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SPLICING OF TIME OPTIMAL CONTROLS

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ABSTRACT. Two controls $u_1(t)$ and $u_2(t)$ are *spliced* (in symbols $(u_1 \cup u_2)(t)$) if $u_2(t)$ is applied after $u_1(t)$; both are defined in finite intervals. We seek conditions under which time optimality of u_1, u_2 imply time optimality of $u_1 \cup u_2$, and extend the results to a finite or infinite number of controls. The theorems (or rather examples, since they are restricted to the right translation semigroup in $L^2(0, \infty)$) are used to construct time optimal controls that do not satisfy Pontryagin's maximum principle.

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1. PRELIMINARIES

The system is

$$y'(t) = Ay(t) + u(t), \quad y(0) = \zeta \tag{1.1}$$

with controls $u(\cdot) \in L^\infty(0, T; E)$. The operator A generates a strongly continuous semigroup $S(t)$ in a Banach space E, called the *state space* of (1.1). The *solution* of (1.1) is the continuous function

$$y(t) = y(t, \zeta, u) = S(t)\zeta + \int_0^t S(t - \sigma)u(\sigma)d\sigma\,. \tag{1.2}$$

A control is *admissible* if

$$\|u(\cdot)\|_{L^\infty(0,T;E)} \le 1\,. \tag{1.3}$$

The control is *time optimal* if it drives the initial point ζ to a point target

$$y(T, \zeta, u) = \bar{y} \tag{1.4}$$

in optimal time T, that is, no other admissible control does the drive in less time. Under restrictions on $\zeta, \bar{y}$, necessary and sufficient conditions for optimality can be given in terms of Pontryagin's maximum principle (1.6) below. The principle requires multipliers [4], [6] Section 2.3 whose construction we outline. When A has a bounded inverse, E^*_{-1} is the completion of the dual E^* in the norm

$$\|y^*\|_{E^*_{-1}} = \|(A^{-1})^*y^*\|_{E^*}\,.$$

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The adjoint semigroup $S(t)^*$ can be extended to an operator $S(t)^* : E^*_{-1} \to E^*_{-1}$. If $S(t)^*$ is strongly continuous, $Z(T) \subseteq E^*_{-1}$ consists of all $z \in E^*_{-1}$ such that $S(t)^* z \in E^*$ $(t > 0)$ and

$$\|z\|_{Z(T)} = \int_0^T \|S(t)^* z\|_{E^*} dt < \infty \,. \tag{1.5}$$

The norm (1.5) makes $Z(T)$ a Banach space. All $Z(T)$ coincide and all norms $\|\cdot\|_{Z(T)}$ are equivalent for $T > 0$. $Z(T)$ is an example of a *multiplier space,* a linear space $\mathcal{Z} \supseteq E^*$ to which $S(t)^*$ can be extended in such a way that $S(t)^* \mathcal{Z} \subseteq E^*$ for $t > 0$ and that the semigroup equation $S(s+t)^* \zeta = S(s)^* S(t)^* \zeta$ is valid for $\zeta \in \mathcal{Z}$ and $s, t > 0$. When A does not have a bounded inverse, E^*_{-1} is the completion of E^* in any of the equivalent norms

$$\|y^*\|_{E^*_{-1},\lambda} = \|((\lambda I - A)^{-1})^* y^*\|_{E^*}$$

(λ large enough). The definition of $Z(T)$ (and of multiplier spaces) doesn't change.

An admissible control $\bar{u}(\cdot) \in L^\infty(0, T; E)$ satisfies *Pontryagin's maximum principle* if

$$\langle S(T-t)^* \eta, \bar{u}(t) \rangle = \max_{\|u\| \le 1} \langle S(T-t)^* \eta, u \rangle \quad \text{a. e. in } 0 \le t < T \,, \tag{1.6}$$

$\langle \cdot\,, \cdot \rangle$ the duality of the spaces E and E^*, η in some multiplier space $\mathcal{Z}$. We call η the *multiplier* and $z(t) = S(T-t)^* \eta \in E^*$ the *costate corresponding to* (or *associated with*) the control $\bar{u}(t)$. We assume that (1.6) is nontrivial, that is, that $S(T-t)^* z$ is not identically zero in the interval $0 \le t < T$, although it may be zero in part of the interval (where (1.6) gives no information on $\bar{u}(t)$). When E is a Hilbert space (1.6) reduces to

$$\bar{u}(t) = \frac{S(T-t)^* \eta}{\|S(T-t)^* \eta\|} \qquad (T - \delta < t \le T) \,, \tag{1.7}$$

where $(T - \delta, T]$ is the maximal interval where $S(T-t)^* \eta \ne 0$; if $\delta = T$ the interval is $0 < t \le T$. We say that $\bar{u}(t)$ is *regular* if it satisfies (1.6) with $\eta = z \in Z(T)$.

Theorem 1.1. *Assume $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ time optimally in the interval $0 \le t \le T$ and that*

$$\bar{y} - S(T)\zeta \in D(A) \,. \tag{1.8}$$

Then $\bar{u}(t)$ is regular.

Theorem 1.2. *Let $\bar{u}(t)$ be a regular control with*

(*a*) $\bar{y} = y(T, \zeta, \bar{u}) \in D(A) \,, \quad \|A\bar{y}\| < 1$, *or*

(*b*) $\zeta \in D(A) \,, \quad \|A\zeta\| < 1 \,, \quad S(T-t)^* z \ne 0 \quad (0 < t \le T)$.

Then $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T, \zeta, \bar{u})$ time optimally in the interval $0 \le t \le T$.

Theorem 1.1 is [6] Theorem 2.5.1. It does *not* guarantee that $S(t)^*z \neq 0$ in the whole interval (see Remark 4.4). Theorem 1.2 is [6] Theorem 2.5.7. We only use it here when $\zeta = 0$ (which fits into (b), with no assumptions on $\bar{y}$).

We say that $\bar{y} \in E$ is *reachable* if there exists an admissible control driving 0 to $\bar{y}$.

Theorem 1.3. (existence of optimal controls). *Let E be a Hilbert space. If $\bar{y}$ is reachable then there exists an optimal control, that is, an admissible control $\bar{u}(\cdot)$ driving 0 to $\bar{y}$ in optimal time.*

Although sufficient for this paper, the result is true in spaces much more general than Hilbert; see [6] Section 3.1. The following result is [6] Theorem 2.1.3 and is totally independent of the maximum principle (1.6).

