \in Z(T)$. To determine the optimal time we use

$$\frac{1}{\omega'(\epsilon)^2} = \frac{1}{\omega'(\epsilon)^2} - \frac{1}{\omega'(T)^2} + \int_T^\infty \bar{y}(x)^2 dx$$

instead of (5.7). We get the same equation (5.9) for the optimal time. Irrespective of boundary conditions we have

$$\left(\frac{\theta(x)}{\theta'(x)}\right)' = 1 - \frac{\theta(x)}{\theta'(x)^2}\theta''(x) = 1 - \frac{\theta(x)}{\theta'(x)^2} \cdot \frac{\bar{y}(x)^2\theta'(x)^3}{2\theta(x)^2} = 1 - \bar{y}(x)^2\frac{\theta'(x)}{\theta(x)}$$

so that $\varphi(x) = \theta(x)/\theta'(x)$ satisfies the differential equation

$$\varphi'(x) = 1 - \frac{\bar{y}(x)^2}{\varphi(x)} \tag{5.13}$$

which, curiously enough, depends only on the target $\bar{y}(x)$ but not at all on the solution $\theta(x)$. Clearly, $\varphi(x)$ is defined in the maximal interval of existence $(0, \mu)$ of $\theta(x)$. For initial conditions (4.4) we have $\varphi(0) = 0$.

6 Small targets

Lemma 6.1 *If $\bar{y}(\cdot) \in \mathcal{R}^2$ satisfies*

$$G(T) = \int_0^T \Big(\frac{\bar{y}(x)}{x}\Big)^2 dx + \frac{1}{T^2}\int_T^\infty \bar{y}(x)^2 dx \leq r^2 \tag{6.1}$$

then $\bar{y}(x)$ is r-reachable in time T. If $r = 1$ (resp. $r < 1$) the optimal driving time T_o from 0 to $\bar{y}$ satisfies

$$T_o \leq T \quad (\textit{resp. } T_o < T)\,. \tag{6.2}$$

Proof. We define a control $u(t,x)$ by $u(t,x) = S(T-\sigma)^* z(x)$ with

$$z(x) = \begin{cases} \dfrac{\bar{y}(x)}{x} & (0 < x \leq T) \\[2ex] \dfrac{\bar{y}(x)}{T} & (x > T) \end{cases} \tag{6.3}$$

Using (2.3) we have $S(T-\sigma)S(T-\sigma)^* z(x) = \chi(T-\sigma, x) z(x)$, thus by (2.6)

$$y(T,x,0,u) = z(x)\int_0^{\min(x,T)} d\sigma = \bar{y}(x)\,.$$

On the other hand,

$$\begin{aligned}\int_0^\infty u(t,x)^2 dx &= \int_t^\infty u(0,x)^2 dx = \int_t^\infty z(x)^2 dx \\ &\leq \int_0^T \Big(\frac{\bar{y}(x)}{x}\Big)^2 dx + \frac{1}{T^2}\int_T^\infty \bar{y}(x)^2 dx \leq r^2\,,\end{aligned}$$

so that $\bar{y}(x)$ is r-reachable in time T and we have the first inequality (6.2). If $r < 1$ the control (6.3) satisfies $\|u(t,\cdot)\|_{L^2(0,T;E)} < 1$, thus is not time optimal [1], [7] Theorem 2.1.3, and the second inequality (6.2) holds as well. This ends the proof.

Remark 6.2 If $G(T)$ is the function (6.1) we have

$$G'(T) = \Big(\frac{\bar{y}(T)}{T}\Big)^2 - \frac{\bar{y}(T)^2}{T^2} - \frac{2}{T^3}\int_T^\infty \bar{y}(x)^2 dx = -\frac{2}{T^3}\int_T^\infty \bar{y}(x)^2 dx \leq 0\,,$$

thus $G(T)$ is decreasing. Accordingly, $G(T) < r$ for T large enough if and only if

$$\int_0^\infty \Big(\frac{y(x)}{x}\Big)^2 dx < r\,. \tag{6.4}$$

Theorem 6.3 *Let $a > 0$. Assume $\bar{y}(x) \in L^2(0,\infty)$ has support in $[0,a]$. Then there exists $\epsilon > 0$ (depending on a) such that if*

$$\int_0^a \left(\frac{\bar{y}(x)}{x}\right)^2 dx \le \epsilon \tag{6.5}$$

then the (unique) control $\bar{u}(t,x)$ driving time optimally from 0 to $\bar{y}$ has a multiplier in $L^2(0,\infty)$, that is, it satisfies (2.9) with $z(\cdot) \in L^2(0,\infty)$.

