Eigenfrequencies for Damped Wave Equations on Zoll Manifolds

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Abstract: The eigenfrequencies associated to a damped wave equation are known to belong to a band parallel to the real axis. Under the assumption of periodicity of the geodesic flow we study the asymptotic distribution of the eigenfrequencies in the band. We show that the set of eigenfrequencies exhibits a cluster structure determined by the Morse index of the closed geodesics and the damping coefficient averaged along the geodesic flow. The asymptotics for the multiplicities of the clusters are also obtained.

1 Introduction and Statement of Results

The purpose of this paper is to study the asymptotic distribution of the eigenfrequencies associated to a damped wave equation on Zoll manifolds, i.e. manifolds all of whose geodesics are closed. In particular we shall show that the eigenfrequencies form clusters determined by the Morse index of the closed geodesics and the damping coefficient averaged along the geodesic flow. In order to describe the results more precisely, we must recall some standard notation and assumptions.

Let M be a smooth compact connected Riemannian manifold of dimension $n \geq 2$, and let Δ be the corresponding Laplacian. In control theory, one is interested in the long time behaviour of the solutions to the Cauchy problem for the wave equation with a damping term,

$$\begin{cases} \left(\partial_t^2 - \Delta + 2a(x)\partial_t\right) u = 0, & (t,x) \in \mathbf{R} \times M, \\ u|_{t=0} = u_0 \in H^1(M), & \partial_t u|_{t=0} = u_1 \in L^2(M). \end{cases} \tag{1.1}$$

Here a is a bounded real-valued function on M, and we shall assume that $a \in C^\infty(M)$. Associated with the evolution problem (1.11) is the propagator $\mathcal{U}(t) = e^{it\mathcal{A}}$ acting in the Hilbert space $H^1 \times L^2$, where the infinitesimal generator $\mathcal{A}$ is the operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\Delta & 2ia(x) \end{pmatrix} : H^1 \times L^2 \to H^1 \times L^2,$$

with the domain $D(\mathcal{A}) = H^2 \times H^1$. Here $H^s = H^s(M)$ is the standard Sobolev space on M.

In the case when $a \geq 0$, which corresponds to actual damping, the energy of the solution $u(x,t)$ to (1.11) is nonincreasing when $t \to \infty$, and relations between the rate of decay of the energy and the spectrum of $\mathcal{A}$ were studied by many people—see Lebeau [11] and references given there.

In this paper we shall only be interested in asymptotic properties of the spectrum of $\mathcal{A}$, and since the inclusion map $D(\mathcal{A}) \to H^1 \times L^2$ is compact, it is true that the spectrum is discrete. We are thus interested in the spectral problem

$$(\mathcal{A} - \tau) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = 0. \tag{1.2}$$

The eigenvalues of $\mathcal{A}$ will also be called the eigenfrequencies. It follows immediately from (1.22) that $\tau \in \mathbf{C}$ is an eigenfrequency precisely when there exists a non-vanishing smooth function u such that

$$\big(-\Delta - \tau^2 + 2i\tau a(x)\big)\, u(x) = 0. \tag{1.3}$$

The real-valuedness of a implies that the set of eigenvalues is symmetric with respect to the reflection in the imaginary axis. When studying large eigenvalues we may therefore restrict the attention to the region $\operatorname{Re}\tau \geq 0$.

The multiplicity $m(\tau_0) \in \{1, 2 \ldots\}$ of an eigenvalue τ_0 is defined as the rank of the spectral projection

$$\Pi_{\tau_0} = \frac{1}{2\pi i} \int_\gamma (\tau - \mathcal{A})^{-1}\, d\tau,$$

where γ is a sufficiently small circle centered at τ_0. We refer to [12] for several equivalent definitions of the multiplicity.

It follows easily from (1.32) that when τ is an eigenvalue, we have that

$$\inf a \leq \operatorname{Im}\tau \leq \sup a, \quad \operatorname{Re}\tau \neq 0,$$

$$2\min\left(\inf a, 0\right) \leq \operatorname{Im}\tau \leq 2\max\left(\sup a, 0\right), \quad \operatorname{Re}\tau = 0.$$

The spectrum is thus confined to a strip parallel to the real axis, and we are interested in the asymptotic distribution of the eigenvalues inside the strip. For general compact manifolds this problem has been studied by Sjöstrand [12], and, to a large extent, our purpose here is to show how the methods of [12] apply to the case of Zoll manifolds. Concerning the background and motivation for the study of Zoll manifolds, we refer to [1] and [7]. In particular, the paper [7] describes a construction of Zoll metrics on the two-dimensional sphere S^2, i.e. Riemannian metrics such that all geodesics are closed with the same period. Let us remark that although very special, Zoll metrics on S^2 provide the largest known class of metrics on surfaces with completely integrable geodesic flows.

