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Lagrangian tori and spectra for non-selfadjoint operators (based on joint works with J. Sjöstrand and S. Vũ Ngọc)

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1   Introduction and statement of results

In the last few years, starting with a pioneering work [25], it has become increasingly clear that non-selfadjoint operators in dimension two share many of the pleasant features of selfadjoint operators in dimension one. In the semiclassical limit ($h \to 0$), it is frequently possible to give complete asymptotic expansions for individual eigenvalues of such operators, in suitable domains in the complex spectral plane. Roughly speaking, the classical Bohr-Sommerfeld rules for real curves in dimension one [9] are replaced by Bohr-Sommerfeld rules for complex curves in dimension two. The purpose of this talk is to describe recent results in this direction obtained in collaboration with Johannes Sjöstrand and San Vũ Ngọc [16], [15].

We shall be concerned with small non-selfadjoint perturbations of selfadjoint h–pseudodifferential operators. Let us remark that various problems of mathematical physics lead to non-selfadjoint operators of this kind—for instance, studying barrier top resonances for semiclassical Schrödinger operators [18] leads one to consider small complex perturbations of the quantum harmonic oscillator [32]. Also, eventually, we expect such operators to be of significance in the study of the scattering poles for a strictly convex analytic obstacle in $\mathbf{R}^3$. See [33] and references given there. Another motivation for being interested in this class of operators comes from the study of spectra for damped wave equations, and by way of introduction, we shall now proceed to recall the formulation of this problem.

Example. Let M be a compact connected C^∞–Riemannian manifold of dimension ≥ 2, and let Δ be the corresponding Laplace-Beltrami operator. Consider the Cauchy

problem for the wave equation with a damping term,

$$\begin{cases} \left(\partial_t^2 - \Delta + a(x)\partial_t\right) v = 0, \quad (t, x) \in \mathbf{R} \times M, \\ v|_{t=0} = v_0 \in H^1(M), \quad \partial_t v|_{t=0} = v_1 \in L^2(M). \end{cases} \tag{1.1}$$

Here $H^1(M)$ is the standard Sobolev space on M, and the damping coefficient a is a bounded non-negative smooth function on M, which does not vanish identically. The study of the energy decay rates for solutions to (1.1), as $t \to \infty$, in relation to the geometry of the underlying manifold and the control (damping) region has a long tradition, and has been pursued in [28], [21], [19]. See also [3].

Here we are only interested in the stationary problem, obtained by setting $v(x, t) = e^{i\tau t}u(x)$ in (1.1):

$$\left(-\Delta + ia(x)\tau - \tau^2\right) u = 0. \tag{1.2}$$

Using (1.2), it is easy to see that the eigenfrequencies $\tau \in \mathbf{C}$, for which (1.2) has a non-vanishing smooth solution, form a discrete set and are confined to a band along the real axis. When studying τ such that $\operatorname{Re}\tau \gg 1$, $\operatorname{Im}\tau = \mathcal{O}(1)$, it is convenient to reformulate the problem semiclassically, and write

$$\tau = \frac{\sqrt{z}}{h}, \quad 0 < h \ll 1, \quad \operatorname{Re}z \sim 1, \quad \operatorname{Im}z = \mathcal{O}(h).$$

We get

$$(\mathcal{P} - z)\, u = 0, \tag{1.3}$$

where

$$\mathcal{P} = -h^2\Delta + iha(x)\sqrt{z}.$$

In this case, the operator in question is therefore an $\mathcal{O}(h)$–perturbation of the semi-classical Laplacian, and we refer to [31] for the general results on the asymptotic distribution of the eigenfrequencies for (1.2), (1.3). Some further results in the case when the geodesic flow on M is periodic have been obtained in [11].

In this talk, we shall be concerned with non-selfadjoint perturbations of selfadjoint h–pseudodifferential operators, for which the strength ε of the perturbation is given as an additional small parameter, $\varepsilon \in \operatorname{neigh}(0, \mathbf{R})$.