Proof. We may assume that $\epsilon < 1$, so that it follows from Lemma 6.1 that we can drive time optimally from 0 to $\bar{y}$ in some time $T_o < a$. We construct the multiplier directly. We take $\delta = a$ in Theorem 4.3, ϵ so small that

$$\epsilon \le \frac{4}{27} \min\left(1, \frac{1}{\delta}\right) \tag{6.6}$$

and apply the combo Theorem 4.3 - Remark 4.4 to show that a solution $\theta(x)$ of (4.4) exists in $0 \le x \le a > T_o$. The function $\theta(x)/\theta'(x)$ is 0 for $x = 0$ and positive in $0 \le x < a$. The left side $F(T)$ of equation (5.9) is positive for $T = 0$ and zero for $T = a$, thus there exists a solution $T_o < a$ of the equation and we have the optimal time and the multiplier, given by (5.10). Since $\omega(x) = x(1 + o(1))$ it follows from (5.10) that

$$z(x) = \frac{\bar{y}(x)}{\omega(x)} = \frac{\bar{y}(x)}{x} \cdot \frac{x}{x(1 + o(1))}$$

near zero, thus $z(\cdot) \in L^2(0,\infty)$. This ends the proof.

7 The initial value problem: uniqueness

Theorem 7.1 *Assume $y(\cdot) \in \mathcal{R}^2$. Then there is no solution of (5.11) satisfying (5.12).*

Proof. Let $\theta(x) > 0$ be a solution of (5.11) with maximal interval of existence $[0,\mu)$. Choose a with $0 < a < \mu$ and define

$$\bar{y}_0(x) = \begin{cases} \bar{y}(x) & 0 < x \le b \\ 0 & x > b \end{cases} \tag{7.1}$$

with $b < a$ such that

$$\int_0^b \left(\frac{y(x)}{x}\right)^2 dx \le \epsilon\,, \tag{7.2}$$

ϵ the parameter in Lemma 6.1 corresponding to the interval $[0,a]$. If we define

$$\theta_0(x) = \begin{cases} \theta(x) & (0 \le x \le b) \\ \theta'(b)(x - b) + \theta(b) & (x > b) \end{cases} \tag{7.3}$$

then $\theta_0(x)$ solves

$$\theta_0''(x) = \frac{\bar{y}_0(x)^2\theta_0'(x)^3}{2\theta_0(x)^2}, \quad \theta_0(0) = 0, \quad \theta_0'(0) = 0 \tag{7.4}$$

with interval of existence $[0, \infty)$. We show that, setting $\varphi(x) = \theta(x)/\theta'(x)$,

$$\lim_{n\to\infty} \varphi(x_n) = 0 \tag{7.5}$$

for a positive sequence $\{x_n\}$ with $x_n \to 0$. By Cauchy-Schwarz and (5.12)

$$\begin{aligned}\int_0^x \sqrt{\varphi(x)}dx &= \int_0^x \sqrt{\theta(x)}\frac{dx}{\sqrt{\theta'(x)}} \\ &\leq \sqrt{\int_0^x \theta(x)dx}\sqrt{\int_0^x \frac{dx}{\theta'(x)}} \leq C\sqrt{\int_0^x \theta(x)dx}\,.\end{aligned} \tag{7.6}$$

Now, $\theta(x) = \theta'(\xi)x$ by the mean value theorem, thus $\theta = xo(1)$ and it follows from (7.6) that

$$\int_0^x \sqrt{\varphi(x)}dx = xo(1)\,. \tag{7.7}$$

If (7.5) is not true then $\varphi(x) \geq \alpha > 0$ near zero, which implies

$$\int_0^x \sqrt{\varphi(x)}dx \geq \sqrt{\alpha}x$$

in contradiction with (7.7), thus (7.5) is proved. Since $\varphi(b) > 0$ and $F(0) > 0$, $F(b) = 0$ equation (5.9) or the optimal time has a solution $T_o < b$ and, using Section 5 we construct a costate $\tilde{z}$ whose associated control drives from 0 to $\bar{y}$ in optimal time T_o. In view of condition (5.12) $\tilde{z} \in Z(T)$; on the other hand, the second initial condition in (5.11) says that $\kappa(0) = \infty$, thus $\tilde{z} \notin L^2(0,\infty)$.

Next, we use Theorem 7.1 to construct a costate $z \in L^2(0, \infty)$ such that its associated control drives 0 to $\bar{y}$ optimally, thus we have equality (1.9) for z and $\tilde{z}$ and it follows from Theorem 1.3 that $z_0 = z$, a contradiction. This ends the proof.