When $p(x, \xi) = \xi^2$ is the principal symbol of $-\Delta$ defined on T^*M, we consider the corresponding Hamilton vector field H_p and recall that the Hamilton flow $\exp(tH_p) : p^{-1}(1) \to p^{-1}(1)$ can be identified with the geodesic flow on the sphere bundle of M. The following periodicity assumption for the flow will be assumed to hold throughout the paper,

$$\exp(\pi H_p)(x, \xi) = (x, \xi), \quad \text{for all } (x, \xi) \in p^{-1}(1). \tag{1.4}$$

Introduce now

$$\langle a \rangle(x, \xi) = \frac{1}{\pi} \int_0^\pi a\left(\exp(tH_p)(x, \xi)\right) dt, \quad \text{on } p^{-1}(1),$$

We notice that $\langle a \rangle(x, \xi)$ is a smooth function on $p^{-1}(1)$ such that $\langle a \rangle(x, -\xi) = \langle a \rangle(x, \xi)$.

The main result of the paper is the following theorem.

Theorem 1.1 *Under the assumption* (1.43) *we have*

1. *There exists a constant $C > 0$ such that all eigenfrequencies τ with $\operatorname{Re} \tau > 0$, except for finitely many values, are contained in the union of the rectangles*

$$I_k = \left[k + \frac{\alpha}{4} - \frac{C}{k}, k + \frac{\alpha}{4} + \frac{C}{k}\right] + i[-\mathcal{O}(1), \mathcal{O}(1)], \quad k = 1, 2, \ldots, \tag{1.5}$$

where $\alpha \in \mathbf{N}$ is the common Maslov index of the closed H_p-orbits in $p^{-1}(1)$. Moreover, if τ is an eigenvalue with $\operatorname{Re} \tau \sim k$, then we have

$$\inf_{p^{-1}(1)} \langle a \rangle - \mathcal{O}\left(\frac{1}{k}\right) \leq \operatorname{Im} \tau \leq \sup_{p^{-1}(1)} \langle a \rangle + \mathcal{O}\left(\frac{1}{k}\right), \quad k \to \infty. \tag{1.6}$$

2. *Assume that $\exp(tH_p) : p^{-1}(1) \to p^{-1}(1)$ has no fixed points when $t \in (0, \pi)$. Then the number of eigenvalues $\tau \in I_k$ is equal to*

$$\frac{2}{(2\pi)^n} \int_{p^{-1}(1)} L_0(d\rho) \left(k + \frac{\alpha}{4}\right)^{n-1} + \mathcal{O}\left(k^{n-3}\right), \tag{1.7}$$

when $k \to \infty$. Here L_0 is the Liouville measure on $p^{-1}(1)$.

Remark. Theorem 1.1 is a well-known and in general optimal result in the self-adjoint case, $a = 0$, where it is due to Colin de Verdière, Guillemin, and Weinstein—see §29.2 of [10] for an exposition of the spectral theory of the Laplacian on Zoll manifolds, and also [14].

The plan of the paper is as follows. In Section 2 we reformulate our problem in the semiclassical language, following [12], which is essential for our methods. The proof of Theorem 1.1 is then carried out in Section 3. The main idea is to subject the operator in question to a similarity transformation in such a way that, after the conjugation, the operator will tend to be normal, at least approximately. We may think of this as an implementation of the averaging method in this context—see [13], [4]. Notice that due to the periodicity of the flow, the full averaging will be achieved in finite time. The first step in the proof is therefore a functional reduction to the case when the Hamilton flow of the principal symbol is periodic with the same period in the relevant part of the phase space. It follows then, by essentially standard arguments, that the wave group is also periodic, after the energy has been localized. We are then able to implement the averaging method alluded to above.

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2 Preliminaries

Following [12], we begin by reformulating the problem in a semiclassical setting. When M is a compact connected Riemannian manifold of dimension $n \geq 2$ and $a \in C^\infty(M, \mathbf{R})$, we consider the eigenvalue problem

$$\left(-\Delta - \tau^2 + 2i\tau a(x)\right) v = 0, \;\; v \neq 0. \tag{2.1}$$

We recall that the eigenvalues to the problem (2.14) are contained in a strip parallel to the real axis. We are only interested in eigenvalues τ with large absolute value, and by the reflection symmetry we may assume that $\operatorname{Re} \tau \gg 1$.

We write $\tau = \sqrt{z}/h$, where $0 < h \ll 1$, and z belongs to the fixed domain $\Omega := (\alpha, \beta) + i(-\gamma, \gamma)$, for some $0 < \alpha < 1 < \beta < \infty$ and $\gamma > 0$. We are then led to the problem

$$(\mathcal{P} - z)\, v = 0, \tag{2.2}$$

where

$$\mathcal{P} = P + ihB(z),$$

with $P = -h^2\Delta$ and $B(z) = 2a(x)\sqrt{z}$. We notice that P is essentially self-adjoint on $C^\infty(M)$, with the domain of the closure being $H^2(M)$, and $B(z)$ is bounded and self-adjoint for real positive z.