Theorem 1.4. *Let $\bar{u}(\cdot)$ be time optimal.*[1] *Then*

$$\|\bar{u}(t)\| = 1 \,. \tag{1.9}$$

Theorem 1.4 implies uniqueness of optimal controls in a Hilbert space. In fact, if two controls $\bar{u}_1(t)$ and $\bar{u}_2(t)$ drive 0 to $\bar{y}$ in optimal time T so does $\bar{u}(t) = (\bar{u}_1(t) + \bar{u}_2(t))/2$; since (1.9) is mandatory, $\bar{u}_1(t) = \bar{u}_2(t)$. The result is valid in strictly convex spaces; see [6] Theorem 3.1.4. We don't know whether strict convexity is necessary for uniqueness, although nonuniqueness examples in spaces not strictly convex exist (see [6], Sections 1.2 and 5.5).

Theorem 1.1 says nothing about time optimal controls driving ζ to $\bar{y}$ *not* satisfying (1.8), in particular about controls driving 0 to $\bar{y} \notin D(A)$. In fact, the main purpose of this paper is to construct time optimal controls that do *not* satisfy the maximum principle (1.6). For these, $y(T, 0, \bar{u}) \notin D(A)$.

2. SPLICING

Let $u_1(\cdot) \in L^\infty(0, T_1; E)$, $u_2(\cdot) \in L^\infty(0, T_2; E)$. The *splice* $u_1 \cup u_2$ of u_1 and u_2 is

$$u(t) = (u_1 \cup u_2)(t) = \begin{cases} u_1(t) & (0 \le t < T_1) \\ u_2(t - T_1) & (T_1 < t \le T_1 + T_2) \end{cases} \tag{2.1}$$

and belongs to $L^\infty(0, T_1 + T_2; E)$. Splicing of n controls $u_j(\cdot) \in L^\infty(0, T_j; E)$ is defined directly, or inductively by

$$u = u_1 \cup u_2 \cup \cdots \cup u_n = (u_1 \cup u_2 \cup \cdots \cup u_{n-1}) \cup u_n \tag{2.2}$$

and $u(\cdot) \in L^\infty(0, T_1 + T_2 + \cdots + T_n; E)$. The definition extends to infinite sequences $\{u_n(\cdot) \in L^\infty(0, T_n; E);\ n = 1, 2, \dots\}$. If $\|u_n(\cdot)\|_{L^\infty(0,T_n;E)} \le C$ then the splice

$$u(t) = \left(\bigcup_{n=1}^{\infty} u_n\right)(t) \tag{2.3}$$

[1] Statements like (1.9) should be qualified by "almost everywhere". We don't bother with this.

belongs to $L^\infty(0,T;E)$, $T=\sum_{n=1}^\infty T_n$. We only use this case when $T<\infty$. Obviously, the finite or infinite splice of admissible controls is admissible. Splicing is associative but not commutative.

In general, splicing optimal controls does not preserve optimality; for the simplest one-dimensional system $y'(t)=u(t)$ the only time optimal controls in $[0,T]$ are $u_1(t)=1$ and $u_{-1}(t)=-1$ and no nontrivial splicing is optimal. One can only hope for examples where splicing preserves optimality, and the results are only for the control system

$$\begin{aligned}\frac{\partial y(t,x)}{\partial t} &= -\frac{\partial y(t,x)}{\partial x}+u(t,x)\,,\\ y(0,x)=\zeta(x)\,,&\quad y(t,0)=0\quad (0\le t,\,x<\infty)\,.\end{aligned}\tag{2.4}$$

This system can be written as (1.1) in the space $E=L^2(0,\infty)$ with

$$Ay(x)=-y'(x)\,,\tag{2.5}$$

domain $D(A)=\{\text{all } y(\cdot)\in L^2(0,\infty) \text{ with } y'(\cdot) \text{ in } L^2(0,\infty) \text{ and } y(0)=0\}$. This operator generates the (isometric) *right translation semigroup*

$$S(t)y(x)=\begin{cases} y(x-t) & (x\ge t)\\ 0 & (x<t)\,.\end{cases}\tag{2.6}$$

We show below that there exists classes $\mathcal{SP}(T)$ of optimal controls[2] such that if $u_2(\cdot)\in\mathcal{SP}(T_2)$ and $u_1(\cdot)$ is an arbitrary optimal control in $[0,T_1]$ then the splice $(u_1\cup u_2)(t)$ is optimal (Theorem 4.1). The result is easily extended to finite splicings (Theorem 4.2). Infinite splicings are not as simple; all we can show (Theorem 5.3) is that there exist inductively defined sequences of optimal controls $\{\bar u_j(\cdot)\}$ ($\bar u_1(\cdot)$ time optimal, $u_j(\cdot)\in\mathcal{SP}(T_j)$ for $j=2,3,\dots$) whose splice is time optimal. Since the results are restricted to the system (2.4) splicing seems little more than a curiosity. However, it can be used to produce time optimal controls not satisfying Pontryagin's maximum principle (1.6). The results are generalizations of the results in [5] and [6] Section 2.7 (where splicing was used in particular cases but not explicitly defined) and we are able to produce time optimal controls "rougher" than the ones therein.

Splicing (although not of optimal controls) is known in control theory; it has been especially used in controllability of systems in Euclidean spaces and Lie groups. For references see [6], Section 6.3.

[2]Each control $u(t)$ is defined in its own interval $[0,T]$, possibly different for different controls. That $u(t)$ is *optimal in* $[0,T]$ means it drives time optimally 0 to the target $\bar y=y(T,0,\bar u)$, that is, the target is not fixed beforehand. Optimality has the same meaning: no other admissible control $u(\cdot)$ can drive 0 to $y(T,0,\bar u)$ in less time. Sometimes we abbreviate "optimal in $[0,T]$" to simply "optimal".

3. THE RIGHT TRANSLATION SEMIGROUP

The adjoint of (2.6) is the *left translation* (and chop-off) semigroup

$$S(t)^*y(x) = \begin{cases} y(x+t) & (x \geq 0) \\ 0 & (x < 0)\,. \end{cases} \tag{3.1}$$

We have

$$S(t)^*S(t) = I\,, \quad S(t)S(t)^*y(x) = \chi_t(x)y(x)\,, \tag{3.2}$$

$\chi_t(x)$ the characteristic function of $[t, \infty)$. Formula (1.2) for the control $u(t)(x) = u(t, x)$ is

$$\begin{aligned} y(t, x, \zeta, u) = y(t, \zeta, u)(x) &= \left(S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)d\sigma\right)(x) \\ &= \zeta(x-t) + \int_0^t u(\sigma, x-(t-\sigma))d\sigma\,, \end{aligned} \tag{3.3}$$

thus the contribution of the control $u(\sigma, x)$ to $y(t, x, \zeta, u)$ is the integral of $u(\sigma, x)$ over the intersection of the positive quadrant $t, x \geq 0$ with the line $(\sigma, x-(t-\sigma))$ joining $(0, x-t)$ with (t, x) as shown in Figure 1.