We now come to describe the precise assumptions on the class of operators that we are going to consider. In what follows, we let M denote $\mathbf{R}^2$ or a compact connected real analytic manifold of dimension 2. We then let $\widetilde{M}$ stand for a complexification of M, so that $\widetilde{M} = \mathbf{C}^2$ in the Euclidean case, and in the manifold case, $\widetilde{M}$ is a Grauert tube of M.

When $M = \mathbf{R}^2$, let

$$P_\varepsilon = P(x, hD_x, \varepsilon; h) \tag{1.4}$$

be the h–Weyl quantization on $\mathbf{R}^2$ of a symbol $P(x, \xi, \varepsilon; h)$ depending smoothly on $\varepsilon \in \operatorname{neigh}(0, \mathbf{R})$ and taking values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4$, with

$$|P(x, \xi, \varepsilon; h)| \leq \mathcal{O}(1) m(\operatorname{Re}(x, \xi)) \tag{1.5}$$

there. Here $m \in C^\infty(\mathbf{R}^4)$ is assumed to be an order function, in the sense that $m > 0$ and

$$m(X) \leq C_0 \langle X - Y \rangle^{N_0} m(Y), \ X, Y \in \mathbf{R}^4, \quad C_0, \ N_0 > 0. \tag{1.6}$$

We also assume that $m \geq 1$. Assume further that

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\varepsilon}(x, \xi) h^j, \ h \to 0, \tag{1.7}$$

in the space of such functions. We make the basic assumption of ellipticity at infinity,

$$|p_{0,\varepsilon}(x, \xi)| \geq \frac{1}{C} m(\operatorname{Re}(x, \xi)), \ |(x, \xi)| \geq C, \tag{1.8}$$

for some $C > 0$.

When M is a compact manifold, we let P_ε be a differential operator on M, such that for every choice of local coordinates, centered at some point of M, it takes the form

$$P_\varepsilon = \sum_{|\alpha| \leq m} a_{\alpha,\varepsilon}(x; h)(hD_x)^\alpha, \tag{1.9}$$

where $a_{\alpha,\varepsilon}(x; h)$ is a smooth function of $\varepsilon \in \operatorname{neigh}(0, \mathbf{R})$ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$. We further assume that

$$a_{\alpha,\varepsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha,\epsilon,j}(x) h^j, \ h \to 0, \tag{1.10}$$

in the space of such functions. The semiclassical principal symbol $p_{0,\varepsilon}$, defined on T^*M, takes the form $p_{0,\varepsilon}(x, \xi) = \sum a_{\alpha,\epsilon,0}(x) \xi^\alpha$, if (x, ξ) are canonical coordinates on T^*M, and as in the Euclidean case, we make the ellipticity assumption

$$|p_{0,\varepsilon}(x, \xi)| \geq \frac{1}{C} \langle \xi \rangle^m, \ (x, \xi) \in T^*M, |\xi| \geq C, \tag{1.11}$$

for some large $C > 0$. Here we assume that M has been equipped with some real analytic Riemannian metric, so that $|\xi|$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ are well-defined.

In what follows we shall write p_ε for $p_{0,\varepsilon}$ and simply p for $p_{0,0}$. Assume that

$$P_{\varepsilon=0} \text{ is formally selfadjoint.} \tag{1.12}$$

In the case when M is compact, we let the underlying Hilbert space be $L^2(M, \mu(dx))$ where $\mu(dx)$ is the Riemannian volume element.

It follows from (1.8), (1.11), and (1.12) that P_ε has a discrete spectrum in some fixed neighborhood of $0 \in \mathbf{C}$, when $h > 0, \varepsilon \geq 0$ are sufficiently small, and that the spectrum in this region is contained in a band

$$|\operatorname{Im} z| \leq \mathcal{O}(\varepsilon). \tag{1.13}$$

We shall also assume that

$$p^{-1}(0) \cap T^*M \text{ is connected,} \tag{1.14}$$

and that the energy level $E = 0$ is non-critical for p, so that $dp \neq 0$ along $p^{-1}(0) \cap T^*M$.