In what follows we shall make use of some calculus of h-pseudodifferential operators ($h - \Psi$DO from now on), and we digress here to recall some relevant notation. Let $S^m(T^*M)$ be the space of functions $a(x,\xi,h)$ on $T^*M \times (0,h_0]$, $h_0 > 0$, which are smooth in (x,ξ) and such that

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h) = \mathcal{O}_{\alpha,\beta}(1)\,(1+|\xi|)^{m-|\beta|}\,, \quad (x,\xi) \in T^*M,\ h \in (0,h_0]. \tag{2.3}$$

When a depends on some additional parameters, we require the symbolic estimates (2.35) to hold uniformly with respect to these parameters. The formula for the classical h-quantization,

$$a(x,hD_x,h)u(x) = \frac{1}{(2\pi h)^n} \iint e^{i\frac{\langle x-y,\xi\rangle}{h}} a(x,\xi,h)u(y)\,dy\,d\xi,$$

then defines a class of $h - \Psi$DO on M, which will be denoted by $\mathcal{E}^m(M)$. Associated to this class is a symbol map, σ, and a short exact sequence,

$$0 \to h\mathcal{E}^{m-1}(M) \to \mathcal{E}^m(M) \to S^m(T^*M)/hS^{m-1}(T^*M) \to 0.$$

For h-dependent symbols, we write $a \in S^m_{\rm cl}$ if there exists $a_0 \in S^m$, independent of h, such that $a - a_0 \in hS^{m-1}$, and we say that a_0 is the principal symbol of a and of the corresponding $h - \Psi$DO. When $A \in \mathcal{E}^m$ is an $h - \Psi$DO with the principal symbol $a_0 \in S^m$, we write $A = \mathrm{Op}_h(a_0)$ and say that A is classical, $A \in \mathcal{E}^m_{\rm cl}$.

We shall finally recall the notion of the essential support, $SE(A)$, of an operator $A \in \mathcal{E}^m(M)$. When $\rho \in T^*M$, we say that $\rho \notin \mathrm{SE}\,(A)$, if the full symbol of A, for some choice of local coordinates near the projection of ρ, is of class $S^{-\infty,m} := \cap_k h^k S^m$ near ρ. It follows from the definition that $\mathrm{SE}\,(A)$ is a closed subset of T^*M, $\mathrm{SE}\,(AB) \subset SE(A) \cap \mathrm{SE}\,(B)$, and $SE(A) = \emptyset \Rightarrow h^{-k}A \in \mathcal{E}^m$ for any k.

3 Proof of Theorem 1.1

Let M be a Zoll manifold and consider the eigenvalue problem (2.24). The assumption (1.43) together with the homogeneity of $p(x,\xi)$ implies that

$$\exp{(TH_p)}(x,\xi) = (x,\xi), \quad \text{for all } (x,\xi) \in p^{-1}(E),\ E > 0, \tag{3.1}$$

with

$$T = T(E) = \pi/\sqrt{E}. \tag{3.2}$$

Notice also that, for $E > 0$, $p^{-1}(E)$ is connected, since M is and the dimension of M is ≥ 2.

We shall first perform a functional reduction to the case when the period T is constant in the energy band in question, see [8]. We can find a smooth real-valued function $f(\lambda)$ such that $f(\lambda) = \sqrt{\lambda}$ when $\lambda \in [\alpha-\varepsilon, \beta+\varepsilon]$, for some $\varepsilon > 0$ small enough,

f is increasing on $[0,\infty)$ and f is equal to a constant plus a C_0^∞ function. It follows then from the semiclassical functional calculus (see [5], [2]), that $Q := f(P)$ is a self-adjoint $h-\Psi$DO of the class $\mathcal{E}^0_{\rm cl}(M)$, with the principal symbol $q(x,\xi) = f(p(x,\xi))$. We also have that the set $\{(x,\xi) \in T^*M; p(x,\xi) \in (\alpha-\varepsilon, \beta+\varepsilon)\} = \{(x,\xi) \in T^*M; q(x,\xi) \in (f(\alpha-\varepsilon), f(\beta+\varepsilon))\}$ has compact closure, and $q(x,\xi) = p^{1/2}(x,\xi)$ on this set.

Since $H_q = f'(p)H_p$, it follows from (3.25) that $\exp{(tH_q)}$ is 2π-periodic for (x,ξ) in a neighbourhood of $q^{-1}([\alpha^{1/2}, \beta^{1/2}])$. Decreasing $\varepsilon > 0$ if necessary, we may assume that

$$\exp{(2\pi H_q)}(x,\xi) = (x,\xi), \quad (x,\xi) \in q^{-1}(E),\ E \in [\alpha^{1/2}-\varepsilon, \beta^{1/2}+\varepsilon]. \tag{3.3}$$

In what follows we shall write $E_1 = \alpha^{1/2}$ and $E_2 = \beta^{1/2}$.