The space $\mathcal{Z}$ of all multipliers consists of all measurable $z(x)$ defined in $x > 0$ and such that

$$\kappa(t, z) = \|S(t)^*z(\cdot)\| = \sqrt{\int_0^\infty z(x+t)^2dx} = \sqrt{\int_t^\infty z(x)^2dx} < \infty\,,$$

for $t > 0$, while $Z(T)$ consists of all $z(\cdot) \in \mathcal{Z}$ satisfying the integrability condition (1.5),

$$\int_0^T \|S(\sigma)^*z(\cdot)\|d\sigma = \int_0^T \kappa(\sigma, z)d\sigma < \infty\,.$$

Since we are in a Hilbert space, (1.7) applies, and any control that satisfies (1.6) is given by

$$\begin{aligned} \bar{u}(t, x) &= \frac{S(T-t)^*z(x)}{\|S(T-t)^*z(\cdot)\|} \\ &= \chi_0(x)\frac{z(x+(T-t))}{\kappa(T-t, z)} \quad (T-\delta < t \leq T)\,, \end{aligned} \tag{3.4}$$

where $(T-\delta, T]$ is the maximal interval where $S(T-t)^*z \neq 0$; if $\delta = T$, the interval in (3.4) is $0 < t \leq T$. Using the second equality (3.2)

$$\begin{aligned} S(T-\sigma)\bar{u}(\sigma, x) &= \frac{S(T-\sigma)S(T-\sigma)^*z(x)}{\|S(T-\sigma)^*z(\cdot)\|} \\ &= \frac{\chi_{T-\sigma}(x)z(x)}{\kappa(T-\sigma, z)} \quad (T-\delta \leq t < T)\,. \end{aligned} \tag{3.5}$$

Consequently, if $\kappa(T-\sigma) \neq 0$ in $0 < \sigma \leq T$, $y(T, x, 0, u)$ is the integral in $0 \leq \sigma \leq T$ of the right side of (3.5). This leads to a simple explicit formula

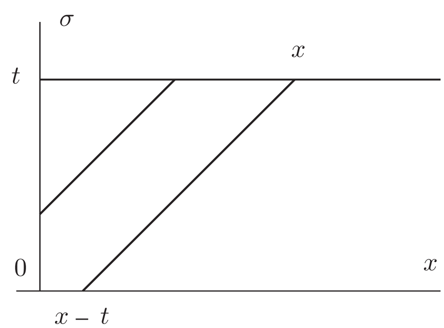

FIGURE 1. Integration lines in formula (3.3).

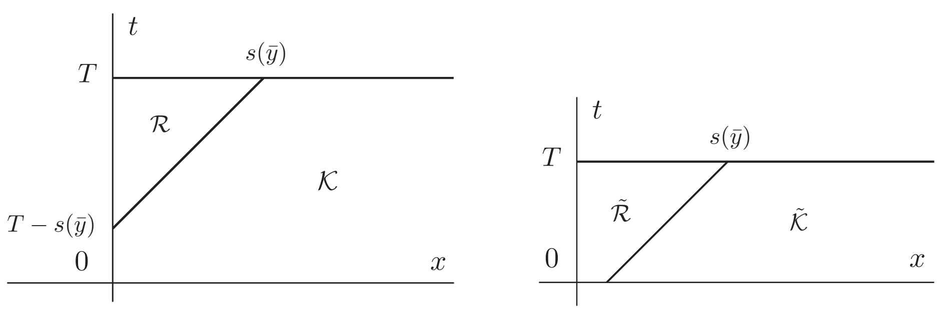

FIGURE 2. Optimal time and optimal control when $s(\bar{y}) < \infty$.

for $y(T, x, 0, u)$ in terms of $z(x)$ ([6] formula (2.6.25)) which we don't need to use here.

Let $\varphi(x)$ be measurable in $x > 0$. Define $s(\varphi(\cdot))$ as the minimum $s \geq 0$ such that support $\varphi(\cdot) \in [0, s]$ (support defined modulo null sets). If the support of $\varphi(\cdot)$ is not contained in any interval $[0, s]$ we set $s(\varphi(\cdot)) = \infty$.

Lemma 3.1. *Assume that $\bar{y}(\cdot) \in L^2(0, \infty)$ is reachable and $s(\bar{y}(\cdot)) < \infty$. Then (a) if T is the optimal driving time from 0 to $\bar{y}(x)$ we have $T \leq s(\bar{y})$, (b) if $\bar{u}(t, x)$ is the optimal control doing the drive, the support of $\bar{u}(t, x)$ is contained in the region $\tilde{\mathcal{R}}$ in Figure 2 right.*

Proof. Assume $T > s(\bar{y})$ so that we have the configuration in Figure 2 left. If the time optimal control $\bar{u}(t, x)$ is not zero in $\mathcal{K}$ we may modify its definition to $\bar{u}(t, x) = 0$ there. This doesn't affect the target and, if anything, improves each norm $\|\bar{u}(t, \cdot)\|_{L^2(0,\infty)}$. The modified control is admissible, thus time optimal; by uniqueness it is equal to $\bar{u}(t, x)$. If $T > s(\bar{y})$ then $\|\bar{u}(t, \cdot)\|_{L^2(0,\infty)} = 0$ in $0 \leq t \leq T - s(\bar{y})$. However, Theorem 1.4 says that a time optimal control must satisfy $\|u(t)\| = 1$, thus we have a contradiction. (b) If the optimal control $\bar{u}(t, x)$ is not zero in $\mathcal{K}$ we modify its definition to $\bar{u}(t, x) = 0$ there and argue as in (a). □

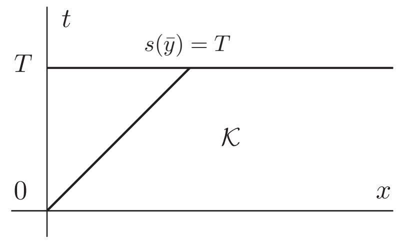

FIGURE 3. Controls in the class $\mathcal{SP}(T)$.