Theorem 7.2 *Assume $y(\cdot) \in \mathcal{R}^2$. Then the solution of the initial value problem* (4.4) *is unique.*

Proof. The argument in Theorem 7.1 works with minor modifications. Given a solution $\theta(x)$ of (4.4) we select a and b in such a way that (7.1)-(7.2) hold, define $\theta_0(x)$ by (7.3) and obtain a solution $\theta_0(x)$ of

$$\theta_0''(x) = \frac{\bar{y}_0(x)^2\theta_0'(x)^3}{2\theta_0(x)^2}, \quad \theta_0(0) = 0, \quad \theta_0'(0) = 1 \tag{7.8}$$

in $[0, \infty)$. Then we proceed to construct the costate z from Section 5; again, $\theta(b)/\theta'(b) > 0$ and equation (5.9) has a solution, thus the control associated with z drives 0 yo $\bar{y}$ in optimal time $T_o < b$. In case (4.4) has a second solution $\tilde{\theta}(x)$ we construct in the same way a second costate $\tilde{z}$ driving 0 to $\bar{y}$ in optimal time. At this point we have (1.9), and Theorem 1.3 says that $z, \tilde{z}$ differ only by multiplication by a nonzero constant. Using (5.4) the same is true of $\omega(x), \tilde{\omega}(x)$ thus of $\theta(x), \tilde{\theta}(x)$. In view of the second initial condition in (4.4) this constant has to be $= 1$, and we are all done.

Proof of Theorem 1.4. If $\bar{y}$ is reachable then the time optimal control (1.13) driving 0 to $\bar{y}$ cannot have an associated costate in $Z(T) \setminus L^2(0, \infty)$, since the $\omega(x)$ given by (3.2) would be one of the forbidden solutions of (4.4).

Proof of Theorem 1.5. The norm optimal problem is *homogeneous;* this means if ρ is the optimal norm driving from 0 to $\bar{y}$ then the optimal norm for driving to $\bar{y}/\rho$ is 1. Since norm optimality and time optimality are equivalent for targets $\bar{y} \in D(A)$ we use the time optimal theory.

Example 7.3 Let

$$z(x) = \frac{e^{1/2x}}{x} \,. \tag{7.9}$$

We have

$$\kappa(x)^2 = \int_x^\infty \frac{e^{1/\xi}}{\xi^2} dx = -\int_x^\infty \big(e^{1/\xi}\big)' d\xi = e^{1/x} - 1 \,,$$

so that

$$\kappa(x) = \sqrt{e^{1/x} - 1} = e^{1/2x}\sqrt{1 - e^{-1/x}}$$

and

$$\theta'(x) = \frac{e^{-1/2x}}{\sqrt{1 - e^{-1/x}}} \implies \theta(x) = \int_0^x \frac{e^{-1/2\xi}}{\sqrt{1 - e^{-1/\xi}}}$$

We have $\omega^{(n)}(0) = 0 \quad (n = 0, 1, 2, 3, \dots)$. It is proved in [8] Section 5 that $\theta(x)$ solves (5.11) with $\bar{y}(\cdot) \in D(A)$. However, (5.12) is not satisfied, thus Theorem 7.1 doesn't apply.

Example 7.4 Let

$$\theta(x) = x^\beta \quad (\beta > 1) \,. \tag{7.10}$$

Then

$$\theta'(x) = \beta x^{\beta - 1} \,, \quad \theta''(x) = \beta(\beta - 1) x^{\beta - 2} \,,$$

so that

$$\frac{\theta'(x)^3}{2\theta(x)^2} = \frac{\beta^3 x^{3\beta - 3}}{2x^{2\beta}} = \frac{\beta^3}{2} x^{\beta - 3} = \beta(\beta - 1)x^{\beta - 2} \cdot \frac{\beta^3}{2\beta(\beta - 1)x} = \frac{\theta''(x)}{\bar{y}(x)^2} \,,$$

thus $\theta(x)$ satisfies (5.11) with

$$\bar{y}(x) = \sqrt{\frac{2\beta(\beta - 1)}{\beta^3}} \sqrt{x} \tag{7.11}$$

which does not belong to $\mathcal{R}^2$, thus we are outside of the scope of Theorem 7.1. The function $\theta(x)$ satisfies both initial conditions (5.11) and (5.12) holds if $\beta < 2$. We have

$$\kappa(x) = \frac{1}{\beta x^{\beta-1}}$$

so that $z(\cdot) \in \mathcal{Z}$ if β satisfies the same condition $\beta < 2$. The costate is

$$z(x) = -(\kappa(x)^2)' = -\Big(\frac{1}{\beta^2 x^{2\beta-2}}\Big)' = \frac{2\beta-2}{\beta^2 x^{2\beta-3}}\,. \tag{7.12}$$