Let now $0 \le \psi \in C_0^\infty((E_1-\varepsilon, E_2+\varepsilon))$ be such that $\psi = 1$ in a neighbourhood of $[E_1, E_2]$, and consider the semiclassical Fourier integral operator

$$U_\psi(t) = e^{-itQ/h}\psi(Q), \quad t \in \mathbf{R}.$$

We are interested here in $U_\psi(2\pi)$, and since $\exp{(2\pi H_q)}(x,\xi) = (x,\xi)$ for all $(x,\xi) \in \operatorname{supp}(\psi(q(x,\xi)))$, we know that this is an $h-\Psi$DO with the principal symbol

$$\psi(q(x,\xi))\exp{(-\pi i\alpha/2)},$$

where $\alpha \in \mathbf{Z}$ is the Maslov index of the trajectory $\{\exp{(tH_q)}(x,\xi),\ t \in [0,2\pi]\}$ in $q^{-1}((E_1-\varepsilon, E_2+\varepsilon))$. (See Chapter 15 of [5] for this essentially well-known fact.) Furthermore, arguing as in [5] (see also [8]), we find that there exists a self-adjoint operator $W \in \mathcal{E}^0$ which commutes with Q and such that

$$e^{-2\pi i/h\left(Q-h\sigma-h^2W\right)}\psi_1(Q) = \psi_1(Q). \tag{3.4}$$

Here we write $\sigma = \alpha/4$, and $0 \le \psi_1 \in C_0^\infty((E_1-\varepsilon, E_2+\varepsilon))$ is such that $\psi_1 = 1$ in a neighbourhood of $[E_1, E_2]$, and the support of ψ_1 is contained in the interior of the set where $\psi = 1$. If we put

$$Q_1 = Q - h^2W, \tag{3.5}$$

we may rewrite (3.46) as

$$e^{-2\pi i/h(Q_1-h\sigma)}\psi_1(Q) = \psi_1(Q). \tag{3.6}$$

Let now $v = v(h)$ be a nontrivial solution of (2.24) for some $z \in \Omega$, such that $||\,v\,|| = ||\,v\,||_{L^2} = 1$. In the following we shall only use that the operator $B(z) \in \mathcal{E}^0_{\rm cl}$ depends analytically on $z \in \Omega$, and it is self-adjoint for z real positive.

We have that the operator in (2.24) is elliptic away from $p^{-1}([\alpha,\beta])$, and it follows therefore that

$$\left(P - Q^2\right)v = \mathcal{O}(h^\infty), \quad \text{in } L^2,$$

uniformly with respect to z and v. Also, considering $0 = \operatorname{Im}((\mathcal{P}-z)v, v)$ we see that

$$\operatorname{Im} z = \mathcal{O}(h). \tag{3.7}$$

We shall therefore write (2.24) in the following form,

$$\big(Q^2 + ihB(\operatorname{Re} z) + F + h^2 R(z) - z\big)\, v = 0, \tag{3.8}$$

where we put $F = P - Q^2$ and $R(z) \in \mathcal{E}^0$. Furthermore, modifying $R = R(z)$ slightly, we may replace Q by Q_1 in (3.87).

We would like to localize the real part of z in (3.87), and the main idea for doing that is to conjugate the operator in (3.87) by an $h-\Psi$DO, bounded and invertible on $L^2(M)$, so that, after the conjugation, the operator $Q_1^2 + ihB(\operatorname{Re} z)$ becomes approximately normal, after the localization in energy. On the principal symbol level this will amount to replacing the principal symbol of $B = B(\operatorname{Re} z)$ by its average along the closed trajectories of $q(x,\xi)$. The construction of the conjugating operator will be similar to the construction in the non-semiclassical setting in [13], see also [10].

Set

$$\hat{B} = \hat{B}(\operatorname{Re} z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-itQ_1/h} B(\operatorname{Re} z) e^{itQ_1/h}\, dt.$$

The operator $\hat{B}$ is a self-adjoint $h - \Psi$DO of order 0. Since

$$\frac{-i}{h} e^{-itQ_1/h} [Q_1, B] e^{itQ_1/h} = \frac{d}{dt} e^{-itQ_1/h} B e^{itQ_1/h},$$

it follows that

$$[Q_1, \hat{B}] = \frac{-h}{2\pi i} \left(e^{-2\pi i Q_1/h} B e^{2\pi i Q_1/h} - B \right),$$

and in view of (3.66) we get that

$$\psi_1(Q)[Q_1, \hat{B}]\psi_1(Q) = 0. \tag{3.9}$$

Here we have also used that Q_1 commutes with $\psi_1(Q)$.

Proposition 3.1 *There exists an elliptic self-adjoint operator $U \in \mathcal{E}^0_{\mathrm{cl}}(M)$ (depending also on* $\operatorname{Re} z$*), and an operator $V = V(\operatorname{Re} z) \in \mathcal{E}^{-1}(M)$ with*

$$\mathrm{SE}\,(V) \cap p^{-1}([\alpha, \beta]) = \emptyset,$$

such that

$$U^{-1}(Q_1^2 + ihB(\operatorname{Re} z))U = Q_1^2 + ih\hat{B}(\operatorname{Re} z) + hV + h^2 R_0(z),$$

where $R_0(z) \in \mathcal{E}^0$.