Lemma 3.2. *Assume $\bar y(x)$ is reachable with a control of the form* (3.4)*. Then*

$$s(z(\cdot)) = s(\bar y(\cdot))\,. \tag{3.6}$$

Proof. Equality (3.6) follows from the far more precise statement

$$z(x) = 0 \quad \Longleftrightarrow \quad \bar y(x) = 0\,. \tag{3.7}$$

To see that (3.7) holds we look at formula (3.5). The target $\bar y(x)$ is obtained integrating in $0 \le \sigma \le T$ the function on the right side, thus $z(x) = 0 \Rightarrow \bar y(x) = 0$. On the other hand, the function $\chi_{T-\sigma}(x)$ (as a function of t) is nonzero for $T - \sigma \le x$ thus $\bar y(x) = 0 \Rightarrow z(x) = 0$ and we are done. □

Lemma 3.1 and Lemma 3.2 have totally different scopes. In Lemma 3.1 $\bar y(x)$ is just reachable by an admissible control (thus by an optimal control in view of Theorem 1.3). In Lemma 3.2, $\bar y(x)$ is reachable by a control of the form (3.4), this control being automatically optimal by Theorem 1.2. The aim of this paper is to produce optimal controls *not* of the form (3.4).

4. FINITE SPLICING

A control $u(t,x)$ in $L^\infty(0,T;L^2(0,\infty))$ belongs to the class $\mathcal{SP}(T)$ if and only if it is admissible and drives 0 in optimal time T to a target $\bar y(x)$ with $s(\bar y(\cdot)) = T$. Examples of controls in $\mathcal{SP}(T)$ are easy to produce with the help of Theorem 1.2 and (3.4): if $z(\cdot) \in Z(T)$ with $s(z(\cdot)) = T$ then the control $\bar u(t,x)$ in (3.4) drives 0 to a target $\bar y(x) = y(T,x,0,\bar u)$ in optimal time T and thus is in $\mathcal{SP}(T)$. Since $Z(T) \supset L^2(0,\infty)$ we may take $z(\cdot) \in L^2(0,\infty)$ for the purposes of this paper. Any control in $\mathcal{SP}(T)$ must be zero in the region $\mathcal{K}$ in Figure 3; if not, we can modify its definition to $\bar u(t,x) = 0$ there and argue as in the proof of Lemma 3.1

The condition $s(z(\cdot)) = s(\bar y(\cdot)) = T$ alone does not guarantee that T is the optimal time to reach $\bar y(\cdot)$; if $s(z(\cdot)) = T$ and $T' < T$ the control

$$\bar u(t,x) = \frac{S(T'-t)^* z(x)}{\|S(T'-t)^* z(\cdot)\|} = \chi_0(x)\frac{z(x + (T'-t))}{\kappa(T'-t, z)} \quad (0 < t \le T')$$

drives 0 to a target $\bar y(\cdot)$ with $s(\bar y(\cdot)) = T$ in time $T' < T$.

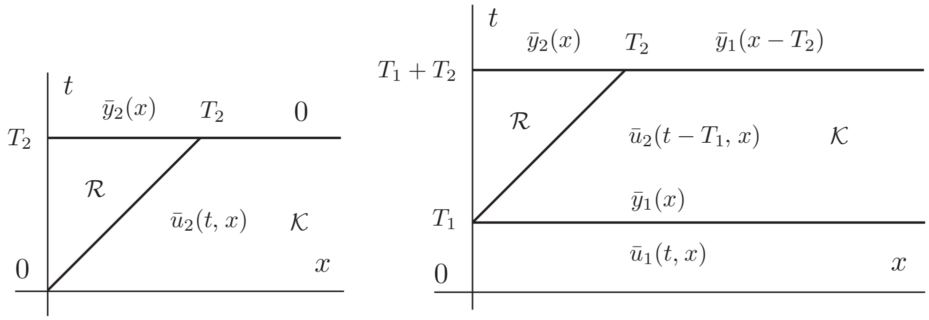

FIGURE 4. Splicing of $\bar{u}_1(t,x)$ and $\bar{u}_2(t,x)$.

Theorem 4.1. *Let the admissible control $\bar{u}_1(\cdot\,,\cdot) \in L^\infty(0,T_1;L^2(0,\infty))$ be optimal in $[0,T_1]$ and let $u_2(\cdot\,,\cdot) \in \mathcal{SP}(T_2)$. Then the splice*

$$(u_1 \cup u_2)(\cdot\,,\cdot) \in L^\infty(0,T_1+T_2;L^2(0,\infty))$$

is optimal in $[0,T_1+T_2]$.

Proof. $\bar{u}_2(t,x)$ is zero in the complement $\mathcal{K}$ of the triangle $\mathcal{R}$ in Figure 4 left, thus the splice $\bar{u}(t,x) = (\bar{u}_1 \cup \bar{u}_2)(t,x)$ is zero in the corresponding region $\mathcal{K}$ of Figure 4 right. If $\bar{u}_2(t,x)$ drives 0 to the target $\bar{y}_2(x)$ in time T_2 and $\bar{u}_1(t,x)$ drives 0 to the target $\bar{y}_1(x)$ in time T_1, formula (3.3) shows that $\bar{u}(t,x)$ drives 0 to the target

$$\bar{y}(x) = \begin{cases} \bar{y}_2(x) & (0 \le x \le T_2) \\ \bar{y}_1(x-T_2) & (T_2 < x) \end{cases} \tag{4.1}$$

in time $T = T_1 + T_2$. It remains to show that this drive is optimal.

Since $\bar{u}(t,x)$ is admissible the target (4.1) is reachable and Theorem 1.3 says that a time optimal control $\tilde{u}(t,x)$ exists. Let $\tilde{T}$ be the optimal driving time, and assume first that $\tilde{T} < T_2$. The control

$$u(t,x) = \begin{cases} \tilde{u}(t,x) & (t,x) \in \tilde{\mathcal{R}} \\ 0 & (t,x) \in \tilde{\mathcal{K}} \end{cases}$$

($\tilde{\mathcal{R}}$, $\tilde{\mathcal{K}}$ the two regions in Figure 5 left) is admissible and drives 0 to $\bar{y}_2(x)$ in time $\tilde{T} < T_2$, which contradicts the optimality of $\bar{u}_2(t,x)$. Accordingly, $\tilde{T} \ge T_2$. If $\tilde{T} < T_1 + T_2$, let $\delta = T_1 + T_2 - \tilde{T}$. The admissible control

$$u_\delta(t,x) = \tilde{u}(t-\delta,x) \quad (\delta \le t \le T_1+T_2)$$

does the drive

$$y(\delta,x,0,u_\delta) = 0\,, \quad y(T_1+T_2,x,0,u_\delta) = \bar{y}(x)\,.$$

The control

$$v(t,x) = \begin{cases} u_\delta(t,x) & (t,x) \in \mathcal{R} \\ 0 & (t,x) \in \mathcal{K} \end{cases} \qquad (T_1 \le t \le T_1+T_2)$$

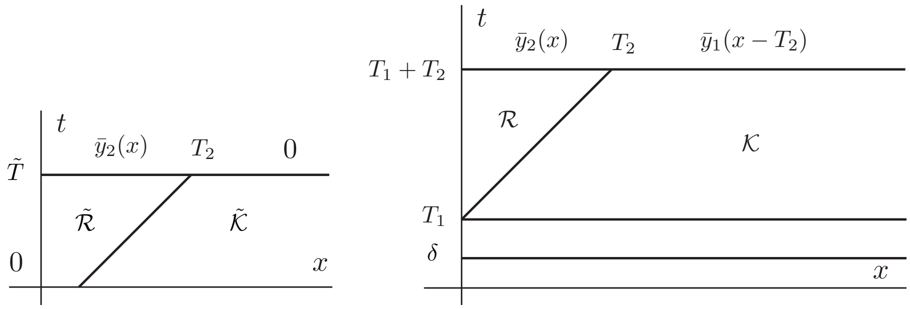

FIGURE 5. Time optimality of the splice.