In what follows we shall write

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2), \tag{1.15}$$

near $p^{-1}(0) \cap T^*M$, and for simplicity we shall assume that the leading perturbation q is real on the real domain. (In the general case, we should simply replace q below by $\operatorname{Re} q$.)

Our goal is to understand the asymptotic behavior of the individual eigenvalues of P_ε in the spectral band (1.13), in the semiclassical limit $h \to 0$. As suggested by the selfadjoint theory [20], [26], in order to do this we should make some assumptions about the classical flow in $p^{-1}(0) \cap T^*M$.

Let

$$H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial \xi}$$

be the Hamilton field of p. The first and the simplest case to consider concerns the situation when the H_p–flow is periodic in an energy shell $p^{-1}(I) \cap T^*M$, where I is a bounded neighborhood of $0 \in \mathbf{R}$, and a series of papers [12]—[14] by Johannes Sjöstrand and the author has addressed this problem. The study of operators with periodic classical flow has a long tradition in the selfadjoint theory (see [34], [5], [17]), and proceeding in the spirit of these works, the starting point of [12]—[14] was to obtain a reduction to a one-dimensional spectral problem, by means of an averaging procedure.

In the first part of this talk, we shall concentrate on the case when the H_p–flow is completely integrable. This requires that we first discuss the precise assumptions on the geometry of the unperturbed energy surface, which we proceed to do now, following [16], [15].

Let us assume that $p^{-1}(0) \cap T^*M$ decomposes into a disjoint union

$$p^{-1}(0) \cap T^*M = \bigcup_{\Lambda \in J} \Lambda, \tag{1.16}$$

where Λ are compact connected H_p–invariant sets. As in [16], we assume that J is a finite connected graph, with S denoting the set of vertices—the graph of a singular Lagrangian foliation of the energy surface. (See also [7] where this point of view is exploited.) Assume that the union of the edges $J \backslash S$ has a natural real analytic structure and that every $\Lambda \in J \backslash S$ is an analytic Lagrangian torus depending analytically on Λ with respect to that structure.

We identify each edge of J analytically with a real bounded interval and this determines a distance on J in the natural way. Assume that we have the continuity property

$$\text{For every } \Lambda_0 \in J \text{ and every } \varepsilon > 0, \exists \delta > 0, \text{ such that if} \tag{1.17}$$
$$\Lambda \in J, \text{ dist}(\Lambda, \Lambda_0) < \delta, \text{ then } \Lambda \subset \{\rho \in p^{-1}(0) \cap T^*M; \text{ dist}(\rho, \Lambda_0) < \varepsilon\}.$$

Now associated with each regular torus $\Lambda \in J \backslash S$ there is a rotation number $\omega(\Lambda) \in \mathbf{RP}^1$, depending analytically on Λ. Quite explicitly, if we represent Λ as

$$\Lambda \simeq \{\xi = 0\} \subset T^*\mathbf{T}^2, \quad \mathbf{T}^2 := \mathbf{R}^2/2\pi\mathbf{Z}^2,$$

using the action-angle coordinates in a neighborhood of Λ, so that $p = p(\xi)$, then we define $\omega(\Lambda) = [p'_{\xi_1}(0) : p'_{\xi_2}(0)]$. Assume that

$$\omega(\Lambda) \text{ is not identically constant on any open edge.}$$

Recall next that q has been introduced in (1.15). For each $\Lambda \in J \backslash S$, we define the torus average $\langle q \rangle_\Lambda$, obtained by integrating $q|_\Lambda$ with respect to the natural smooth measure on Λ, and assume that $\langle q \rangle_\Lambda$ depends analytically on $\Lambda \in J \backslash S$ and is not identically constant on any open edge. Also, assume that $\langle q \rangle_\Lambda$ extends continuously to all of J.