Proof: We introduce the operator $B(t) = B(t, \operatorname{Re} z) = e^{-itQ_1/h} B e^{itQ_1/h}$ and write

$$\begin{aligned} i(\hat{B} - B) &= \frac{i}{2\pi} \int_0^{2\pi} \left(\int_0^t \frac{d}{ds} B(s)\, ds \right) dt \qquad (3.10)\\ &= \frac{i}{2\pi} \int_0^{2\pi} \left(\int_0^t \frac{1}{ih} [Q_1, B(s)]\, ds \right) dt \\ &= \frac{1}{2\pi h} \int_0^{2\pi} \left(\int_0^t [Q_1, B(s)]\, ds \right) dt = \frac{1}{h}[Q_1, S], \end{aligned}$$

where
$$S = S(\operatorname{Re} z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^t B(s)\,ds\right)dt \in \mathcal{E}^0_{\mathrm{cl}}(M)$$
is a self-adjoint operator.

Since the operator Q_1 is elliptic near $p^{-1}([\alpha,\beta])$, we can construct a microlocal parametrix $T \in \mathcal{E}^0$ for Q_1, so that

$$\mathrm{SE}\,(TQ_1 - I) \cap p^{-1}([\alpha,\beta]) = \emptyset, \quad \text{and} \quad \mathrm{SE}\,(Q_1 T - I) \cap p^{-1}([\alpha,\beta]) = \emptyset. \tag{3.11}$$

We can also take T to be self-adjoint in view of the self-adjointness of Q_1. We then compute

$$\begin{aligned}
[Q_1^2, TS] &= [Q_1^2, T]S + T[Q_1^2, S] = [Q_1^2, T]S + TQ_1[Q_1, S] + T[Q_1, S]Q_1 \\
&= [Q_1^2, T]S + 2TQ_1[Q_1, S] + T[[Q_1, S], Q_1] \\
&= [Q_1^2, T]S + 2[Q_1, S] + 2[Q_1, S](TQ_1 - I) + h^2 R_1,
\end{aligned}$$

where $R_1 = R_1(z) \in \mathcal{E}^{-2}(M)$. Since

$$[Q_1^2, T] = 2Q_1[Q_1, T] + h^2 R_2,$$

with $R_2 \in \mathcal{E}^{-2}$, it follows that

$$[Q_1^2, TS] = 2[Q_1, S] + 2[Q_1, S](TQ_1 - I) + 2Q_1 S[Q_1, T] + h^2 R_3(z), \tag{3.12}$$

where $R_3 \in \mathcal{E}^{-2}(M)$. A similar computation now shows that

$$[Q_1^2, ST] = 2[Q_1, S] + 2[Q_1, S](Q_1 T - I) + 2Q_1 S[Q_1, T] + h^2 R_4(z), \quad R_4 \in \mathcal{E}^{-2}(M). \tag{3.13}$$

Put now
$$G = \frac{1}{4}\left(ST + TS\right).$$

Then $G \in \mathcal{E}^0_{\mathrm{cl}}(M)$ is a self-adjoint operator, and from (3.128) and (3.138) we get

$$[Q_1^2, G] = [Q_1, S] + \frac{1}{2}[Q_1, S](TQ_1 - I) + \frac{1}{2}[Q_1, S](Q_1 T - I) + Q_1 S[Q_1, T] + h^2 R_5(z), \tag{3.14}$$

with $R_5 \in \mathcal{E}^{-2}$. In view of (3.118) we conclude that

$$[Q_1^2, G] = [Q_1, S] + hV + h^2 R_5(z), \tag{3.15}$$

where $V \in \mathcal{E}^{-1}$ is such that

$$\mathrm{SE}\,(V) \cap p^{-1}([\alpha,\beta]) = \emptyset.$$

We now set $U = e^G \in \mathcal{E}^0_{\mathrm{cl}}$, and conjugate the operator in (3.87) by means of U. We have
$$U^{-1} Q_1^2 U = Q_1^2 + U^{-1}[Q_1^2, U] = Q_1^2 - ihY, \tag{3.16}$$

where $Y \in \mathcal{E}_{\mathrm{cl}}^{-1}$ has the principal symbol

$$e^{-g}H_{q^2}(e^g) = H_{q^2}g,$$

where $g \in S^0$ is the principal symbol of G. It follows that

$$U^{-1}Q_1^2U = Q_1^2 - ih\mathrm{Op}_h(H_{q^2}g) = Q_1^2 + [Q_1^2, G] + h^2R_6(z),$$

with $R_6 \in \mathcal{E}^{-2}(M)$. Using (3.158) and (3.107) we then get

$$\begin{aligned} U^{-1}(Q_1^2 + ihB)U &= Q_1^2 + ihB + [Q_1^2, G] + h^2R_7(z) \\ &= Q_1^2 + ihB + [Q_1, S] + hV + h^2R_8(z) \\ &= Q_1^2 + ih\hat{B} + hV + h^2R_8(z), \end{aligned} \tag{3.17}$$