($\mathcal{R}, \mathcal{K}$ the regions in Figure 5 right) is admissible and does the drive

$$y(T_1, x, 0, v) = 0\,, \quad y(T_1 + T_2, x, 0, v) = \bar{y}_2(x)$$

while $\bar{u}_2(t - T_1, x)$ does the same drive optimally; it follows that $v(t, x)$ is also optimal and, by uniqueness of optimal controls,[3]

$$v(t, x) = u_\delta(t, x) = \bar{u}_2(t - T_1, x) \quad (T_1 \le t \le T_1 + T_2) \tag{4.2}$$

which shows that $u_\delta(t, x) = 0$ in $\mathcal{K}$. By Formula (3.3) (Figure 1) we must have

$$y(T_1, x, 0, u_\delta) = \bar{y}_1(x)\,. \tag{4.3}$$

Accordingly, $u_\delta(t + \delta, x) = \tilde{u}(t, x)$ drives 0 to $\bar{y}_1(x)$ in time $T_1 - \delta$, which contradicts the optimality of $\bar{u}_1(t, x)$. □

Theorem 4.2. *Let $\bar{u}_1(\cdot\,,\cdot) \in L^\infty(0, T_1; L^2(0, \infty))$ be optimal in $[0, T_1]$ and let $\bar{u}_j(\cdot, \cdot) \in \mathcal{SP}(T_j)$, $j = 2, \dots, n$. Then the splice*

$$\bar{v}_n(t, x) = (\bar{u}_1 \cup \bar{u}_2 \cup \cdots \cup \bar{u}_n)(\cdot, \cdot) \in L^\infty(0, T_1 + T_2 + \dots T_n; L^2(0, \infty))$$

is optimal in $[0, T_1 + T_2 + \dots T_n]$.

Proof. Theorem 4.1 and induction: use formula (2.2). □

Remark 4.3. We can figure out the target hit by $\bar{v}_n(t, x)$ arguing as in Theorem 4.1. Let $\bar{y}_j(x) = y(T_j, x, 0, \bar{u}_j)$ be the target hit by the control $\bar{u}_j(t, x)$ in optimal time T_j, and define $S_j = T_1 + \cdots + T_j$ $(j = 1, \dots n - 1)$, $T = T_1 + T_2 + \cdots + T_n$. Then

$$\begin{aligned} &y(T, x, 0, \bar{v}_n) \\ &= \begin{cases} \bar{y}_1(x - (T - S_1)) & (T - S_1 < x) \\ \bar{y}_j(x - (T - S_j)) & (T - S_j \le x < T - S_{j-1}\,,\ j = 2, \dots, n - 1) \\ \bar{y}_n(x) & (0 \le x < T_n)\,. \end{cases} \end{aligned} \tag{4.4}$$

[3]Time optimality can be defined for an arbitrary interval $[a, b]$. Results are the same.

This formula reads from right to left as j increases. We can put it in table form:

Interval	Target $\bar{y}(x)$
$[0, T_n)$	$\bar{y}_n(x)$
$[T_n, T_n + T_{n-1})$	$\bar{y}_{n-1}(x - T_n)$
$[T_n + T_{n-1}, T_n + T_{n-1} + T_{n-2})$	$\bar{y}_{n-2}(x - T_n - T_{n-1})$
....	....
$T_n + \cdots + T_3, T_n + \cdots + T_2)$	$\bar{y}_2(x - T_n - \cdots - T_3)$
$[T_n + \cdots + T_2, \infty)$	$\bar{y}_1(x - T_n - \cdots - T_2)$

Remark 4.4. The finite splices of optimal controls in Theorem 4.2 have a seemingly conflictual relationship with Theorem 1.1. Assume the optimal controls $\bar{u}_j(\cdot\,,\cdot)$ in the splice are produced from formula (3.4) and let $z_j(x)$ be the multiplier for each control. It is proved in [6] Section 2.6 that $z_j(x)$ can be chosen in such a way that $\bar{y}_j(x) = (T_j, x, 0, \bar{u}_j)$ is continuously differentiable in $[0, T_j]$ with $\bar{y}_j(0) = \bar{y}_j'(0) = \bar{y}_j(T_j) = 0 = \bar{y}_j'(T_j) = 0$. In view of the definition (2.5) of the infinitesimal generator A and of formula (4.4) the target $\bar{y}(T, x, 0, \bar{v}_n)$ belongs to $D(A)$. In fact, the different pieces of the target are smooth and match (together with their derivatives) at the divisory points of (4.4) and the target is zero for $x = 0$. Theorem 1.1 applies and $\bar{v}(t, x)$ satisfies the maximum principle (1.6) in its Hilbert space version (1.7), which seems to clash with the discontinuous nature of $\bar{v}_n(t, x)$. The explanation lies in the fact that Theorem 1.1 does *not* guarantee that the costate $z(t) = S(T - t)^* z$ in (1.6) is nonzero in the entire control interval $[0, T]$, just in an interval $(T - \delta, T]$. The multiplier provided by Theorem 1.1 for the control $\bar{v}(t, x)$ is the multiplier $z_n(\cdot)$ (extended $= 0$ for $x \geq T_n$) for the last piece $\bar{u}_n(t, x)$ of the splice $\bar{v}_n(t, x)$, and the costate $z(t) = S(T - t)^* z_n$ vanishes outside of the interval $(T - T_n, T]$. In fact, the control $\bar{v}_n(t, x)$ "switches costates" every time t crosses one of the divisory points $S_j = T_1 + \cdots + T_j$ in the interval $[0, T]$. For more details on this see [6] Section 2.7.