We introduce

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(tH_p) \, dt, \quad T > 0, \tag{1.18}$$

and define the compact intervals $Q_\infty(\Lambda) \subset \mathbf{R}$, $\Lambda \in J$,

$$Q_\infty(\Lambda) = \left[\lim_{T \to \infty} \inf_\Lambda \langle q \rangle_T, \lim_{T \to \infty} \sup_\Lambda \langle q \rangle_T\right]. \tag{1.19}$$

Notice that when $\Lambda \in J \backslash S$ and $\omega(\Lambda) \notin \mathbf{Q}$ then the restriction of the H_p–flow to Λ is ergodic so that $Q_\infty(\Lambda) = \{\langle q \rangle_\Lambda\}$. In the rational case, we write $\omega(\Lambda) = \frac{m}{n}$, where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$ are relatively prime, and with $k(\omega(\Lambda)) := |m| + |n|$, we recall from [16] that

$$Q_\infty(\Lambda) \subset \langle q \rangle_\Lambda + \mathcal{O}\left(\frac{1}{k(\omega(\Lambda))^\infty}\right)[-1, 1]. \tag{1.20}$$

Using suitable phase space exponential weights corresponding to the averaging along the H_p–flow together with the sharp Gårding inequality, one then shows that

$$\frac{1}{\varepsilon}\operatorname{Im}\left(\operatorname{Spec}(P_\varepsilon)\cap\{z;|\operatorname{Re}z|\le\delta\}\right)\subset\left[\inf\bigcup_{\Lambda\in J}Q_\infty(\Lambda)-o(1),\sup\bigcup_{\Lambda\in J}Q_\infty(\Lambda)+o(1)\right],$$

as ε, h, $\delta\to 0$. This establishes the first rough bound on the location of the spectrum inside the band (1.13).

Let $\alpha>0$, $d>0$. In what follows we shall say that the torus $\Lambda\in J\backslash S$ is (α,d)–Diophantine if

$$\left|\omega(\Lambda)-\frac{p}{q}\right|\ge\frac{\alpha}{q^{2+d}},\quad p\in\mathbf{Z},\quad q\in\mathbf{N}.\tag{1.21}$$

Here we view $\omega(\Lambda)$ as an element of $\mathbf{R}$.

The following result has been established in [16].

Theorem 1.1 (The Diophantine Case.) *When $\alpha>0$ and $d>0$, define*

$$\mathcal{G}_{\alpha,d}=\bigcup_{\Lambda\in J}Q_\infty(\Lambda)\setminus\mathcal{B}_{\alpha,d},\tag{1.22}$$

where $\mathcal{B}_{\alpha,d}$ is the following set:

$$\left(\bigcup_{\operatorname{dist}(\Lambda,S)<\alpha}Q_\infty(\Lambda)\right)\bigcup\left(\bigcup_{\Lambda\in J\backslash S,\,|d_\Lambda\omega(\Lambda)|<\alpha}Q_\infty(\Lambda)\right)\bigcup\left(\bigcup_{\Lambda\in J\backslash S,\,|d_\Lambda\langle q\rangle_\Lambda|<\alpha}Q_\infty(\Lambda)\right)$$
$$\bigcup\left(\bigcup_{\Lambda\in J\backslash S,\,\Lambda\,is\,not\,(\alpha,d)-Diophantine}Q_\infty(\Lambda)\right).$$

When $F_0\in\mathcal{G}_{\alpha,d}$, write

$$\langle q\rangle_\Lambda^{-1}(F_0)=\{\Lambda_1,\dots\Lambda_L\},$$

where $\Lambda_j\in J\backslash S$, $1\le j\le L$, are (α,d)–Diophantine tori $\subset p^{-1}(0)\cap T^*M$. Let $S_j\in\mathbf{R}^2$ be the actions and $k_j\in\mathbf{Z}^2$ be the Maslov indices of the fundamental cycles $\alpha_{k,j}$ in Λ_j, defined by $\kappa_j(\alpha_{k,j})=\{x\in\mathbf{T}^2;x_k=0\}$, $k=1,2$, where

$$\kappa_j:\operatorname{neigh}(\Lambda_j,T^*M)\to\operatorname{neigh}(\xi=0,T^*\mathbf{T}^2)\tag{1.23}$$

is the canonical transformation given by the action-angle variables near Λ_j, $1\le j\le L$.