This completes the proof. □

If v is a nontrivial solution of (2.24) such that $||\, v\, || = 1$, we have

$$\left(U^{-1}\left(Q_1^2 + ihB(\mathrm{Re}\, z) + F + h^2R\right)U - z\right)U^{-1}v = 0, \tag{3.18}$$

and it follows from Proposition 3.1 that

$$\left(Q_1^2 + ih\hat{B} + hV + F_1 + h^2R_9(z) - z\right)u = 0, \tag{3.19}$$

where F_1 is the conjugate of F by means of U, and $u = U^{-1}v$. In particular we have that the principal symbol of F_1 is $p - q^2$ and

$$\mathrm{SE}\,(F_1) \cap p^{-1}([\alpha, \beta]) = \emptyset.$$

We see therefore that

$$||\, F_1u\, || = \mathcal{O}(h^\infty)||\, u\, ||, \quad ||\, Vu\, || = \mathcal{O}(h^\infty)||\, u\, ||.$$

Using this in (3.199) we get

$$||\, \left(Q_1^2 + ih\hat{B}(\mathrm{Re}\, z) - z\right)u\, ||^2 \leq \mathcal{O}(h^4)||\, u\, ||^2. \tag{3.20}$$

On the other hand, if C and D are bounded self-adjoint operators, we have that

$$||\, (C + iD)u\, ||^2 = ||\, Cu\, ||^2 + ||\, Du\, ||^2 + i([C, D]u, u), \tag{3.21}$$

and applying this to the left-hand side of (3.209) we obtain

$$||\, (Q_1^2 - \mathrm{Re}\, z)u\, ||^2 + ih([Q_1^2, \hat{B}]u, u) \leq \mathcal{O}(h^4)||\, u\, ||^2. \tag{3.22}$$

Now (3.97) implies that

$$\psi_1(Q)[Q_1^2, \hat{B}]\psi_1(Q) = 0, \tag{3.23}$$

and using (3.239) together with the fact that $\|\psi_1(Q)u - u\| \leq \mathcal{O}(h^\infty)\|u\|$ we see that

$$ih([Q_1^2, \hat{B}]u, u) = \mathcal{O}(h^\infty)\|u\|^2 + ih([Q_1^2, \hat{B}]\psi_1(Q)u, \psi_1(Q)u) = \mathcal{O}(h^\infty)\|u\|^2.$$

From (3.229) we get

$$\|(Q_1^2 - \mathrm{Re}\, z)u\| \leq \mathcal{O}(h^2)\|u\|,$$

and by the spectral theorem we conclude that

$$\mathrm{dist}\,(\mathrm{Re}\, z, \mathrm{Spec}\,(Q_1)^2) = \mathcal{O}(h^2), \tag{3.24}$$

where $\mathrm{Spec}\,(Q_1^2)$ denotes the spectrum of Q_1^2.

On the other hand it follows from (3.66) that

$$\mathrm{Spec}\,(Q_1) \cap [E_1, E_2] \subset \{h(\sigma + k); k \in \mathbf{Z}\}, \tag{3.25}$$

for all sufficiently small h. Combining (3.2410) together with (3.2510) we get the following semiclassical version of the localization result (1.53):

Theorem 3.2 *There exist $C_0 > 0$, $h_0 > 0$ such that*

$$\mathrm{Spec}\,(\mathcal{P}) \cap \Omega \subset \cup_k I_k(h), \quad k = 1, 2, \ldots, \quad k \sim \frac{1}{h}, \tag{3.26}$$

for all $h \in (0, h_0]$, with

$$I_k(h) = \left[h^2 \left(k + \frac{\alpha}{4}\right)^2 - C_0 h^2, h^2 \left(k + \frac{\alpha}{4}\right)^2 + C_0 h^2\right] + i[-\mathcal{O}(h), \mathcal{O}(h)].$$

To derive (1.53) from Theorem 3.2, it suffices to apply the semiclassical reduction of Section 2. The relation between the eigenvalues z of $\mathcal{P}$ and the original eigenfrequencies τ is given by $z = (h\tau)^2$, $\mathrm{Im}\,\tau = \mathcal{O}(1)$, $\mathrm{Re}\,\tau \sim h^{-1}$, so that

$$\mathrm{Re}\, z = (h\mathrm{Re}\,\tau)^2 + \mathcal{O}(h^2), \quad \mathrm{Im}\, z = 2h(\mathrm{Re}\,\tau)h(\mathrm{Im}\,\tau). \tag{3.27}$$

Now (1.53) follows immediately from (3.2610).