5. INFINITE SPLICING

If $y \in E =$ state space of (1.1) we denote by $\mathcal{T}(y)$ the optimal driving time from 0 to y. If y is reachable and E is a Hilbert space, Theorem 1.3 says that $\mathcal{T}(y)$ exists and is finite. If y is not reachable, that is, if 0 cannot be driven to y by any admissible control in any time we define $\mathcal{T}(y) = \infty$. The first result below is in [4], [6] Lemma 2.7.3; the second result is specific to the system (2.4).

Lemma 5.1. *Let E be a Hilbert space and let $\{y_n\} \subset E$ be a sequence with $y_n \to y \in E$. Then*

$$\mathcal{T}(y) \leq \liminf_{n\to\infty} \mathcal{T}(y_n)\,. \tag{5.1}$$

Lemma 5.2. *Assume* $|y_1(x)| \le |y_2(x)|$. *Then*

$$\mathcal{T}(y_1(\cdot)) \le \mathcal{T}(y_2(\cdot))\,. \tag{5.2}$$

Proof. We may assume that $\mathcal{T}(y_2(\cdot)) < \infty$. If $\bar{u}_2(t,x)$ is the control driving 0 to $y_2(x)$ optimally we define a new control $u_1(t,x)$ multiplying $\bar{u}_2(t,x)$ by the quotient $y_1(x)/y_2(x)$ over each integration line $(\sigma, x-(T-\sigma))$ in Formula (3.3) (Figure 1); precisely

$$u_1(\sigma, x-(T-\sigma)) = \phi(x)\bar{u}_2(\sigma, x-(T-\sigma))$$
$$(0 \le x < \infty,\ 0 \le \sigma \le T,\ x+(T-\sigma) \ge 0)$$

where $\phi(x) = 0$ if $y_2(x) = 0$, $\phi(x) = y_1(x)/y_2(x)$ if $y_2(x) \ne 0$. We have $|\phi(x)| \le 1$, thus $|u_1(\sigma,x)| \le |\bar{u}_2(\sigma,x)|$ and it follows that $u_1(\sigma,x)$ is admissible. Formula (3.3) shows that $y(T,x,0,u_1) = y_1(x)$, so that $y_1(x)$ is reachable in time $T = \mathcal{T}(y_2(\cdot))$. The optimal driving time is $\le T$, thus inequality (5.2) holds and we are done. □

The infinite sequence to be spliced is $\{\bar{u}_n(t,x)\}$, where the first control $\bar{u}_1(\cdot,\cdot) \in L^\infty(0,T_1;L^2(0,\infty))$ is time optimal in $[0,T_1]$ and $u_n(\cdot\,,\cdot) \in \mathcal{SP}(T_n)$ $(n = 2,3,\dots)$. The sequence is far from arbitrary; in fact it will be constructed inductively, each $\bar{u}_n(t,x)$ (or, rather, its support) depending on the choice of the preceding terms $\bar{u}_1(t,x),\dots,\bar{u}_{n-1}(t,x)$. The procedure implies $T = \sum_{n=1}^\infty T_n < \infty$, thus the splice is defined in a finite interval $[0,T]$. The primary objects are not the T_n but two sequences $\{S_n; n = 0,1,2,\dots\}$ and $\{S^n; n = 0,1,2,\dots\}$ such that

$$0 < S_0 < S_1 < S_2 < \dots \qquad S^0 > S^1 > S^2 > \dots$$
$$S_n < S^n \quad (n = 0,1,2,\dots)\,. \tag{5.3}$$

In addition, the sequences $\{S_n\}$ and $\{S^n\}$ satisfy

$$S_{n+1} - S_n \le \alpha_n\,, \quad S^n - S_n \le \alpha_n \quad (n = 0,1,2,\dots)\,, \tag{5.4}$$

where the summable positive series $\{\alpha_n; n = 0,1,\dots\}$ is given in advance and totally arbitrary. The T_n are obtained from the S_n by

$$T_n = S_n - S_{n-1} \quad (n = 1,2,\dots)\,, \tag{5.5}$$

hence

$$T = \sum_{n=1}^\infty T_n = \sum_{n=1}^\infty (S_n - S_{n-1}) \le \sum_{n=1}^\infty \alpha_{n-1} = \sum_{n=0}^\infty \alpha_n = \alpha\,. \tag{5.6}$$

The sequences begin with $S_0 = 0$ and $S^0 > 0$ satisfying the second condition (5.4) for $n = 0$, that is, $S^0 - S_0 \le \alpha_0$. For the inductive step, we assume that the S^j, the S_j and the $\bar{u}_j(t,x)$ have been constructed for $j \le n$. Define $\bar{v}_n(t,x)$ as the finite splice

$$\bar{v}_n(t,x) = (\bar{u}_1 \cup \bar{u}_2 \cup \dots \cup \bar{u}_n)(t,x)\,. \tag{5.7}$$

By Theorem 4.2 it drives 0 in optimal time S_n to the target $y(S_n,x,0,\bar{v}_n)$. If we define

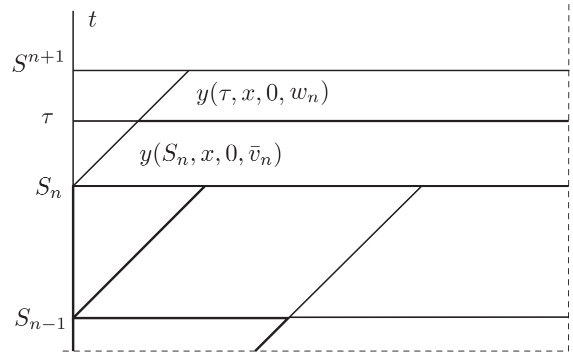

FIGURE 6. Choice of S^{n+1} from $\bar{v}_n(t,x)$.

$$w_n(t,x) = \begin{cases} \bar{v}_n(t,x) & (0 \le t \le S_n) \\ 0 & (S_n < t) \end{cases} \tag{5.8}$$

we have (Figure 6)

$$y(\tau, x, 0, w_n) = \begin{cases} 0 & (0 \le x < \tau - S_n) \\ y(S_n, x - (\tau - S_n)), 0, \bar{v}_n) & (\tau - S_n \le x) \end{cases} \tag{5.9}$$

for $\tau > S_n$, so that

$$||y(\tau, \cdot\,, 0, w_n) - y(S_n, \cdot\,, 0, \bar{v}_n)||_{L^2(0,\infty)} \to 0 \quad \text{as } \tau \to S_n\ .$$

Accordingly, Lemma 5.1 can be applied and we can select $S^{n+1} > S_n$ with

$$\mathcal{T}(y(\tau, \cdot\,, 0, w_n)) \ge \mathcal{T}(y(S_n, \cdot\,, 0, \bar{v}_n)) - \frac{1}{n} \quad (S_n \le \tau \le S^{n+1})\,. \tag{5.10}$$