Assume that $\varepsilon=\mathcal{O}(h^{\delta_0})$, $\delta_0>0$, satisfies $\varepsilon\ge h^K$, where $K\gg 1$ is fixed. Then the eigenvalues of P_ε in

$$|\operatorname{Re}z|<\frac{\varepsilon^\delta}{\mathcal{O}(1)},\quad|\operatorname{Im}z-\varepsilon F_0|<\frac{\varepsilon^{1+\delta}}{\mathcal{O}(1)},\quad\delta>0,\tag{1.24}$$

are given by

$$P_j^{(\infty)}\left(h\left(k - \frac{k_j}{4}\right) - \frac{S_j}{2\pi}, \varepsilon; h\right) + \mathcal{O}(h^\infty),$$

for $k \in \mathbf{Z}^2$, $1 \le j \le L$. Here $P_j^{(\infty)}(\xi, \varepsilon; h)$ is smooth in $\xi \in \mathrm{neigh}(0, \mathbf{R}^2)$ and $\varepsilon \in \mathrm{neigh}(0, \mathbf{R})$, and real-valued for $\varepsilon = 0$. We have

$$P_j^{(\infty)}(\xi, \varepsilon; h) \sim \sum_{\ell=0}^\infty h^\ell p_{j,\ell}^{(\infty)}(\xi, \varepsilon), \quad 1 \le j \le L,$$

and

$$p_{j,0}^{(\infty)}(\xi, \varepsilon) = p(\xi) + i\varepsilon\langle q\rangle(\xi) + \mathcal{O}(\varepsilon^2).$$

Here p and q have been expressed in terms of the action-angle coordinates near Λ_j, given by κ_j in (1.23), and $\langle q\rangle$ is the torus average of q in these coordinates.

Remarks.

- The eigenvalues of P_ε in the spectral window (1.24), for each $F_0 \in \mathcal{G}_{\alpha,d}$, form a superposition of a finite number of slightly distorted lattices, each Diophantine torus contributing its own lattice of quasi-eigenvalues. It would be interesting to see whether such lattices could be observed numerically.

- The measure of the "bad" set $\mathcal{B}_{\alpha,d} \subset \cup_{\Lambda \in J} Q_\infty(\Lambda)$ is small when $\alpha > 0$ is small enough, provided that the measure of the set

$$\bigcup_{\Lambda \in \omega^{-1}(\mathbf{Q}) \cup S} Q_\infty(\Lambda)$$

  is sufficiently small.

- Theorem 1.1 applies to the spectral problem (1.2), (1.3) for the damped wave equation on an analytic strictly convex surface of revolution, say. In this case, $\varepsilon \sim h$.

Remark. Using the isoenergetic KAM theorem [2], it is possible to show that the result of Theorem 1.1 is stable with respect to small real perturbations of the leading symbol of $P_{\varepsilon=0}$, p, destroying the complete integrability of the classical flow. We refer to Section 7 of [16] for a detailed discussion of this important point, and just remark here that the required smallness of the real perturbation of p depends only on the Diophantine parameter α in (1.21), provided that $d > 0$ is kept fixed.

Remark. The second main result of [16] establishes that, suitably extended, Theorem 1.1 is valid also in the range

$$h^{\frac{1}{3}-\delta} < \varepsilon \le \varepsilon_0 \ll 1, \quad \delta > 0.$$

We now come to describe the second main result of this talk [15], dealing with the contributions to the spectrum of P_ε coming from the flow–invariant Lagrangian tori that are rational.