Remark. It follows from (3.2510) that the spectrum of P in $[E_1, E_2]$ consists of clusters of width $\mathcal{O}(h^2)$ separated by a distance of the order of h. It is precisely due to the form of the perturbation, $ihB(z)$, where $B(z)$ is, up to an error of order $\mathcal{O}(h)$, a self-adjoint operator, that the cluster structure of the spectrum of $\mathcal{P}$ persists.

Remark. In Theorem 3.2 we have obtained the result giving the optimal width of the clusters, $\mathcal{O}(h^2)$. It may be interesting to remark that a slightly weaker result giving the width $\mathcal{O}(h^{3/2})$ can be obtained by a direct application of the methods of Section 2 of [12]. For the sake of completeness, and also, for future reference, we shall now briefly outline the argument. Arguing as in Section 2 of [12], we find that after a conjugation of $P + ihB(\operatorname{Re} z)$ by a suitable bounded invertible $h - \Psi$DO (which also depends on $\operatorname{Re} z$), very similar to the operator U in Proposition 3.1, we can replace the principal symbol of B, $b = b(\operatorname{Re} z)$, by its average along the closed H_p-orbits in $p^{-1}(\operatorname{Re} z)$. After the conjugation the operator becomes

$$P + ih\hat{B} + h^2 R(z),$$

where $\hat{B} \in \mathcal{E}^1$ has the principal symbol $\hat{b} = \hat{b}(\operatorname{Re} z) = \langle b \rangle_T$ on $p^{-1}(\operatorname{Re} z)$, and $R(z) \in \mathcal{E}^1$. Here

$$\langle b \rangle_T(x,\xi) = \frac{1}{T}\int_0^T b(\exp{(tH_p)}(x,\xi))\,dt, \quad (x,\xi) \in p^{-1}(\operatorname{Re} z),$$

and $T = T(\operatorname{Re} z)$ is the common period of the closed H_p-orbits in $p^{-1}(\operatorname{Re} z)$. Therefore, $H_p\hat{b} = 0$ on $p^{-1}(\operatorname{Re} z)$, and we then have

$$H_p\hat{b} = k(p - \operatorname{Re} z), \quad k \in S^0.$$

If K is an $h - \Psi$DO with the principal symbol k, it follows that

$$\frac{i}{h}[P, \hat{B}] - K(P - \operatorname{Re} z) \in h\mathcal{E}^1(M). \tag{3.28}$$

Now if z is an eigenvalue of $\mathcal{P}$ we have

$$(P + ih\hat{B} + h^2 R(z) - z)u = 0, \tag{3.29}$$

for some $u \in L^2(M)$, $||\, u \,|| = 1$. We know that $\operatorname{Im} z = \mathcal{O}(h)$, and a standard observation based on the ellipticity of the operator in (3.2911) for large ξ shows that $||\, Au \,|| \leq \mathcal{O}(1)$ when $A \in \mathcal{E}^2$. In particular, $\hat{B}u = \mathcal{O}(1)$ in L^2, and it follows that

$$(P - \operatorname{Re} z)u = \mathcal{O}(h) \ \text{ in } \ L^2. \tag{3.30}$$

An application of (3.2811) then shows

$$||\, [P, \hat{B}]u \,|| \leq \mathcal{O}(h^2)||\, u \,||,$$

and applying (3.219) to (3.2911) we get that

$$||\, (P - \operatorname{Re} z)u \,|| \leq \mathcal{O}(h^{3/2})||\, u \,||,$$

so that $\operatorname{dist}(\operatorname{Spec}(P), \operatorname{Re} z) = \mathcal{O}(h^{3/2})$. To conclude, we only have to use the fact that under the assumption (1.43), it is true that

$$\operatorname{Spec}(P) \cap [\alpha, \beta] \subset \cup_k [h^2(k + \alpha/4)^2 - \mathcal{O}(h^2), h^2(k + \alpha/4)^2 + \mathcal{O}(h^2)], \quad k \in \mathbf{Z},$$

in view of (3.46).

We now come to the proof of the estimate for the imaginary parts (1.63). This comes almost directly from [12], and we shall therefore only indicate how to extract it from there. In doing so we shall continue to work with (3.2911).

Choose a real-valued $\tilde{b} = \tilde{b}_{\mathrm{Re}\, z} \in S^0$ such that $\tilde{b}(x,\xi) = \hat{b}(x,\xi,\mathrm{Re}\, z)$ on $p^{-1}(\mathrm{Re}\, z)$, and

$$\sup \tilde{b} = \sup_{p^{-1}(\mathrm{Re}\, z)} \hat{b}(\mathrm{Re}\, z), \tag{3.31}$$

and

$$\inf \tilde{b} = \inf_{p^{-1}(\mathrm{Re}\, z)} \hat{b}(\mathrm{Re}\, z). \tag{3.32}$$