In fact, if this choice is not possible there exists a sequence $\{\tau_k\}$, $\tau_k \to S_n$ such that

$$\mathcal{T}(y(S_n, \cdot\,, 0, \bar{v}_n)) \ge \liminf_{k\to\infty} \mathcal{T}(y(\tau_k, \cdot\,, 0, w_n)) + \frac{1}{n}$$

contradicting (5.1). Since $\mathcal{T}(y(S_n, \cdot\,, 0, \bar{v}_n)) = S_n$, (5.10) becomes

$$\mathcal{T}(y(\tau, \cdot\,, 0, w_n)) \ge S_n - \frac{1}{n} \quad (S_n \le \tau \le S^{n+1})\,. \tag{5.11}$$

Moving S^{n+1} to the left if necessary we may assume that

$$S^{n+1} - S_n \le \alpha_{n+1} \tag{5.12}$$

and then select S_{n+1} with

$$S_n < S_{n+1} < S^{n+1} \tag{5.13}$$

which, in view of (5.12) gives the second condition (5.4). The control $\bar{u}_{n+1}(t,x)$ added at this step to the splice is an arbitrary control in $\mathcal{SP}(T_{n+1})$, where $T_{n+1} = S_{n+1} - S_n$. This control exists, since formula (3.4) applied to $z(\cdot)$ with $s(z(\cdot)) = T$ (T arbitrary) produces a control in $\mathcal{SP}(T)$. The sequences $\{S_n\}$ and $\{S^n\}$ constructed by repeated application of this inductive step satisfy the three conditions (5.3) and the two conditions (5.4). We then define

$$T = \sup_n S_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} S^{n+1} = \inf_n S^{n+1}$$

and splice the chosen sequence $\bar{u}_n(t,x)$,

$$\bar{u} = \bigcup_{n=1}^{\infty} \bar{u}_n$$

obtaining an admissible control $\bar{u}(\cdot\,,\,\cdot) \in L^\infty(0,T;L^2(0,\infty))$. This control drives 0 to the target

$$\begin{aligned} \bar{y}(x) &= y(T,x,0,\bar{u}) \\ &= \begin{cases} \bar{y}_1(x-(T-S_1)) & (T-S_1 < x) \\ \bar{y}_j(x-(T-S_j)) & (T-S_j \le x < T-S_{j-1}\,,\ j=2,3,\dots) \end{cases} \end{aligned} \tag{5.14}$$

where $\bar{y}_j(x) = y(T_j,x,0,\bar{u}_j)$ has support in $[0,T_j]$ for $j \ge 2$; the support of $\bar{y}_1(x)$ is unrestricted. (this formula is the generalization of (4.4) to infinite splices). Consider now a second splice where each $\bar{u}_j(t,x)$ $(j \ge n+1)$ is replaced by $\xi_j(t,x) = 0$ in the same interval $[0,T_j]$,

$$w_n = \left(\bigcup_{j=1}^{n} \bar{u}_n\right) \cup \left(\bigcup_{j=n+1}^{\infty} \xi_j\right).$$

This control is the same defined in (5.8). It drives 0 to a target $y(T,x,0,w_n)$ which can be described by (5.14) with the only difference that $\bar{y}_j(x) = 0$ for $j \ge n+1$; precisely

$$\begin{aligned} \bar{y}(x) &= y(T,x,0,\bar{u}) \\ &= \begin{cases} \bar{y}_1(x-(T-S_1)) & (T-S_1 \le x) \\ \bar{y}_j(x-(T-S_j)) & (T-S_j \le x < T-S_{j-1}\,,\ j=2,3,\dots,n) \\ 0 & (0 \le x < T-S_n) \end{cases} \end{aligned} \tag{5.15}$$

which can be written in the form

$$y(T,x,0,w_n) = \chi(x) y(T,x,0,\bar{u}) \tag{5.16}$$

where $\chi(x)$ is the characteristic function of the interval $[0, T-S_n]$. It follows from (5.16) that

$$|y(T,x,0,w_n)| \le |y(T,x,0,\bar{u})|$$

hence, using Lemma 5.2 in the first inequality and taking (5.11) into account in the second,

$$\mathcal{T}(\bar{y}(\cdot)) \ge \mathcal{T}(y(T,\cdot\,,0,w_n)) \ge S_n - \frac{1}{n} \to T \quad \text{as } n \to \infty\,. \tag{5.17}$$

Taking limits, we deduce that $\mathcal{T}(\bar{y}(\cdot)) \ge T$. Since $\bar{u}(t,x)$ drives 0 to $\bar{y}(x)$ in time T, it follows that $\mathcal{T}(\bar{y}(\cdot)) = T$ and that $\bar{u}(t,x)$ is optimal. We have proved

Theorem 5.3. *The infinite splice* (2.3) *of the controls* $\{\bar{u}_n(t,x); n = 1, 2, \dots\}$ *is time optimal.*

Remark 5.4. The function $\mathcal{T}(y)$ is a version of the *value function* used in the Hamilton-Jacobi approach to optimal control, although the usual value function is defined as the time needed to drive a target $y \in E$ to 0 (see [1] [2] [3]). Lemma 5.1 is related to existing results on smoothness of the value function.

6. APPLICATIONS: STRONGLY SINGULAR TIME OPTIMAL CONTROLS

A control is called *singular* if does not satisfy the maximum principle (1.6) with any multiplier $z \in Z(T)$. If a singular control satisfies (1.6) with a multiplier $\eta \in \mathcal{Z}$ it is called *weakly singular;* if it does *not* satisfy (1.6) with *any* multiplier η, it is called *strongly singular.* The control $\bar{u}(t,x)$ constructed in Theorem 5.3 using controls $\bar{u}_n(\cdot\,,\cdot) \in \mathcal{SP}(T_n)$ of the form (3.4) as building blocks is strongly singular, that is, it does not satisfy the maximum principle (1.6) with any multiplier η. In fact, if it did, it would have itself the form (3.4) with $z(\cdot) \in \mathcal{Z} =$ space of all multipliers (Section 3) in some interval $[T-\delta, T]$. This is impossible; controls of the form (3.4) are either $= 0$ or $\neq 0$ on lines $x + (T-t) = c$, a quality that $\bar{u}(t,x)$ conspicuously lacks. However, $\bar{u}(t,x)$ is time optimal in any subinterval[4] $[0, T']$, $T' < T$ and is regular in each of these subintervals. This is the class of controls constructed in [5] [6] Theorem 2.7.4.