Theorem 1.2 (The rational case.) *Let $\Lambda_0 \subset p^{-1}(0) \cap T^*M$ be an invariant Lagrangian torus such that $\omega(\Lambda_0) \in \mathbf{Q}$ and assume that*

$$d_{\Lambda=\Lambda_0}\omega(\Lambda) \neq 0.$$

Let $F_0 \in Q_\infty(\Lambda_0)$ be such that

$$\langle q \rangle_{\Lambda_0} \geq F_0 + \frac{1}{\mathcal{O}(1)}, \tag{1.25}$$

and assume that

$$F_0 \notin \bigcup_{\Lambda \notin \text{neigh}(\Lambda_0)} Q_\infty(\Lambda). \tag{1.26}$$

Assume furthermore that $\varepsilon = \mathcal{O}(h^{\delta_0})$, $\delta_0 > 0$, satisfies $\varepsilon \gg h$. Then the number of eigenvalues of P_ε in the rectangle

$$|\text{Re}\, z| < \frac{\varepsilon}{\mathcal{O}(1)}, \quad \left|\frac{\text{Im}\, z}{\varepsilon} - F_0\right| < \frac{1}{\mathcal{O}(1)} \tag{1.27}$$

does not exceed

$$\mathcal{O}\left(\frac{\varepsilon^{3/2}}{h^2}\right). \tag{1.28}$$

Remark. Let $F_0 \in \mathcal{G}_{\alpha,d}$, with the latter set defined in (1.22). Then it follows from Theorem 1.1 that the number of eigenvalues of P_ε in a domain of the form

$$|\text{Re}\, z| \leq \frac{\varepsilon}{\mathcal{O}(1)}, \quad \left|\frac{\text{Im}\, z}{\varepsilon} - F_0\right| \leq \frac{\varepsilon^\delta}{\mathcal{O}(1)}, \quad \delta > 0,$$

is

$$\sim \frac{\varepsilon^{1+\delta}}{h^2} \gg \frac{\varepsilon^{3/2}}{h^2},$$

if $\delta > 0$ is small enough. The result of Theorem 1.2 may therefore be interpreted as saying that the contribution to the spectrum coming from a rational region in this case is much weaker than that of the Diophantine tori. (Notice however that here the assumption (1.25) is made.) When trying to go further and obtain more complete results about the distribution of individual eigenvalues, beyond the counting estimate (1.28), we run into pseudospectral difficulties, typical for non-selfadjoint spectral problems [8].

Remark. It would be very interesting and natural to try to understand the distribution of eigenvalues of P_ε near the energy levels $(0, F_0) \in \mathbf{C}$, where $F_0 \in \cup_{\Lambda \in J} Q_\infty(\Lambda)$ corresponds to a combination of both the Diophantine and the rational tori. Effectively,

this should require estimating and exploiting the tunnel effect between the Diophantine and rational regions. The work in this direction is currently in progress [15].

2   Some ideas of the proofs of Theorems 1.1 and 1.2

Referring to [16] and [15] for the complete details of the proofs, here we shall merely indicate the main steps along the way.

The starting point of the proof of Theorem 1.1 is a formal quantum Birkhoff normal form construction for P_ε near a fixed Diophantine torus, say $\Lambda_1 \subset p^{-1}(0) \cap T^*M$. Introducing the action-angle coordinates near Λ_1 and passing to the torus model, we may assume that the operator P_ε is defined microlocally near $\xi = 0$ in $T^*\mathbf{T}^2$ and has the leading symbol

$$p_\varepsilon(x,\xi) = p(\xi) + i\varepsilon q(x,\xi) + \mathcal{O}(\varepsilon^2), \tag{2.1}$$

with $p(\xi) = a \cdot \xi + \mathcal{O}(\xi^2)$, and $a = (a_1, a_2) \in \mathbf{R}^2$ such that $\omega(\Lambda_1) = [a_1 : a_2]$ satisfies the Diophantine condition (1.21). One then constructs a holomorphic canonical transformation κ_ε defined in a fixed complex neighborhood of the zero section of $T^*\mathbf{T}^2$, such that

$$p_\varepsilon \circ \kappa_\varepsilon = p^{(N)}(\xi, \varepsilon) + r_{N+1}(x, \xi, \varepsilon),$$

where

$$p^{(N)}(\xi, \varepsilon) = p(\xi) + i\varepsilon \langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2), \quad \langle q \rangle(\xi) = \frac{1}{(2\pi)^2} \int q(x, \xi)\, dx \tag{2.2}$$

is independent of x, and $r_{N+1}(x, \xi, \varepsilon) = \mathcal{O}((\xi, \varepsilon)^{N+1})$. Here $N \in \mathbf{N}$ is arbitrarily large but fixed.