We have

$$\hat{b}(\mathrm{Re}\, z) = \tilde{b} + \tilde{k}(p - \mathrm{Re}\, z),$$

with $\tilde{k} \in S^{-1}$. Let now $\tilde{B}$ be a self-adjoint $h - \Psi\mathrm{DO}$ with principal symbol $\tilde{b}$, and $\tilde{K}$ an $h - \Psi\mathrm{DO}$ with principal symbol $\tilde{k}$. Then we have that

$$\begin{aligned} & 0 = \mathrm{Im}\,((P + ih\hat{B} + h^2 R(z) - z)u, u) = h(\hat{B}u, u) - \mathrm{Im}\, z||\, u\,||^2 \\ + \quad & \mathcal{O}(h^2)||\, u\,||^2 = h(\tilde{B}u, u) + h(\tilde{K}(P - \mathrm{Re}\, z)u, u) - \mathrm{Im}\, z||\, u\,||^2 \\ + \quad & \mathcal{O}(h^2)||\, u\,||^2 = h(\tilde{B}u, u) - \mathrm{Im}\, z||\, u\,||^2 + \mathcal{O}(h^2)||\, u\,||^2. \end{aligned} \tag{3.33}$$

Here we have also used (3.3011). An application of the semiclassical version of the sharp Gårding inequality (see [5]), shows that

$$(\inf \tilde{b} - \mathcal{O}(h))||\, u\,||^2 \leq (\tilde{B}u, u) \leq (\sup \tilde{b} + \mathcal{O}(h))||\, u\,||^2,$$

and combining (3.3112) and (3.3212) with (3.3312) we conclude that

$$h \inf_{p^{-1}(\mathrm{Re}\, z)} \langle b \rangle_T - \mathcal{O}(h^2) \leq \mathrm{Im}\, z \leq h \sup_{p^{-1}(\mathrm{Re}\, z)} \langle b \rangle_T + \mathcal{O}(h^2). \tag{3.34}$$

In order to derive (1.63) we recall that the period T in (3.3412) is given by $T = T(\mathrm{Re}\, z) = \pi/\sqrt{\mathrm{Re}\, z}$ and that $b(x,\xi,\mathrm{Re}\, z) = 2a(x)\sqrt{z}$. These observations, combined with (3.3412), (3.2710), and the homogeneity properties of the H_p-flow imply that when τ is an eigenfrequency such that $h \sim (\mathrm{Re}\, \tau)^{-1}$, we have

$$(h\mathrm{Re}\, \tau) \inf_{p^{-1}(1)} \langle a \rangle - \mathcal{O}(h) \leq (h\mathrm{Re}\, \tau)\mathrm{Im}\, \tau \leq (h\mathrm{Re}\, \tau) \sup_{p^{-1}(1)} \langle a \rangle + \mathcal{O}(h),$$

and this completes the proof of (1.63).

When proving (1.73) we set

$$\mathcal{P}_t = P + ihtB(z), \quad t \in [0,1].$$

It follows by inspection that the proof of (3.2610) applies to $\mathcal{P}_t$ with all constants uniform in t. We can therefore find a closed rectangular-shaped curve $\gamma_k(h)$ with dist $(z, \mathrm{Spec}\,(\mathcal{P}_t)) \geq h/C_0$, $z \in \gamma_k(h)$, $t \in [0,1]$, with $C_0 > 0$ sufficiently large, and such that $\mathrm{Spec}\,(\mathcal{P}_t) \cap \mathrm{int}(\gamma_k(h)) = \mathrm{Spec}\,(\mathcal{P}_t) \cap I_k(h)$. Here the rectangles $I_k(h)$ have been defined in Theorem 3.2 and $\mathrm{int}(\gamma_k(h))$ denotes the interior of $\gamma_k(h)$. When f is an analytic function in a neighbourhood of the closure of the interior of $\gamma_k(h)$, then according to Section 5 of [12] we have that

$$\sum_{\lambda \in \mathrm{Spec}(\mathcal{P}_t) \cap \mathrm{int}(\gamma_k(h))} f(\lambda) = \mathrm{tr}\left(\frac{1}{2\pi i} \int_{\gamma_k(h)} f(z)(1 - \partial_z \mathcal{P}_t)(z - \mathcal{P}_t)^{-1}\, dz\right). \tag{3.35}$$

Here the right-hand side depends continuously on t, and taking $f = 1$ we get that

$$\#\left(\mathrm{Spec}\,(\mathcal{P}) \cap \mathrm{int}(\gamma_k(h))\right) = \#\left(\mathrm{Spec}\,(P) \cap \mathrm{int}(\gamma_k(h))\right).$$

The latter number is given by the expression (1.73) in view of the results of [13] and [3], see also [8].

Remark. We refer to [10] and [3] for a proof of the fact that there exists a polynomial R such that the multiplicity of the k-th cluster I_k in Theorem 1.1 is equal to $R(k + \alpha/4)$, for k sufficiently large.

Remark. The analysis of the present paper makes it possible to prove that we have a sufficiently good control over the resolvent of $\mathcal{P}$ along all of the contour $\gamma_k(h)$, see [9]. We hope to return to this observation and analyze the trace integrals (3.3513) for more general functions f, in order to describe the distribution of the eigenfrequencies in the clusters more precisely.

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