We can apply the same idea to construct optimal controls that are "more singular" than the ones in [5], [6]. We prove first an addendum to Theorem 5.3 having to do with the assumptions on $\bar{u}_1(\cdot\,,\cdot)$. In Theorem 5.3 $\bar{u}_1(\cdot\,,\cdot)$ is just assumed to be an optimal control in $[0, T_1]$ (the other controls $\bar{u}_n(\cdot\,,\cdot)$ are in $\mathcal{SP}(T_n)$). We assume now that $u_1(\cdot\,,\cdot)$ belongs to the class $\mathcal{SP}(T_1)$ as well.

Theorem 6.1. *Assume that all controls* $\{\bar{u}_n(t,x); n = 1, 2, \dots\}$ *in Theorem 5.3 belong to* $\mathcal{SP}(T_n)$. *Then the infinite splice* $\bar{u}(t,x)$ *belongs to* $\mathcal{SP}(T)$.

Proof. We know from Theorem 5.3 that $\bar{u}(t,x)$ drives 0 in optimal time T to a target $\bar{y}(x)$ described by formula (5.14); all we have to show is that

$$s(\bar{y}(\cdot)) = T\,. \tag{6.1}$$

By the assumption on $\bar{u}_1(t,x)$ we have $s(\bar{y}_1(\cdot)) = s(y(T_1, x, 0, \bar{u}_1)) = T_1$, in particular, support $\bar{y}_1(\cdot) \subseteq [0, T_1]$. This means the support of $y_1(\cdot - (T - S_1))$ in the first line of (5.14) is contained in $[T - S_1, T] = [T - T_1, T]$, so that

[4] The *optimality principle* says that if $\bar{u}(\cdot)$ is time optimal in $[0, T]$ then it is time optimal in any subinterval $[a, b] \subseteq [0, T]$. The proof is immediate; if we can improve the driving time in $[a, b]$ then (via a time translation) we can improve the driving time in $[0, T]$. The time translation works because the system (1.1) does not depend explicitly on time.

support $\bar{y}(\cdot) \subseteq [0, T]$. If support $\bar{y}(\cdot) \subseteq [0, T - \epsilon]$ for $\epsilon > 0$ we reverse this implication and deduce that $s(\bar{y}_1(\cdot))) \le T - \epsilon$, a contradiction. □

We call $\mathcal{SP}_0(T) \subseteq \mathcal{SP}(T)$ the set of all controls of the form (3.4) with $s(z(\cdot)) = T$. The set $\mathcal{SP}_1(T) \subseteq \mathcal{SP}(T)$ consists of all infinite splices produced under the auspices of Theorem 6.1 with building blocks $\bar{u}_n(\cdot\,,\cdot) \in \mathcal{SP}_0(T_n)$.

Lemma 6.2. *$\mathcal{SP}_1(T) \neq \emptyset$ for all $T > 0$.*

Proof. A result like this is needed because the optimal time T in Theorems 5.3 and 6.1 is not chosen in advance; it is a byproduct of the theorem, precisely, it depends on the choice of the S_n. Given $T > 0$ arbitrary we construct first a control $\bar{v}_1(\cdot\,,\cdot) \in \mathcal{SP}_1(T')$ for some $T' \le T$. This is possible since the series $\{\alpha_n\}$ and its sum α in (5.6) are completely arbitrary; we may take $\alpha = T$. If $T' = T$ we are done. Otherwise, we take a control $\bar{v}_0(\cdot\,,\cdot) \in \mathcal{SP}_0(T)$ with optimal time $T - T'$ and define

$$\bar{u} = \bar{v}_0 \cup \bar{v}_1\,,$$

that is, we add the building block $\bar{v}_0$ at the beginning of the splice. By Theorem 6.1 the control $\bar{u}$ belongs to $\mathcal{SP}_1((T - T') + T') = \mathcal{SP}_1(T)$. □

Theorem 6.3. *The controls $\bar{u}_n(t, x)$ making up the infinite splice in Theorem* 6.1 *can be taken in $\mathcal{SP}_1(T_n)$.*

Proof. We go over the proof of Theorem 5.3 as modified in Theorem 6.1, especially over the inductive choice of $\bar{u}_{n+1}(t, x)$ in the paragraph following (5.11). The only qualification on $\bar{u}_{n+1}(t, x)$ is "an arbitrary control in $\mathcal{SP}(T_{n+1})$ with $T_{n+1} = S_{n+1} - S_n$" where S_{n+1} has been chosen on the basis of (5.11). By Lemma 6.2 we can select $\bar{u}(\cdot\,,\cdot)$ in $\mathcal{SP}_1(T_{n+1})$ for arbitrary T_{n+1}. The rest of the proof of Theorem 5.3 remains the same. □

The set of all splices produced by Theorem 6.1 with building blocks $\bar{u}_n(\cdot\,,\cdot) \in \mathcal{SP}_1(T_n)$ is $\mathcal{SP}_2(T) \subseteq \mathcal{SP}(T)$. Controls in this class are strongly singular not only in the whole interval $[0, T]$ but in each subinterval $[0, S_n,]$ $\{S_n\}$ the sequence in Theorem 5.3 with $S_n \to T$.

Exploiting the same idea we prove first that $\mathcal{SP}_2(T) \neq \emptyset$ for all $T > 0$ and then use Theorem 6.1 to construct controls whose building blocks are in $\mathcal{SP}_2(T_n)$. This is the class $\mathcal{SP}_3(T)$; the intervals where the controls are strongly singular are $[0, S_{mn}]$ where $S_{mn} \to S_n$ as $m \to \infty$ and $S_n \to T$. The classes $\mathcal{SP}_n(T)$ are inductively defined in a similar way for all n and contain optimal controls which are strongly singular in more and more subintervals of $[0, T]$.

A time optimal control $\bar{u}(\cdot)$ for (1.1) is called *hypersingular* in $[0, T]$ if it is strongly singular in any interval $[a, b] \subseteq [0, T]$ (for an example, see [7]). Obviously, the controls in any class $\mathcal{SP}_n(T)$ fall short of hypersingular. To produce a hypersingular control taking $\bar{u}_n(\cdot\,,\cdot) \in \mathcal{SP}_n(T)$ and then taking limits as $n \to \infty$ won't work since the controls in the sequence would have support in a shrinking band $0 \le t \le T$, $0 \le x \le \epsilon_n \to 0$. We don't know if the construction in this paper can be modified to give hypersingular controls, in fact we don't know if hypersingular controls for (2.4) exist at all.

The toy control system (2.4) (one of the simplest involving partial differential equations) is the carrier of a number of interesting control theory examples and counterexamples; for information on this see [6], [9], [10]. For a recent survey on regular, singular and strongly singular controls for the system (1.1) and related topics see [10].

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