Metric graphs, stable polynomials and Fourier quasicrystals
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Spectra of Laplacians on metric graphs, also known as *quantum graphs*, have been intensively studies in recent years due to possible applications to nano-physics. Let us restrict our studies to finite metric graphs formed from compact intervals $e_n,\ n = 1, 2, \ldots, N$, of lengths ℓ_n and Laplace operators $-\frac{d^2}{dx^2}$ with standard vertex conditions:

- the function is continuous at the vertex (continuity condition);
- the sum of outgoing first derivates at the vertex is equal to zero (Kirchhoff condition).

One of the most interesting results is the trace formula connecting the spectrum $\lambda_j = k_j^2$ to geometric and topologic properties of the metric graph [2–4, 9]:

$$\underbrace{\sum_{k_n \neq 0} \left(\delta(k-k_n) + \delta(k+k_n)\right)}_{\text{spectral information}} = \underbrace{\underbrace{1-\beta_1}_{=\chi} \delta(k) + \frac{\mathcal{L}}{\pi} + \frac{1}{\pi} \sum_{p \in \mathcal{P}} l(\text{prim}\,(p)) S_{\text{v}}(p) \cos k\ell(p)}_{\text{geometric/topologic information}}$$

where

- $\mathcal{L} = \sum_{n=1}^{N} \ell_n$ - the total length of the graph;
- χ – Euler characteristic of Γ;
- β_1 – number of independent cycles in Γ;
- $\mathcal{P}$ – the set of closed oriented paths p on Γ;
- $\ell(p)$ – length of the closed path p;
- $S_{\text{v}}(p)$ – product of all vertex scattering coefficients along the path p.

This formula is a direct generalisation of the classical Poisson summation formula and coincides with it if the graph is just one interval with Neumann conditions at the end points. In contrast to similar formulas for Riemanian manifolds the obtained trace formula is exact.

It appears that this formula is extremely interesting for Fourier analysis since it provides explicit examples of crystalline measure, which can be defined following Y. Meyer as [7]:

A discrete measure μ is **crystalline** *if it is a tempered distribution and if the measure itself and its Fourier transform $\hat{\mu}$ are sums of delta functions with discrete supports:*

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \qquad \hat{\mu} = \sum_{s \in S} b_s \delta_s.$$

Collecting all delta function on the left hand side the trace formula can be written as:

$$\sum_{k_n \neq 0} \left(\delta(k-k_n) + \delta(k+k_n)\right) - \chi\delta(k) = \frac{\mathcal{L}}{\pi} + \frac{1}{\pi} \sum_{p \in \mathcal{P}} l(\text{prim}\,(p)) S_{\text{v}}(p) \cos k\ell(p).$$

One may obtain similar summation formulas starting from multivariate stable polynomials [5]. The supports of the corresponding measures are described as zeroes of trigonometric polynomials. If the multivariate polynomials in addition are symmetric, *i.e.* invariant under involution $z_j \mapsto 1/z_j$, then the trigonometric polynomials have only real zeroes and the corresponding measures are crystalline measures.

It appears that all one-dimensional crystalline measures are given by real rooted trigonometric polynomials [8]. It was proven recently that all such polynomials can be obtained using our construction via multivariate stable polynomials [1]. Spectral properties of Laplacians on metric graphs are further described in [3, 6].

This is a joint work with Peter Sarnak.

References

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