8 Numerics

Example 8.1 We reconstruct a (known) costate from its target. Let

$$z(x) = \frac{1}{1+x}$$

so that

$$\kappa(x) = \sqrt{\int_x^\infty \frac{dx}{(1+x)^2}} = \frac{1}{\sqrt{1+x}} \qquad \Longleftrightarrow \qquad \theta'(x) = \sqrt{1+x}$$

and

$$\theta(x) = \begin{cases} \dfrac{2}{3}((1+x)^{3/2}-1) & (0 \le x \le T) \\ \\ \dfrac{2}{3}((1+T)^{3/2}-1) & (x \ge T) \end{cases}$$

where T is the (optimal) driving time. The target is

$$\bar{y}(x) = \begin{cases} \dfrac{2}{3}\dfrac{((1+x)^{3/2}-1)}{1+x} & (0 \le x \le T) \\ \\ \dfrac{2}{3}\dfrac{((1+T)^{3/2}-1)}{1+T} & (x \ge T) \end{cases}$$

We take $T = 1$ and obtain the costate from the target solving (4.4) and applying the methods of Section 5. Since *MATHEMATICA's* `NDSolve` will not solve (for good reason!) with initial condition $\theta(0) = 0$ we take $\theta(0) = 10^{-8}$, $\theta'(0) = 1$ in the interval $0 \le x \le 1$. If $\eta(x)$ is the solution and we set

$$\theta(x) = \begin{cases} x & (0 \le x \le 10^{-8}) \\ \eta(x - 10^{-8}) & (10^{-8} \le x \le 1) \end{cases} \tag{8.1}$$

we obtain one of the approximations used in Section 4. Since 10^{-8} is very small, we don't bother to make the shift in (8.1) and set $\theta(x) = \rho(x)$. We have

$\kappa(0) = 1$, thus $\theta(x) = \omega(x)$. The graph of both sides of the equation (5.9) is in Figure 3. As expected, the curves join at $T = 1$; in fact, the evaluations coincide within the decimals shown,

$$F(1) = 0.742921\,, \qquad \frac{\theta(1)^2}{\theta'(1)^2} = 0.742921\,.$$

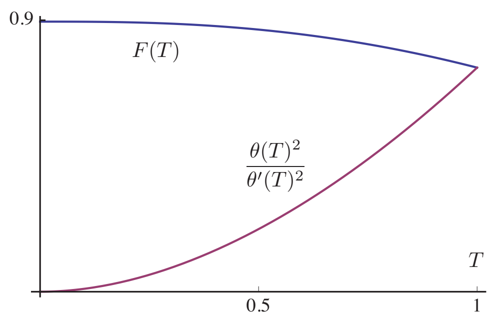

Figure 3

The graphs of the computed $\theta(x), \theta'(x)$ visually coincide with their exact values. So does the graph of the computed costate $\tilde{z}(x)$ and the original costate $z(x) = 1/(1+x)$. Figure 4 (the plot of $z(x) - \tilde{z}(x)$) shows some noise in the range $[-10^{-6}, 10^{-6}]$; as expected, the discrepancy is larger near the origin.

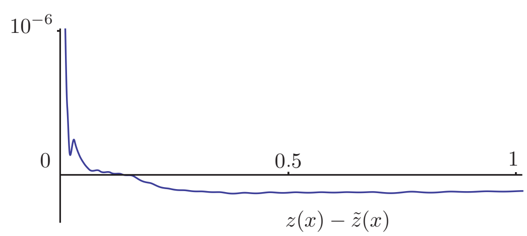

Figure 4

Example 8.2 We compute the time optimal control driving 0 to

$$\bar{y}(x) = \frac{x}{1+x^2}$$

We have

$$\int_0^\infty \left(\frac{\bar{y}(x)}{x}\right)^2 dx = \frac{\pi}{4} = 0.785398$$

thus Lemma 6.1 and Corollary 6.2 guarantee that $\bar{y}$ is 1-reachable. The function $G(T)$ in (6.1) evaluates to 0.955632 for $T = 1.9$ thus we have a bound (not too precise, as we shall see) for the optimal time. We solve (4.4) again with initial conditions $\theta(0) = 10^{-8}$, $\theta'(0) = 1$ in the interval $0 \le x \le 2$. The first graph in Figure 5 is that of $\theta(x)$ together with x, the second that of $\theta'(x)$ with 1 so that both inequalities (5.1) can be seen in action.