On the operator level, one next gets a reduction of P_ε to an operator of the form

$$P^{(N)}(hD_x, \varepsilon; h) + R_{N+1}(x, hD_x, \varepsilon; h), \tag{2.3}$$

where the full symbol of $P^{(N)}(hD_x.\varepsilon; h)$ is independent of x and $R_{N+1}(x, \xi, \varepsilon; h) = \mathcal{O}\left((h, \xi, \varepsilon)^{N+1}\right)$. The operator in (2.3) acts on the space $L^2_\theta(\mathbf{T}^2)$ of microlocally defined Floquet periodic functions on $\mathbf{T}^2$, the elements u of which satisfy

$$u(x - \nu) = e^{i\theta \cdot \nu} u(x), \quad \nu \in 2\pi \mathbf{Z}^2, \quad \theta = \frac{S_1}{2\pi h} + \frac{k_1}{4} \in \mathbf{R}^2.$$

The principal symbol of $P^{(N)}(hD_x, \varepsilon; h)$ is $p^{(N)}(\xi, \varepsilon)$ given by (2.2).

Remark. The construction of the quantum Birkhoff normal form around flow–invariant Diophantine tori and Cantor families of such tori, as well as the study of the associated quasimodes, has a long tradition in the selfadjoint theory—see [20], [4], [27], [6].

The main part of the proof then consists of justifying the preceding formal construction and showing that in the case when $F_0 \in \mathcal{G}_{\alpha,d}$, the formal quasi-eigenvalues coming from the Birkhoff normal form construction near the Diophantine tori Λ_j, $1 \leq j \leq L$, give *all* of the actual eigenvalues of P_ε modulo $\mathcal{O}(h^\infty)$, in a region of the form (1.24). The crucial step is given by the proof of the following result.

Proposition 2.1 *Let $F_0 \in \mathcal{G}_{\alpha,d}$. There exists an IR–manifold $\Lambda \subset T^*\widetilde{M}$, ε–close to T^*M and equal to T^*M outside a compact set, such that away from a ε^δ–neighborhood of $\cup_{j=1}^L \Lambda_j$ in Λ, $\delta > 0$, we have*

$$|\operatorname{Re} P_\varepsilon| \geq \frac{\varepsilon^\delta}{\mathcal{O}(1)} \quad or \quad |\operatorname{Im} P_\varepsilon - \varepsilon F_0| \geq \frac{\varepsilon^{1+\delta}}{\mathcal{O}(1)}. \tag{2.4}$$

For each j with $1 \leq j \leq L$, there exists an elliptic Fourier integral operator

$$U_j = \mathcal{O}(1) : H(\Lambda) \to L^2_\theta(\mathbf{T}^2), \tag{2.5}$$

such that microlocally near Λ_j,

$$U_j P_\varepsilon = \left(P_j^{(N)}(hD_x, \varepsilon; h) + R_{N+1,j}(x, hD_x, \varepsilon; h) \right) U_j.$$

Here $P_j^{(N)}(hD_x, \varepsilon; h) + R_{N+1,j}(x, hD_x, \varepsilon; h)$ is defined as in (2.3) when $j = 1$.

Remark. The h–dependent Hilbert space $H(\Lambda)$, occurring in (2.5), is naturally associated to Λ and is defined using the techniques of [29], [30], by modifying the exponential weight on the FBI–Bargmann transform side.

It follows from Proposition 2.1 that the entire spectral problem for P_ε becomes microlocalized to a small neighborhood of the union of the Diophantine tori, where one has a good description of the operator thanks to the Birkhoff normal form. With Proposition 2.1 available, Theorem 1.1 follows by solving an appropriate globally well-posed Grushin problem for P_ε in the space $H(\Lambda)$.

Remark. To reach sufficiently small but h–independent values of ε, the possibility of which has been alluded to in a remark in Section 1, one needs to combine the Birkhoff normal form construction for p_ε described above with the additional idea due to [25] of working with cohomological equations of $\overline{\partial}$–type on the standard 2–torus, thereby liberating ourselves from Diophantine conditions. See also [32].

As the proof of Theorem 1.2 is somewhat more technical, in the remainder of this talk, we shall only indicate briefly the main ingredients in the proof of this result.

The starting point in the proof is a microlocal normal form construction for P_ε near $\Lambda_0 \simeq \mathbf{T}^2$. As opposed to the Diophantine case, in the present situation one relies upon secular perturbation theory [22], and carries out successive averagings of the lower order symbols of P_ε along the family of closed H_p–orbits comprising the rational torus Λ_0. In the simplest case, a local model for the leading symbol of P_ε near $\xi = 0$ in $T^*\mathbf{T}^2$, after the secular reduction, is given by

$$p_\varepsilon(x, \xi) = \xi_2 + \xi_1^2 + \mathcal{O}(\varepsilon) + \mathcal{O}((\varepsilon, \xi_1)^\infty), \tag{2.6}$$

where the $\mathcal{O}(\varepsilon)$–term in (2.6) is independent of x_2. Using the assumptions (1.25) and (1.26), it is then possible to show that, roughly speaking, only the phase space region

$$\xi_1 = \mathcal{O}(\varepsilon^{1/2}), \quad \xi_2 = \mathcal{O}(\varepsilon) \tag{2.7}$$

can contribute to the spectrum of P_ε in the domain (1.27). To handle the region in (2.7), one constructs a trace class operator K (acting on a suitable h–dependent Hilbert space associated to an appropriate IR–deformation of T^*M), of trace class norm $\mathcal{O}(\varepsilon^{3/2})h^{-2}$, such that the operator $P_\varepsilon + i\varepsilon K - z$ becomes invertible for z in the domain (1.27), with an $\mathcal{O}(\varepsilon^{-1})$–control on the norm of the inverse. One then gets the result of Theorem 1.2 by working with perturbation determinant estimates, in the spirit of the classical theory of non-selfadjoint operators as described in [10], [23].

We would finally like to mention that the trace class operator K above is constructed as a Toeplitz operator on the FBI–Bargmann transform side, and when estimating its trace class norm, we use the following general estimate, which may be perhaps of some independent interest.

Proposition 2.2 *Let $\Phi_0(x)$ be a real strictly plurisubharmonic quadratic form on $\mathbf{C}^n$, and let $\Phi \in C^\infty(\mathbf{C}^n)$ be strictly plurisubharmonic and such that $\Phi - \Phi_0$ is bounded and $\sup \left|\frac{\partial \Phi}{\partial x} - \frac{\partial \Phi_0}{\partial x}\right|$ small enough. Assume also that $\nabla^k \Phi \in L^\infty(\mathbf{C}^n)$ for each $k \geq 2$. When $L_\Phi^2 := L^2(\mathbf{C}^n; e^{-\frac{2\Phi}{h}} L(dx))$, where $L(dx)$ is the Lebesgue measure on $\mathbf{C}^n$, we let H_Φ stand for the holomorphic subspace, and introduce the orthogonal projection*

$$\Pi_\Phi : L_\Phi^2 \to H_\Phi.$$

Let $p \in C_0^\infty(\mathbf{C}^n)$. Then the Toeplitz operator

$$\mathrm{Top}(p) = \Pi_\Phi p \Pi_\Phi = \mathcal{O}(1) : H_\Phi \to H_\Phi$$

is of trace class and we have

$$\|\,\mathrm{Top}(p)\,\|_{\mathrm{tr}} \leq \frac{\mathcal{O}(1)}{h^n}. \tag{2.8}$$

Proposition 2.2 is a simple consequence of the asymptotic description of the Bergman projector Π_Φ, established in [24], in the spirit of [1].

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