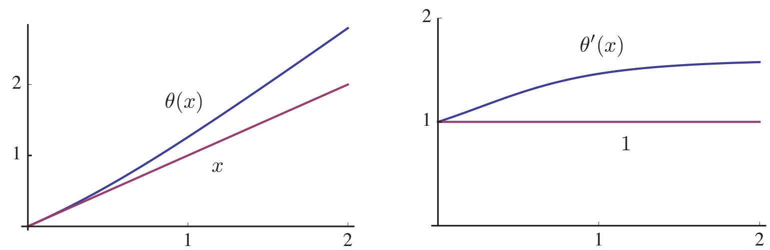

Figure 5

Figure 6 shows the graph of both sides of (5.9); the optimal time (calculated with *MATHEMATICA's* `FindRoot`) is

$$T_o = 0.942178\,.$$

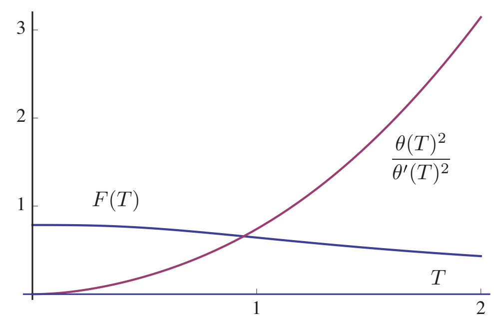

Figure 6

Figure 7 shows the costate $z(x)$ from formula (5.10)

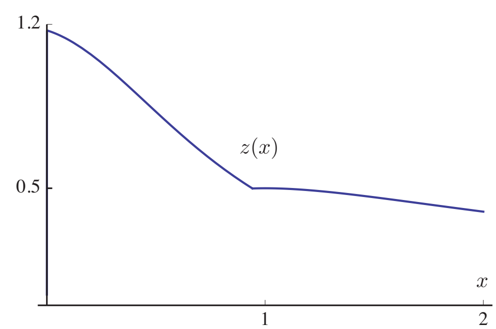

Figure 7

Example 8.3 We compute the optimal control driving 0 to

$$\bar{y}(x) = \frac{1.69\,x}{1 + x^2}\,.$$

The integral (6.4) evaluates to $1.69\pi/4 = 2.24318$ thus Lemma 6.1 and Remark 6.2 don't help here. We solve (4.4) in the interval $0 \le x \le 5$. Figure 8 shows the graph of both sides of Equation (5.9); the optimal time is

$$T_o = 2.76692$$

We omit the rest of the graphs since they are (qualitatively) similar to those in Example 8.2.

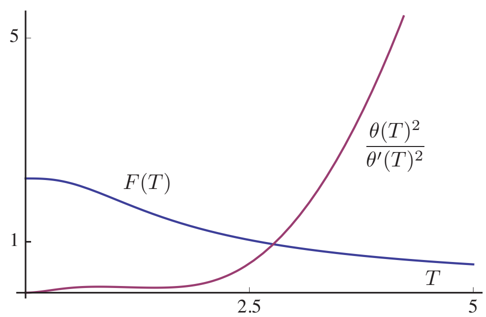

Figure 8

Example 8.4 Same as Example 8.3, this time with target

$$\bar{y}(x) = \frac{1.695\,x}{1 + x^2}\,.$$

`NDSolve` indicates that $\theta(x)$ blows up at $\mu = 1.67443$. Figure 9 shows both sides of Equation (5.9). The lack of intersection signals that the target $\bar{y}(x)$ is not 1-reachable in any time $T > 0$.

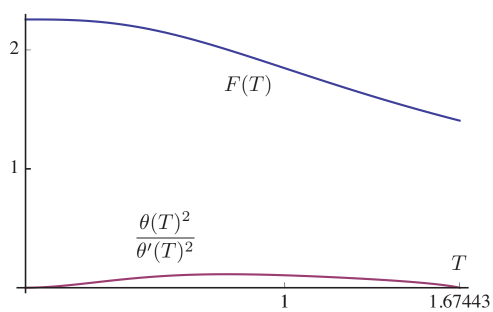

Figure 9

The *MATHEMATICA 7* notebook with the computations for Examples 8.1 to 8.4 can be downloaded at `http://www.math.ucla.edu/~hof/strong.nb`.

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